Average-Case Hardness in Proof Complexity
(a biased survey)

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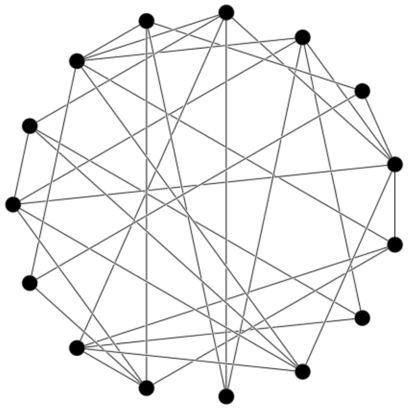
Planted clique problem

Erdős-Rényi random graph $G \sim \mathcal{G}(n, 1/2)$

w.h.p. largest clique has size $\omega(G) \approx 2 \log n$



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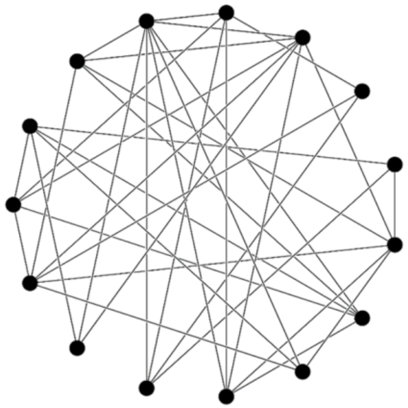

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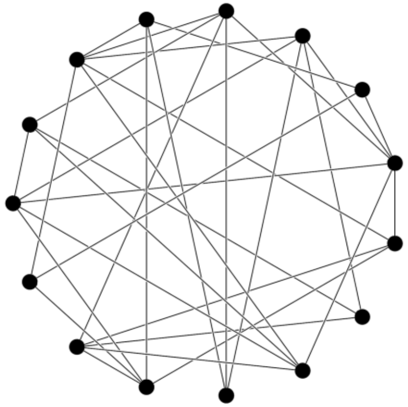

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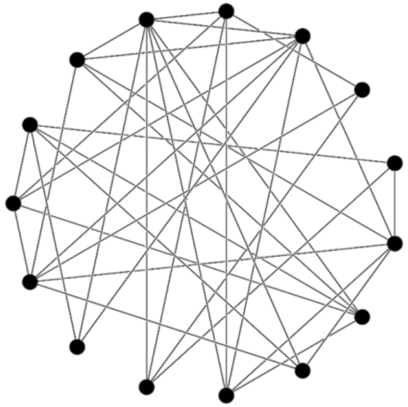

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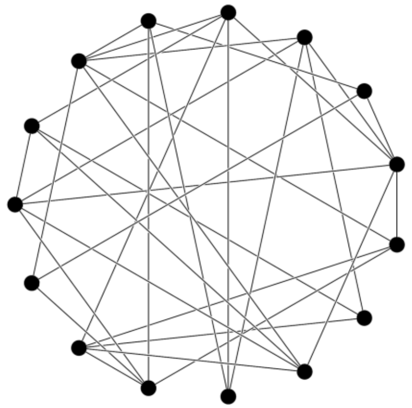

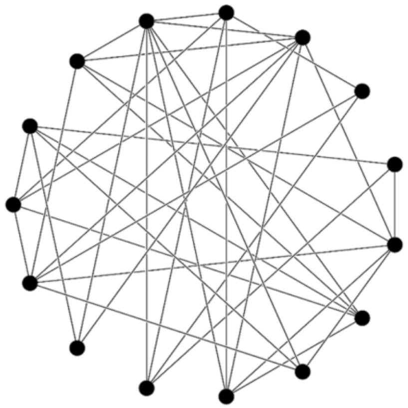

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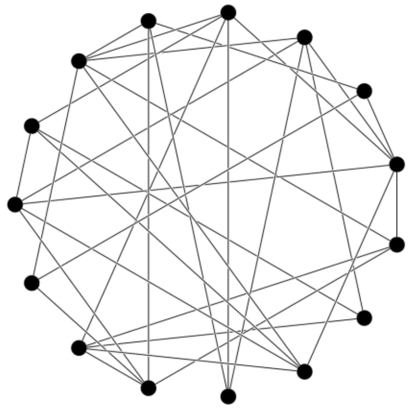

