Nonlocal-interaction equation on graphs: gradient flow structure and continuum limit

Joint works with F. S. Patacchini, A. Schlichting, and D. Slepčev

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Department MATHEMATIK

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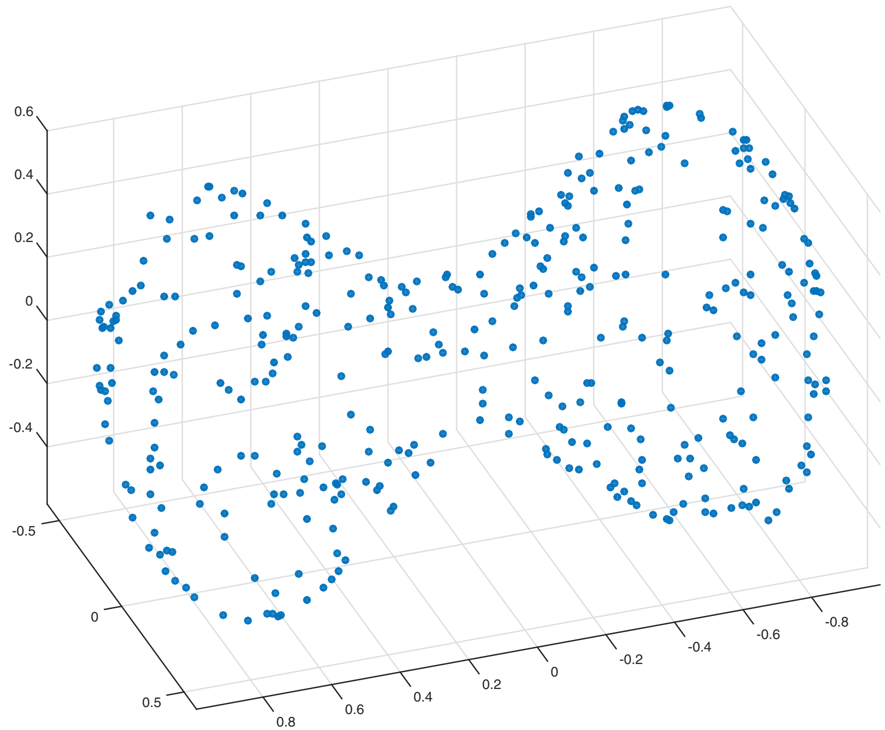
- $X = \{x_1, x_2, ..., x_n\}$ random sample i.i.d. according to $\mu \in \mathcal{M}^+(\mathbb{R}^d)$
  $\Rightarrow$ empirical measure $\mu^n = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$