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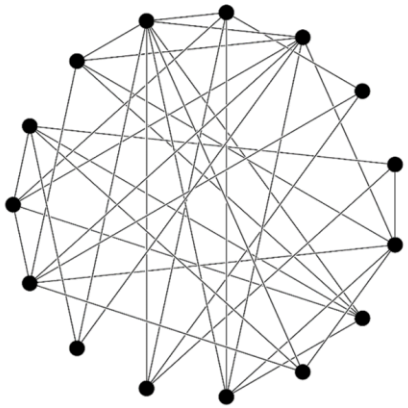

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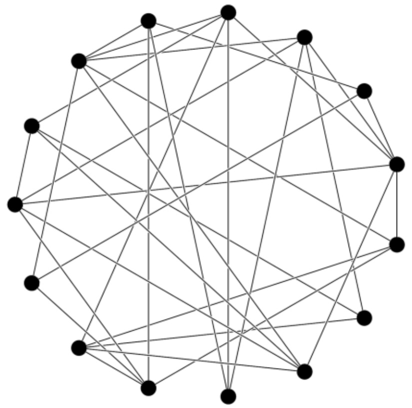

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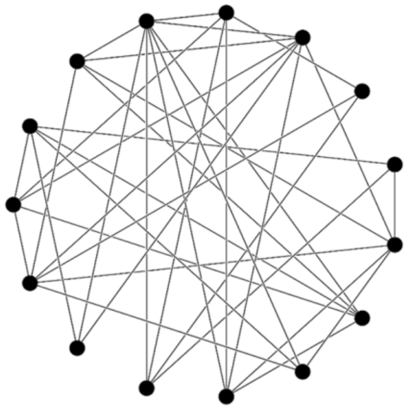

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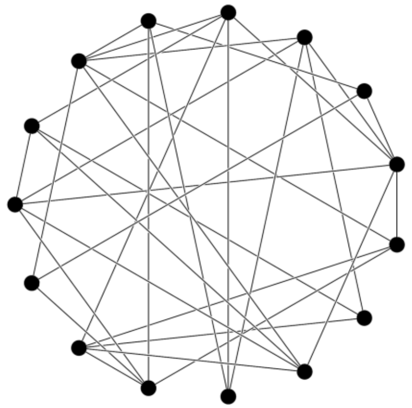

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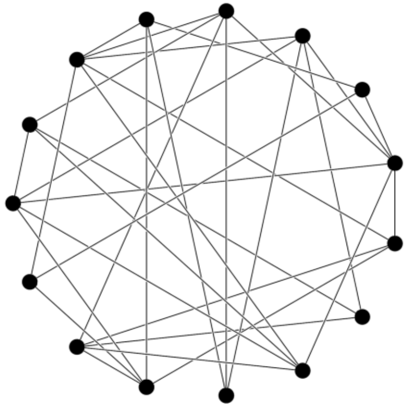

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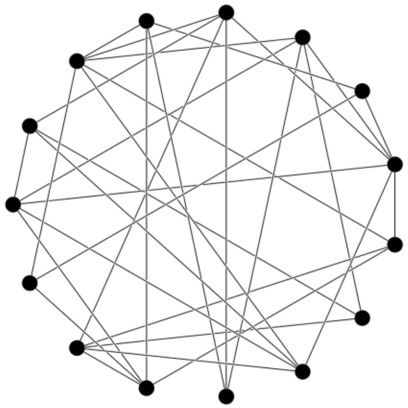

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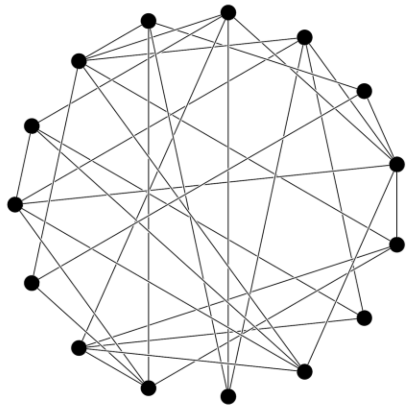

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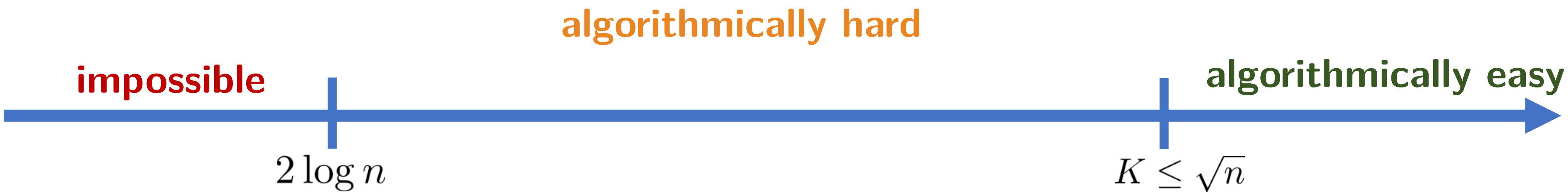