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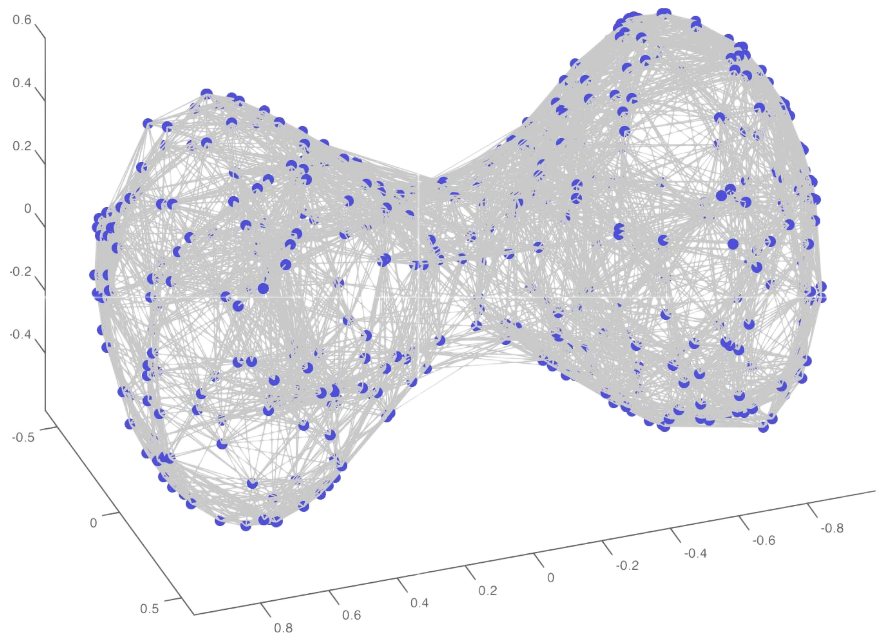
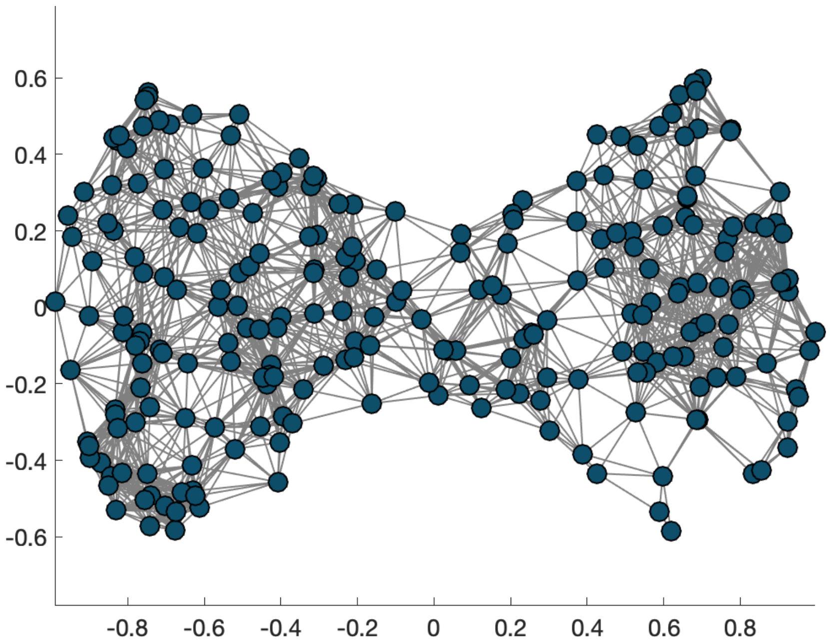
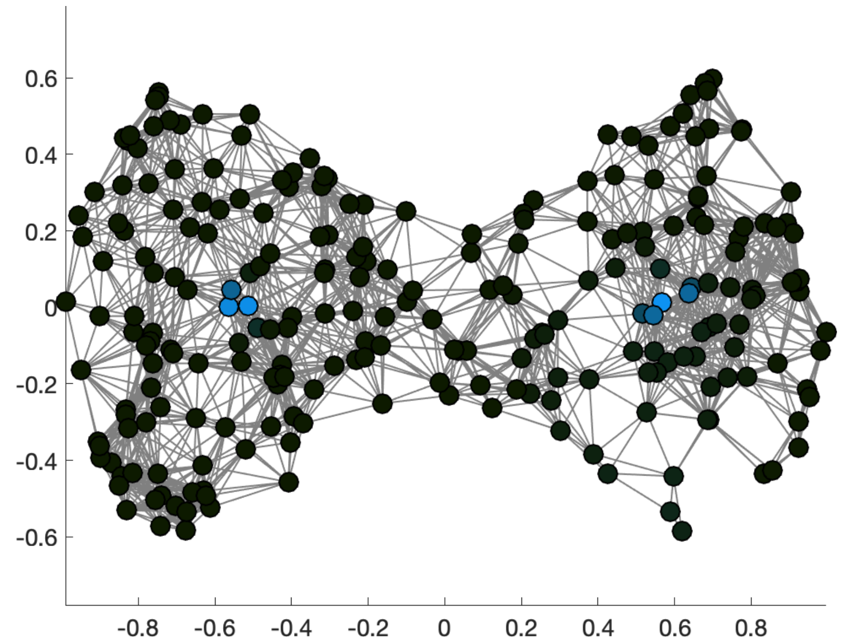
  $\Rightarrow$ empirical measure $\mu^n = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$
- a symmetric weight function $\eta : D \to [0, \infty)$ with $D := (\mathbb{R}^d \times \mathbb{R}^d) \setminus \{x = y\}$
  $\Rightarrow (\mu^n, \eta)$ defines an undirected discrete weighted graph



Dynamics driven by interaction energies on graphs





Video

Dynamics driven by interaction energies on graphs

$$\mathcal{E}_X(\rho) = \frac{1}{2} \sum_{x \in X} \sum_{y \in X} K_{x,y} \rho_x \rho_y \tag{1}$$

On $\mathbb{R}^d$

$$\dot{x}_i = - \sum_{j=1}^{n} \rho_j \nabla_x K(x_i, x_j) \tag{2}$$

On finite graphs

$$\frac{d\rho_x}{dt} = - \sum_{y \in X} j_{x,y} \eta(x, y) \tag{3}$$

$$j_{x,y} = I\big(\rho_x, \rho_y\big) v_{x,y} \tag{4}$$

Goals

- Define gradient flow of interaction energy on graph (μ, η)
- Dynamics stable under graph limit $n \to \infty$ (discrete-to-continuum)
- Dynamics stable for local limit: $\mu =$ Leb$(\mathbb{R}^d)$, $\eta^\varepsilon(x, y) = \varepsilon^{-d} \eta\left(\frac{x-y}{\varepsilon}\right)$
  $\Rightarrow$ limit $\varepsilon \to 0$ should give $\partial_t \rho = \nabla \cdot (\rho \nabla K * \rho)$

Dynamics driven by interaction energies on graphs

General framework

- $\mathbb{R}^d$ set of possible vertices, $\mathbb{R}^d \times \mathbb{R}^d \setminus \{x = y\}$ set of possible edges
- $\eta : \mathbb{R}^d \times \mathbb{R}^d \setminus \{x = y\} \to [0, \infty)$ symmetric weight function
- $G := \{\mathbb{R}^d \times \mathbb{R}^d \setminus \{x = y\} | \eta(x, y) > 0\}$ set of edges
- $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ set of vertices
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Evolution of interest

Gradient descent of the energy $\mathcal{E} : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ given by

$$\mathcal{E}(\rho) = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(x, y) \, d\rho(x) \, d\rho(y),$$

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Continuum setting: NLIE

$\partial_t \rho = \nabla \cdot (\rho \nabla K * \rho)$ is a Wasserstein gradient flow for $\mathcal{E}$[a]

[a] J.A. Carrillo, M. Di Francesco, A. Figalli, T. Laurent, D. Slepčev - Duke Math. J. 156 (2011)

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What is the analogue of the NLIE on a graph?

Related Literature (not exhaustive!)

- [Maas '11] / [Mielke '11] / [Chow, Huang, Li, Zhou '12]
  Diffusion on graphs as gradient flows of the entropy
  $\Rightarrow$ Wassertein metric on a finite graph
- [Erbar '14] Jump processes $-(-\Delta)^{\alpha/2}$ for $\alpha \in (0, 2)$
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Gradient flows for free energies/(relative) entropies:

$$\mathcal{F}^\sigma(\rho) = \sigma \int \rho(x) \log \rho(x)\, dx + \frac{1}{2} \iint K(x, y)\, d\rho(x)\, d\rho(y)$$

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What if $\sigma = 0$?

Nonlocal metrics introduced above do not have a clear/well-defined limit for $\sigma \to 0$!

What is a suitable metric for gradient structure of interaction energies?

Nonlocal continuity equation

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$$\partial_t \rho_t(x) + (\overline{\nabla} \cdot j_t)(x) = \partial_t \rho_t(x) + \int_{\mathbb{R}^d} j_t(x,y)\, \eta(x,y)\, dy = 0$$

Flux: defined on the edges!
Velocity: jump rate $\Rightarrow v_t : G \to \mathbb{R}$ nonlocal (antisymmetric) vector field [edge-based quantity]
Density: vertex-based quantity

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Upwind interpolation: density along edges = density at the source

Set $(a)_+ = \max\{0, a\}$ and $(a)_- = \max\{0, -a\}$ and define

$$j_t(x,y) = \rho(x) v_t(x,y)_+ - \rho(y) v_t(x,y)_-$$

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Nonlocal continuity equation

For $\rho_t \ll \mu$

$$\partial_t \rho_t(x) + \int_{\mathbb{R}^d} \left(\rho_t(x) v_t(x,y)_+ - \rho_t(y) v_t(x,y)_-\right) \eta(x,y)\, d\mu(y) = 0 \qquad \text{(NCE)}$$

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Benamou-Brenier

$$W_2^2(\rho_0, \rho_1) = \inf \left\{ \frac{1}{2} \int_0^1 \int_{\mathbb{R}^d} |v_t(x)|^2 \rho_t(x) \, dx \, dt \mid (\rho_t, v_t) \in \mathsf{CE}(\rho_0, \rho_1) \right\}$$

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Upwind nonlocal transportation metric: Benamou-Brenier

$$\inf_{(\rho,v)\in \mathrm{CE}(\rho_0,\rho_1)} \left\{ \frac{1}{2} \int_0^1 \iint_G \left(|v_t(x,y)_+|^2 \rho_t(x) + |v_t(x,y)_-|^2 \rho_t(y) \right) \eta(x,y)\, d\mu(x)\, d\mu(y)\, dt \right\}$$

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Nonlocal interaction equation on graphs: NL^{2}IE

If $v_t = -\overline{\nabla} \frac{\delta \mathcal{E}}{\delta \rho} = -\overline{\nabla} K * \rho_t$

$$\partial_t \rho_t(x) + \int_{\mathbb{R}^d} \left(\rho_t(x) \overline{\nabla}(K * \rho_t)(x,y)_- - \rho_t(y) \overline{\nabla}(K * \rho_t)(x,y)_+ \right) \eta(x,y)\, d\mu(y) = 0$$

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Note that:

- ρ might contain atoms, even if μ is Lebesgue!
  $\Rightarrow$ measure valued framework
- Benamou-Brenier functional is not jointly convex in (ρ_t, v_t)
  $\Rightarrow$ flux variables

Action

Definition

For $\mu \in \mathcal{M}^+(\mathbb{R}^d)$, $\rho \in \mathcal{P}(\mathbb{R}^d)$ and $\boldsymbol{j} \in \mathcal{M}(G)$, consider $\lambda \in \mathcal{M}(G)$ such that $\rho \otimes \mu, \mu \otimes \rho, |\boldsymbol{j}| \ll |\lambda|$. We define

$$\mathcal{A}(\mu;\rho,\boldsymbol{j}) = \frac{1}{2}\iint_G \left(\alpha\left(\frac{d\boldsymbol{j}}{d|\lambda|}, \frac{d(\rho\otimes\mu)}{d|\lambda|}\right) + \alpha\left(-\frac{d\boldsymbol{j}}{d|\lambda|}, \frac{d(\mu\otimes\rho)}{d|\lambda|}\right)\right)\eta\, d|\lambda|. \tag{1}$$

Hereby, the lower semicontinuous, convex, and positively one-homogeneous function $\alpha\colon \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ is defined, for all $j \in \mathbb{R}$ and $r \geq 0$, by

$$\alpha(j,r) := \begin{cases} \frac{(j_+)^2}{r} & \text{if } r > 0, \\ 0 & \text{if } j \leq 0 \text{ and } r = 0, \\ \infty & \text{if } j > 0 \text{ and } r = 0, \end{cases} \tag{2}$$

with $j_+ = \max\{0, j\}$. If the measure μ is clear from the context, we write $\mathcal{A}(\rho,\boldsymbol{j})$ for $\mathcal{A}(\mu;\rho,\boldsymbol{j})$.