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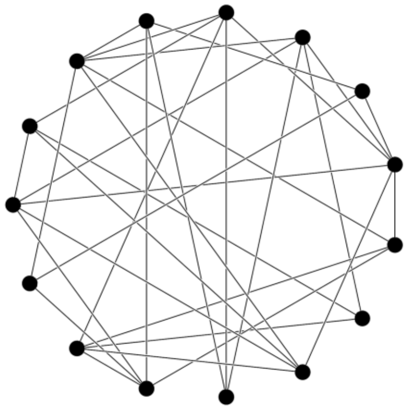

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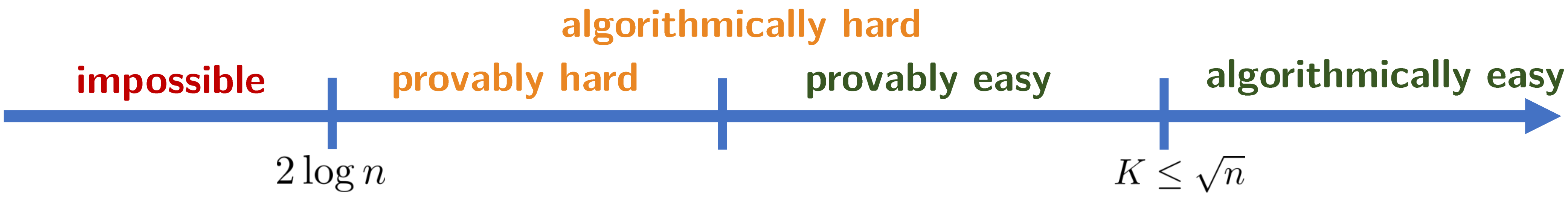

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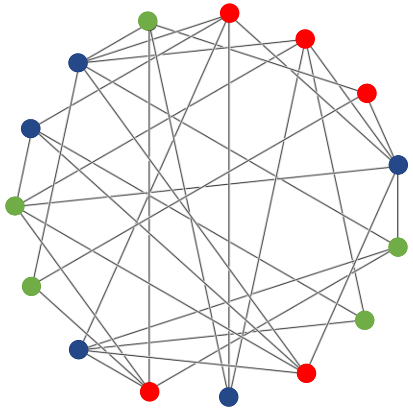

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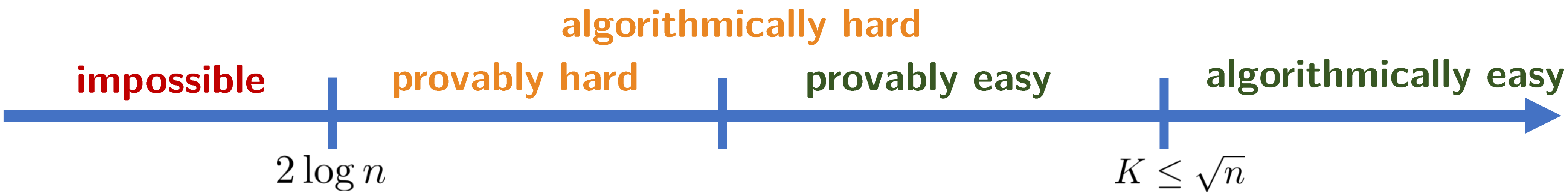

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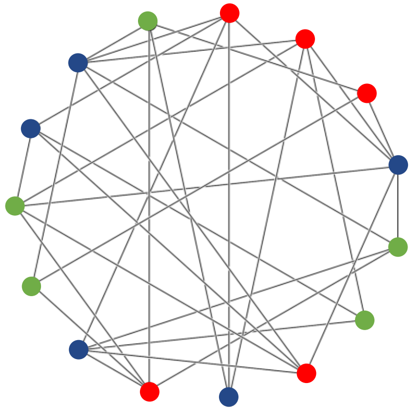