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If $\rho \ll \mu$ and $\boldsymbol{j} \ll \mu\otimes\mu$

$$\mathcal{A}(\mu;\rho,\boldsymbol{j}) = \frac{1}{2}\iint_G \left(\frac{(j(x,y)_+)^2}{\rho(x)} + \frac{(j(x,y)_-)^2}{\rho(y)}\right)\eta(x,y)\, d\mu(x)\, d\mu(y)$$

Lemma (Finite action $\Rightarrow$ upwind flux)

Let $\mu \in \mathcal{M}^+(\mathbb{R}^d)$, $\rho \in \mathcal{P}(\mathbb{R}^d)$ and $\boldsymbol{j} \in \mathcal{M}(G)$ be such that $\mathcal{A}(\mu;\rho,\boldsymbol{j}) < \infty$. Then there exists a measurable $v\colon G \to \mathbb{R}$ such that

$$d\boldsymbol{j}(x,y) = v(x,y)_+ \, d\rho(x)\, d\mu(y) - v(x,y)_- \, d\mu(x)\, d\rho(y), \tag{3}$$

and it holds

$$\mathcal{A}(\mu;\rho,\boldsymbol{j}) = \frac{1}{2}\iint_G \left(|v(x,y)_+|^2 + |v(y,x)_-|^2\right) \eta(x,y)\, d\rho(x)\, d\mu(y). \tag{4}$$

In particular, if $v \in \mathcal{V}^{\mathrm{as}}$, then

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Corollary (Antisymmetric vector fields have lower action)

Let $\mu \in \mathcal{M}^+(\mathbb{R}^d)$, $\rho \in \mathcal{P}(\mathbb{R}^d)$ and $\boldsymbol{j} \in \mathcal{M}(G)$ be such that $\mathcal{A}(\mu; \rho, \boldsymbol{j}) < \infty$. Then there exists an antisymmetric flux $\boldsymbol{j}^{\mathrm{as}} \in \mathcal{M}^{\mathrm{as}}_{\eta\gamma_1}$ such that

$$\overline{\nabla} \cdot \boldsymbol{j} = \overline{\nabla} \cdot \boldsymbol{j}^{\mathrm{as}},$$

with lower action:

$$\mathcal{A}(\mu; \rho, \boldsymbol{j}^{\mathrm{as}}) \leq \mathcal{A}(\mu; \rho, \boldsymbol{j}).$$

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Lemma (Lower semicontinuity of the action)

The action is lower semicontinuous with respect to the narrow convergence in $\mathcal{M}^+(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d) \times \mathcal{M}(G)$. That is, if $\mu^n \rightharpoonup \mu$ in $\mathcal{M}(\mathbb{R}^d)$, $\rho^n \rightharpoonup \rho$ in $\mathcal{P}(\mathbb{R}^d)$, and $\boldsymbol{j}^n \rightharpoonup \boldsymbol{j}$ in $\mathcal{M}(G)$, then

$$\liminf_{n \to \infty} \mathcal{A}(\mu^n; \rho^n, \boldsymbol{j}^n) \ge \mathcal{A}(\mu; \rho, \boldsymbol{j}) \, .$$

Nonlocal continuity equation: measure-valued flux form

A pair $(\rho_t, \boldsymbol{j}_t)_{t\in[0,T]} \in \mathrm{CE}_T$ iff $(\rho_t, \boldsymbol{j}_t) \in \mathcal{P}(\mathbb{R}^d) \times \mathcal{M}(G)$ for all $t \in [0,T]$ satisfies

$$\partial_t \rho_t + \overline{\nabla} \cdot \boldsymbol{j}_t = 0 \qquad \text{in } \mathcal{D}'((0,T) \times \mathbb{R}^d),$$

i.e.

$$\int_0^T \int_\Omega \partial_t \varphi_t(x)\, d\rho_t(x)\, dt + \frac{1}{2} \int_0^T \iint_G \overline{\nabla} \varphi_t(x,y)\, \eta(x,y)\, d\boldsymbol{j}_t(x,y)\, dt = 0\,.$$

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$\left|\overline{\nabla}\varphi(x,y)\right| \le \|\varphi\|_{C^1(\Omega)}(2 \wedge |x-y|) \Rightarrow$ well-defined under integrability condition

$$\int_0^T \iint_G (2 \wedge |x-y|)\eta(x,y)\, d|\boldsymbol{j}_t|(x,y)\, dt < +\infty \tag{IC}$$

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- Existence of measure valued narrowly continuous solutions
- Uniformly boundedness of second order moments

Compactness

Compactness of solutions to NCE

Let μ^n satisfy **moment bound** and **local blow-up control** and $\mu^n \rightharpoonup \mu$. Let $(\rho^n, \boldsymbol{j}^n) \in \mathrm{CE}_T$ for each $n \in \mathbb{N}$ such that

$$\sup_{n\in\mathbb{N}} M_2(\rho_0^n) < \infty \quad \text{and} \quad \sup_n \int_0^T \mathcal{A}(\mu^n; \rho_t^n, \boldsymbol{j}_t^n)\, dt < +\infty.$$

Then, there exists $(\rho, \boldsymbol{j}) \in \mathrm{CE}_T$ such that, up to pass to a subsequence,

$$\begin{aligned} \rho_t^n &\rightharpoonup \rho_t \quad \text{for all } t \in [0,T], \\ \boldsymbol{j}^n &\rightharpoonup \boldsymbol{j} \quad \text{in } \mathcal{M}_{\mathrm{loc}}(G \times [0,T]), \end{aligned}$$

with $\rho_t \in \mathcal{P}_2(\mathbb{R}^d)$ for any $t \in [0,T]$. Moreover, the action is lower semicontinuous

$$\liminf_{n\to\infty} \int_0^T \mathcal{A}(\mu^n; \rho_t^n, \boldsymbol{j}_t^n)\, dt \geq \int_0^T \mathcal{A}(\mu; \rho_t, \boldsymbol{j}_t)\, dt.$$

Nonlocal upwind transportation quasi-metric

Definition

For $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ satisfying Assumptions moment bound and local blow-up control, and $\rho_0, \rho_1 \in \mathcal{P}_2(\mathbb{R}^d)$, the nonlocal upwind transportation cost between ρ_0 and ρ_1 is defined by

$$\mathcal{T}_\mu(\rho_0, \rho_1)^2 = \inf \left\{ \int_0^1 \mathcal{A}(\mu; \rho_t, \boldsymbol{j}_t)\, dt : (\rho, \boldsymbol{j}) \in \mathrm{CE}(\rho_0, \rho_1) \right\}. \tag{6}$$

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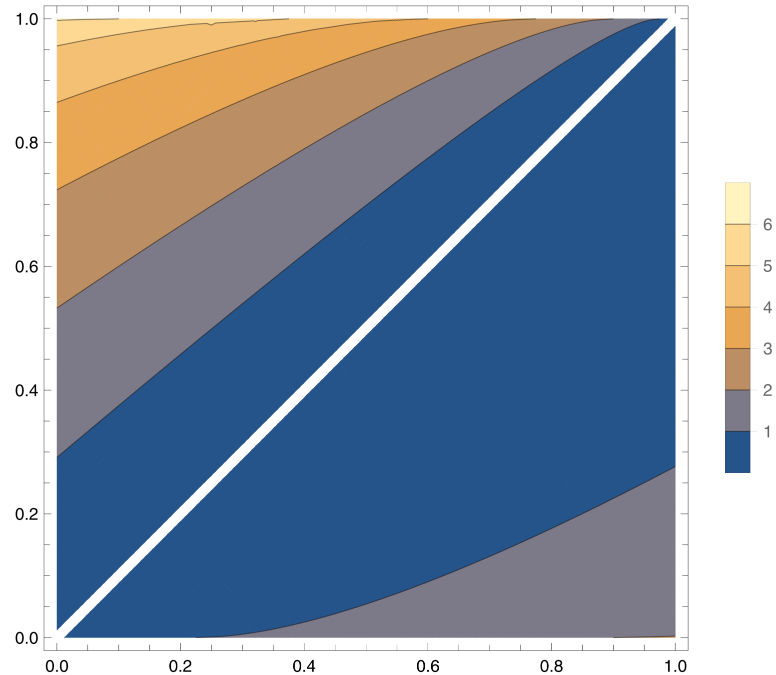
- For $\rho \in \mathcal{P}_2(\mathbb{R}^d)$ it holds that $\boldsymbol{j} \in T_\rho\mathcal{P}_2(\mathbb{R}^d)$ if and only if $\boldsymbol{j}^+ \ll \gamma_1$, $\boldsymbol{j}^- \ll \gamma_2$, and $v^+ := \frac{d\boldsymbol{j}^+}{d\gamma_1}$ and $v^- := \frac{d\boldsymbol{j}^-}{d\gamma_2}$ satisfy, for $v := v^+ - v^-$, the relation

$$v \in \overline{\left\{\overline{\nabla}\varphi : \varphi \in C_c^\infty(\mathbb{R}^d)\right\}}^{L^2(\eta\hat{\gamma}^v)}, \qquad \text{where} \qquad d\hat{\gamma}^v = \chi_{\{v>0\}}\,d\gamma_1 + \chi_{\{v<0\}}\,d\gamma_2.$$

Two-point space

$\Omega := \{0,1\}$, with $\eta(0,1) = \eta(1,0) = \alpha > 0$, $\mu(0) = p > 0$ and $\mu(1) = q > 0$. Let $\rho, \nu \in \mathcal{P}_2(\Omega)$ such that $\rho = \rho_0\delta_0 + \rho_1\delta_1$ and $\nu = \nu_0\delta_0 + \nu_1\delta_1$. It holds

$$\mathcal{T}(\rho,\nu) = \begin{cases} \frac{2}{\sqrt{\alpha p}}\left(\sqrt{\rho_1} - \sqrt{\nu_1}\right) & \text{if } \rho_0 < \nu_0, \\ \frac{2}{\sqrt{\alpha q}}\left(\sqrt{\rho_0} - \sqrt{\nu_0}\right) & \text{if } \nu_0 < \rho_0. \end{cases}$$



$(\mathrm{NL}^2\mathrm{IE})$ as gradient flow w.r.t. the quasi-metric $\mathcal{T}$

For $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ satisfying moment bound and local blow-up control, and $\rho \ll \mu$,

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Nonlocal-interaction energy

$$\mathcal{E}(\rho) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y)\, d\rho(x)\, d\rho(y),$$

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$\mathcal{T}$ is a quasi-metric $\Rightarrow$ underlying structure of $\mathcal{P}_2(\mathbb{R}^d)$ is Finslerian
$\Rightarrow T_\rho \mathcal{P}_2(\mathbb{R}^d)$ is not a Euclidean space, but rather a manifold in its own right!