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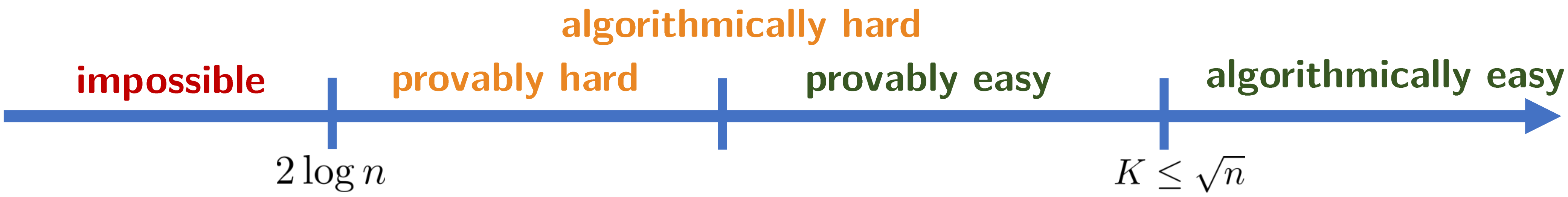

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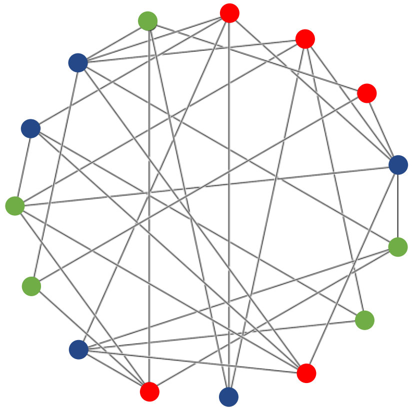

Focus on $k = \omega(G) + 1$

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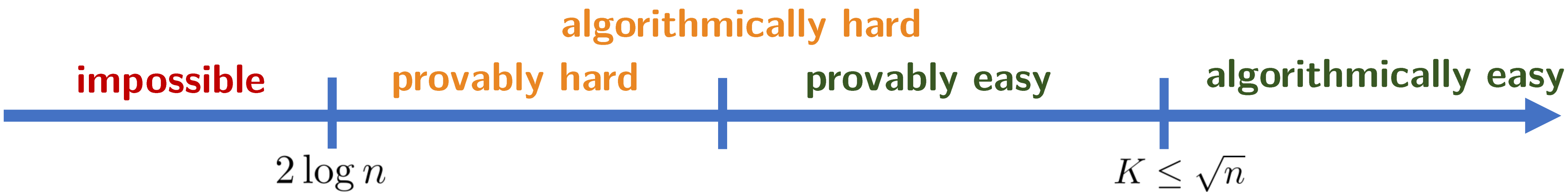

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states that G has a k clique

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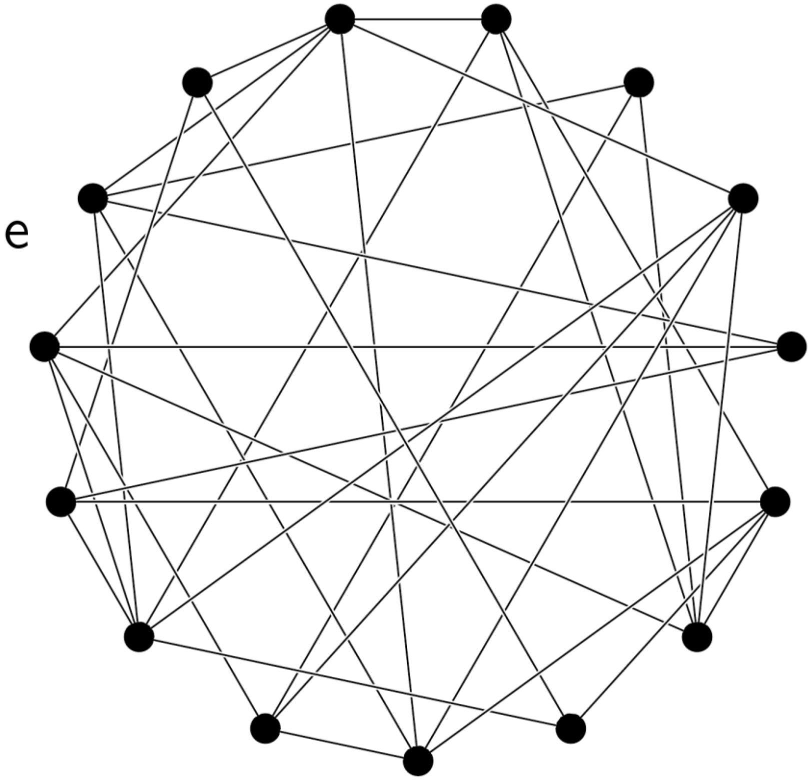