Gradient descent in Finsler geometry[1]

[1] [Ohta-Sturm '09, '12] and [Agueh '12]

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Inner product

$\boldsymbol{j} \in T_\rho \mathcal{P}_2(\mathbb{R}^d)$, we define an inner product $g_{\rho,\boldsymbol{j}} : T_\rho \mathcal{P}_2(\mathbb{R}^d) \times T_\rho \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ by

$$g_{\rho,\boldsymbol{j}}(\boldsymbol{j}_1, \boldsymbol{j}_2) = \frac{1}{2} \iint_G j_1(x,y)\, j_2(x,y)\, \eta(x,y) \left(\frac{\chi_{\{j>0\}}(x,y)}{\rho(x)} + \frac{\chi_{\{j<0\}}(x,y)}{\rho(y)} \right) d\mu(x)\, d\mu(y),$$

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Gradient vector: $\mathrm{Diff}_\rho\, \mathcal{E}[\boldsymbol{j}] = g_{\rho, \mathrm{grad}\, \mathcal{E}(\rho)}\big(\mathrm{grad}\, \mathcal{E}(\rho), \boldsymbol{j}\big)$ for all $\boldsymbol{j} \in T_\rho \mathcal{P}_2(\mathbb{R}^d)$

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$$g_{\rho,\boldsymbol{j}}(\boldsymbol{j}_1, \boldsymbol{j}_2) = \frac{1}{2} \iint_G j_1(x,y)\, j_2(x,y)\, \eta(x,y) \left(\frac{\chi_{\{j>0\}}(x,y)}{\rho(x)} + \frac{\chi_{\{j<0\}}(x,y)}{\rho(y)} \right) d\mu(x)\, d\mu(y),$$

Goal: direction of steepest discent from ρ!

Gradient vector: $\mathrm{Diff}_\rho\, \mathcal{E}[\boldsymbol{j}] = g_{\rho, \mathrm{grad}\, \mathcal{E}(\rho)}(\mathrm{grad}\, \mathcal{E}(\rho), \boldsymbol{j})$ for all $\boldsymbol{j} \in T_\rho \mathcal{P}_2(\mathbb{R}^d)$

Direction steepest descent is in general NOT $-\,\mathrm{grad}\, \mathcal{E}(\rho)$

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[1] [Ohta-Sturm '09, '12] and [Agueh '12]

Gradient descent in Finsler geometry[1]

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Gradient flows in $(\mathcal{P}_2(\mathbb{R}^d),\mathcal{T})$: $\partial_t\rho_t = \overline{\nabla}\cdot\mathrm{grad}^-\,\mathcal{E}(\rho)$

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Nonlocal interaction energy

$$\mathrm{grad}^-\, \mathcal{E}(\rho)(x,y) = -\overline{\nabla}(K * \rho)(x,y) \left(\rho(x) \chi_{\{-\overline{\nabla}K*\rho>0\}}(x,y) + \rho(y)\chi_{\{-\overline{\nabla}K*\rho<0\}}(x,y) \right)$$

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Towards the variational characterisation for $(\mathrm{NL^2IE})$

Towards the variational characterisation for $(\mathrm{NL}^2\mathrm{IE})$

Euclidean case: $\dot{x}(t) = -\nabla_x F(x(t))$

$$F(x(s)) - F(x(t)) = \int_s^t -\nabla F(x(z)) \cdot x'(z)\,dz \le \int_s^t |\nabla F(x(z))| \cdot |x'(z)|\,dz$$
$$\le \int_s^t \left(\frac{1}{2}|\nabla F(x(z))|^2 + \frac{1}{2}|x'(z)|^2\right)\,dz$$

Towards the variational characterisation for $(\mathrm{NL}^2\mathrm{IE})$

For any $\rho \in \mathrm{AC}\big([0,T];(\mathcal{P}_2(\mathbb{R}^d),\mathcal{T})\big)$, $\exists!$ antisymmetric $(w_t)_{t\in[0,T]}$ such that $(\rho_t,\boldsymbol{j}_t)_{t\in[0,T]} \in \mathrm{CE}_T$ and

$$d\boldsymbol{j}_t(x,y) = w_t(x,y)_+\,d\rho(x)\,d\mu(y) - w_t(x,y)_-\,d\mu(x)\,d\rho(y)$$

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Finsler product

$$\widehat{g}_{\rho,w}(u,v) = \frac{1}{2}\iint_G u(x,y)\,v(x,y)\,\eta(x,y)\big(\chi_{\{w>0\}}(x,y)\,d\gamma_1(x,y)+\chi_{\{w<0\}}(x,y)\,d\gamma_2(x,y)\big).$$

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Chain rule from CE

$$\frac{d}{dt}\int\varphi(x)\,d\rho_t(x) = \frac{1}{2}\iint\overline{\nabla}\varphi(x,y)\eta(x,y)\,d\boldsymbol{j}_t(x,y) = \widehat{g}_{\rho_t,w_t}\big(w_t,\overline{\nabla}\varphi\big)$$

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One-sided Cauchy–Schwarz inequality

$$\widehat{g}_{\rho,w}(w,v) \le \sqrt{\widehat{g}_{\rho,v}(v,v)\,\widehat{g}_{\rho,w}(w,w)},$$

Chain rule

Interaction potential

(K1) $K \in C(\mathbb{R}^d \times \mathbb{R}^d)$;

(K2) K is symmetric, i.e., $K(x,y) = K(y,x)$ for all $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$;

(K3) $|K(x,y) - K(x',y')| \le L\left(|(x,y) - (x',y')| \vee |(x,y) - (x',y')|^2\right).$

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Proposition

For all $\rho \in \mathrm{AC}\big([0,T];(\mathcal{P}_2(\mathbb{R}^d),\mathcal{T})\big)$ and $0 \le s \le t \le T$ we have the chain-rule identity

$$\mathcal{E}(\rho_t) - \mathcal{E}(\rho_s) = \frac{1}{2}\int_s^t \iint_G \overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}(\rho_\tau)(x,y)\,\eta(x,y)\,d\boldsymbol{j}_\tau(x,y)\,d\tau$$
$$= \int_s^t \widehat{g}_{\rho_\tau,w_\tau}\left(w_\tau, \overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}(\rho_\tau)\right)d\tau,$$

where $(w_t)_{t\in[0,T]}$ is the antisymmetric vector field associated to $(\rho,\boldsymbol{j}) \in \mathrm{CE}_T$.

Curve of maximal slope

$$
\begin{aligned}
\mathcal{E}(\rho_T) - \mathcal{E}(\rho_0) &= \int_0^T \widehat{g}_{\rho_\tau, w_\tau}\left(w_\tau, \overline{\nabla}\frac{\delta \mathcal{E}}{\delta \rho}(\rho_\tau)\right) d\tau = -\int_0^T \widehat{g}_{\rho_\tau, w_\tau}\left(w_\tau, -\overline{\nabla}\frac{\delta \mathcal{E}}{\delta \rho}(\rho_\tau)\right) d\tau \\
&\geq -\int_0^T \sqrt{\widehat{g}_{\rho, -\overline{\nabla}\frac{\delta \mathcal{E}}{\delta \rho}}\left(-\overline{\nabla}\frac{\delta \mathcal{E}}{\delta \rho}, -\overline{\nabla}\frac{\delta \mathcal{E}}{\delta \rho}\right)} \sqrt{\widehat{g}_{\rho_t, w_\tau}(w_\tau, w_\tau)}\, d\tau \\
&\geq -\frac{1}{2}\int_0^T \widehat{g}_{\rho, -\overline{\nabla}\frac{\delta \mathcal{E}}{\delta \rho}}\left(-\overline{\nabla}\frac{\delta \mathcal{E}}{\delta \rho}, -\overline{\nabla}\frac{\delta \mathcal{E}}{\delta \rho}\right) d\tau - \frac{1}{2}\int_0^T \widehat{g}_{\rho_t, w_\tau}(w_\tau, w_\tau)\, d\tau,
\end{aligned}
$$

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\mathcal{E}(\rho_T) - \mathcal{E}(\rho_0) &= \int_0^T \widehat{g}_{\rho_\tau, w_\tau}\left(w_\tau, \overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}(\rho_\tau)\right) d\tau = -\int_0^T \widehat{g}_{\rho_\tau, w_\tau}\left(w_\tau, -\overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}(\rho_\tau)\right) d\tau \\
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&\geq -\frac{1}{2}\int_0^T \widehat{g}_{\rho, -\overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}}\left(-\overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}, -\overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}\right) d\tau - \frac{1}{2}\int_0^T \widehat{g}_{\rho_t, w_\tau}(w_\tau, w_\tau)\, d\tau,
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Local slope & De Giorgi Functional

For any $\rho \in \mathrm{AC}([0,T]; (\mathcal{P}_2(\mathbb{R}^d), \mathcal{T}))$, the De Giorgi functional at ρ is defined as

$$
\mathcal{G}_T(\rho) := \mathcal{E}(\rho_T) - \mathcal{E}(\rho_0) + \frac{1}{2}\int_0^T \left(\mathcal{D}(\rho_\tau) + |\rho_\tau'|^2\right) d\tau \geq 0,
$$

Curve of maximal slope

$$\begin{aligned}\mathcal{E}(\rho_T)-\mathcal{E}(\rho_0) &= \int_0^T \widehat{g}_{\rho_\tau,w_\tau}\left(w_\tau, \overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}(\rho_\tau)\right) d\tau = -\int_0^T \widehat{g}_{\rho_\tau,w_\tau}\left(w_\tau, -\overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}(\rho_\tau)\right) d\tau \\ &\geq -\int_0^T \sqrt{\widehat{g}_{\rho,-\overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}}\left(-\overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}, -\overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}\right)}\sqrt{\widehat{g}_{\rho_t,w_\tau}(w_\tau,w_\tau)}\, d\tau \\ &\geq -\frac{1}{2}\int_0^T \widehat{g}_{\rho,-\overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}}\left(-\overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}, -\overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}\right) d\tau - \frac{1}{2}\int_0^T \widehat{g}_{\rho_t,w_\tau}(w_\tau,w_\tau)\, d\tau,\end{aligned}$$