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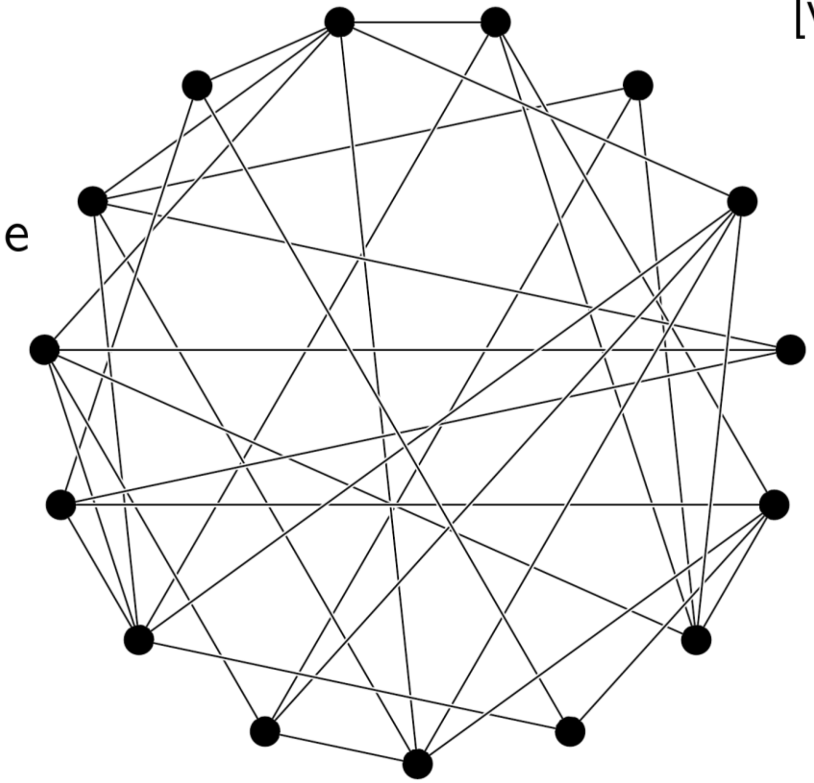

Variable x_{vi} = [vertex v is ith member of clique]

There are k clique members

$$\bigvee_{v \in V} x_{vi} \qquad \forall i \in [k]$$

A vertex can only be once in clique

$$\overline{x}_{vi} \vee \overline{x}_{vi'} \qquad \forall v \in V,\ \forall i \neq i' \in [k]$$

Non-neighbors are not both in clique

$$\overline{x}_{ui} \vee \overline{x}_{vi'} \qquad \forall (u, v) \notin E,\ \forall i, i' \in [k]$$

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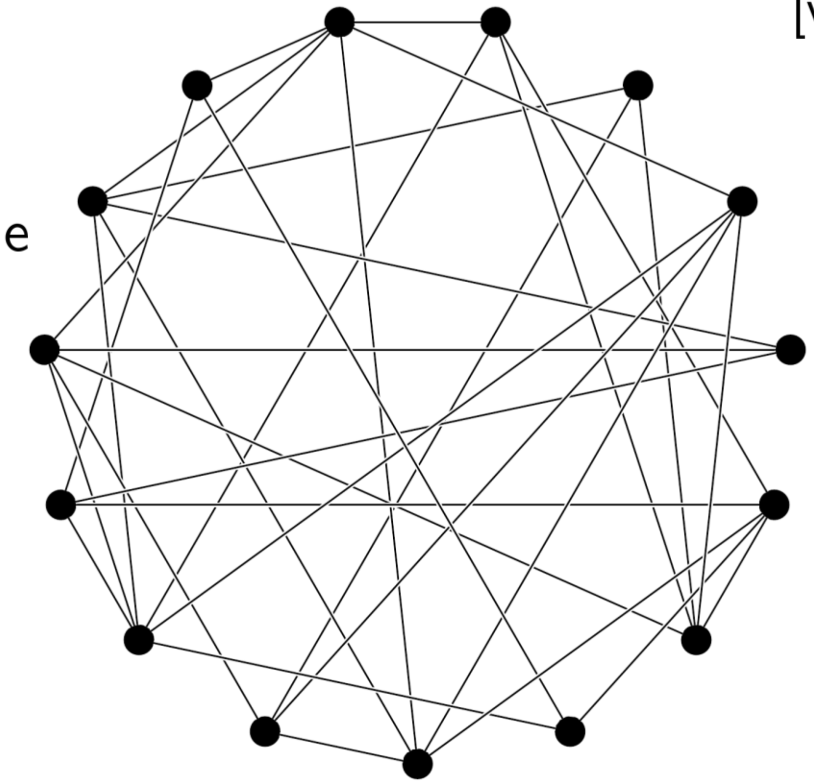