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$$\begin{aligned}\mathcal{D}(\rho) :&= \widehat{g}_{\rho,-\overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}}\left(-\overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}, -\overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}\right) \\ &= -\iint_G \left|\overline{\nabla}\frac{\delta\mathcal{E}}{\delta\rho}(x,y)_-\right|^2 \eta(x,y)\, d\rho_\tau(x)\, d\mu(y)\end{aligned}$$

Variational characterisation of $(\mathrm{NL^2IE})$

Nonlocal-nonlocal interaction equation

$$\partial_t \rho + \overline{\nabla} \cdot \boldsymbol{j} = 0,$$

where the flux $\boldsymbol{j}$ is given by

$$d\boldsymbol{j}(x,y) = \overline{\nabla}(K*\rho)(x,y)_- \eta(x,y)\, d\rho(x)\, d\mu(y) - \overline{\nabla}(K*\rho)(x,y)_+ \eta(x,y)\, d\rho(y)\, d\mu(x)\ .$$

Theorem

A curve $(\rho_t)_{t\in[0,T]} \subset \mathcal{P}_2(\mathbb{R}^d)$ is a weak solution to $(\mathrm{NL^2IE})$ if and only if ρ belongs to $\mathrm{AC}([0,T];(\mathcal{P}_2(\mathbb{R}^d),\mathcal{T}))$ and is a curve of maximal slope for $\mathcal{E}$ with respect to $\sqrt{\mathcal{D}}$, that is, satisfies

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What about existence?

- Minimisers exist by direct method, however not necessarily global!
- Possibility: minimising movement scheme in quasi-metric spaces
- Instead: Show existence via finite-dimensional approximation and stability

Stability with respect to graph approximations

Stability of gradient flows

Let $(\mu^n)_n \subset \mathcal{M}^+(\mathbb{R}^d)$ and suppose that $(\mu^n)_n$ narrowly converges to μ. Assume that the base measures μ^n and μ satisfy **(MB)** and **(LBC)** uniformly in n, and let the interaction potential K satisfy **(K1)–(K3)**. Suppose that ρ^n is a gradient flow of $\mathcal{E}$ with respect to μ^n for all $n \in \mathbb{N}$, that is,

$$\mathcal{G}_T(\mu^n; \rho^n) = 0 \quad \text{for all } n \in \mathbb{N},$$

such that $(\rho_0^n)_n$ satisfies $\sup_{n\in\mathbb{N}} M_2(\rho_0^n) < \infty$ and $\rho_t^n \rightharpoonup \rho_t$ as $n \to \infty$ for all $t \in [0,T]$ for some curve $(\rho_t)_{t\in[0,T]} \subset \mathcal{P}_2(\mathbb{R}^d)$. Then, $\rho \in \mathrm{AC}([0,T]; (\mathcal{P}_2(\mathbb{R}^d), \mathcal{T}_\mu))$ and ρ is a gradient flow of $\mathcal{E}$ with respect to μ, that is,

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Corollary

Existence of weak solution to $(\mathrm{NL}^2\mathrm{IE})$ via finite-dimensional approximation.

Strong measure solutions and nonlocal conservation laws

More general flux

$$\mathrm{d}\boldsymbol{j}^\Phi[\mu;\rho,v] = \Phi\left(\frac{\mathrm{d}(\rho\otimes\mu)}{\mathrm{d}\lambda}, \frac{\mathrm{d}(\mu\otimes\rho)}{\mathrm{d}\lambda}; v\right)\mathrm{d}\lambda. \tag{7}$$

Nonlocal conservation law

A curve $\rho : [0,T] \to \mathcal{M}^+_{TV}(\mathbb{R}^d)$ is said to be a strong solution to the nonlocal conservation law

$$\partial_t\rho + \overline{\nabla}\cdot\boldsymbol{j}^\Phi[\mu;\rho,v(\rho)] = 0, \tag{NCL}$$

provided that, for any $A \in \mathcal{B}(\mathbb{R}^d)$, it holds that

1. $(\rho_t)_{t\in[0,T]} \in \mathrm{AC}([0,T];\mathcal{M}_{TV}(\mathbb{R}^d))$;
2. $t \mapsto \overline{\nabla}\cdot\boldsymbol{j}^\Phi[\mu;\rho_t,v_t(\rho_t)][A] \in L^1([0,T])$;
3. ρ satisfies

$$\rho_t[A] + \int_0^t \overline{\nabla}\cdot\boldsymbol{j}^\Phi[\mu;\rho_s,v_s(\rho_s)][A]\mathrm{d}s = \rho_0[A] \qquad \text{for all } t\in[0,T]. \tag{8}$$

$\Longrightarrow$ fixed point argument

Final remarks: open questions / future works

- convexity - contractivity - stability
  $\Rightarrow$ in Finslerian geometry these become different concepts [Ohta-Sturm '12]

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$$\mathcal{F}^\sigma(\rho) = \sigma \int \log \rho(x)\, d\rho(x) + \frac{1}{2} \int K(x,y)\, d\rho(x)\, d\rho(y)$$

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- **A. E., F. S. Patacchini, A. Schlichting, D. Slepčev**, *Nonlocal-interaction equation on graphs: gradient flow structure and continuum limit* - ARMA (2021).
- **A. E., F. S. Patacchini, A. Schlichting, D. Slepčev**, *Strong solutions to nonlocal conservation laws on graphs* - in preparation.

Thank you for your attention!