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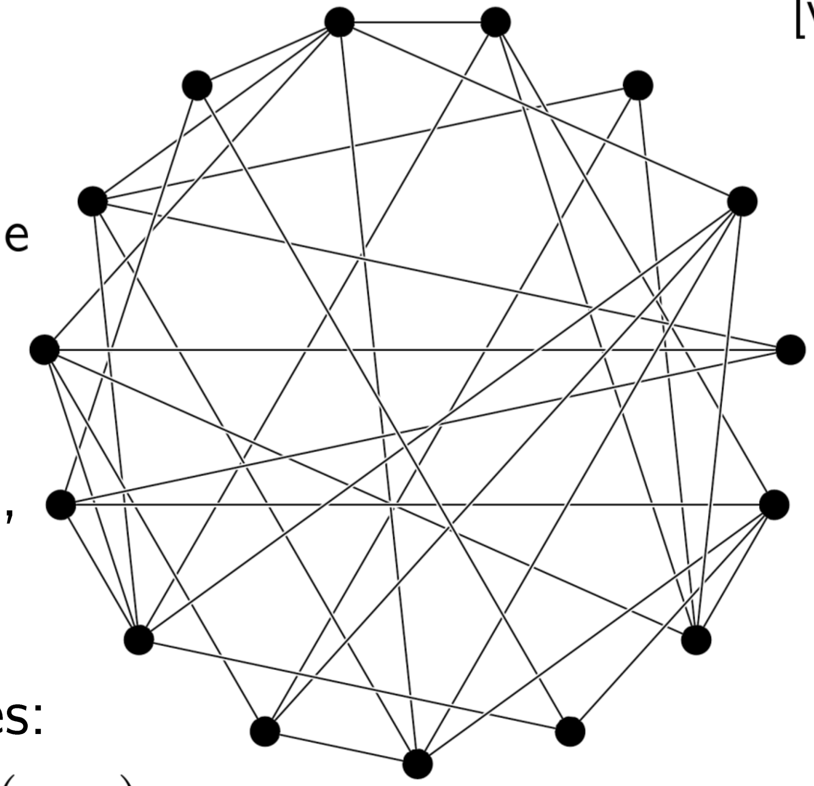
- Prove Clique(G, k) is unsatisfiable
- Lower bound size of refutation?
- Can we show that "brute-force", size $n^{\Theta(k)}$ refutation is optimal?
- Natural candidate hard instances: Erdős-Rényi random graph $G \sim \mathcal{G}(n, p)$
  p close to k-clique threshold



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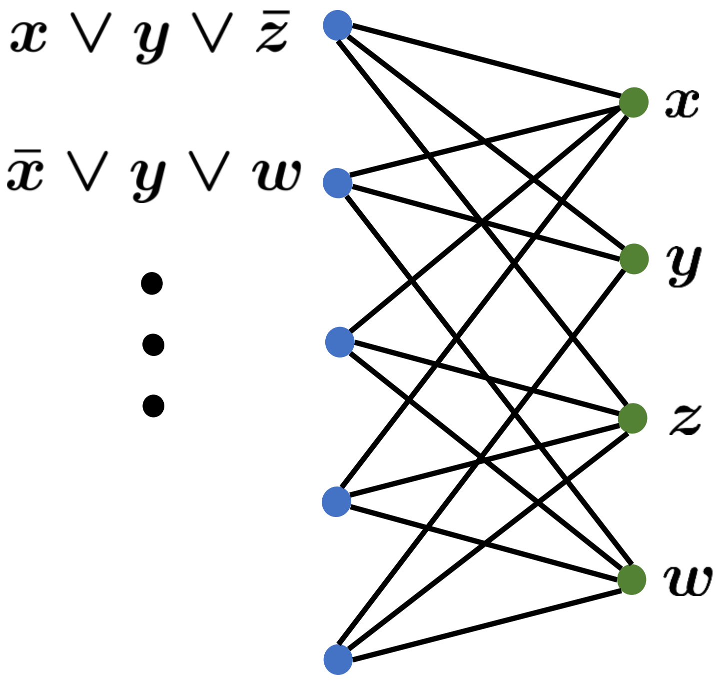

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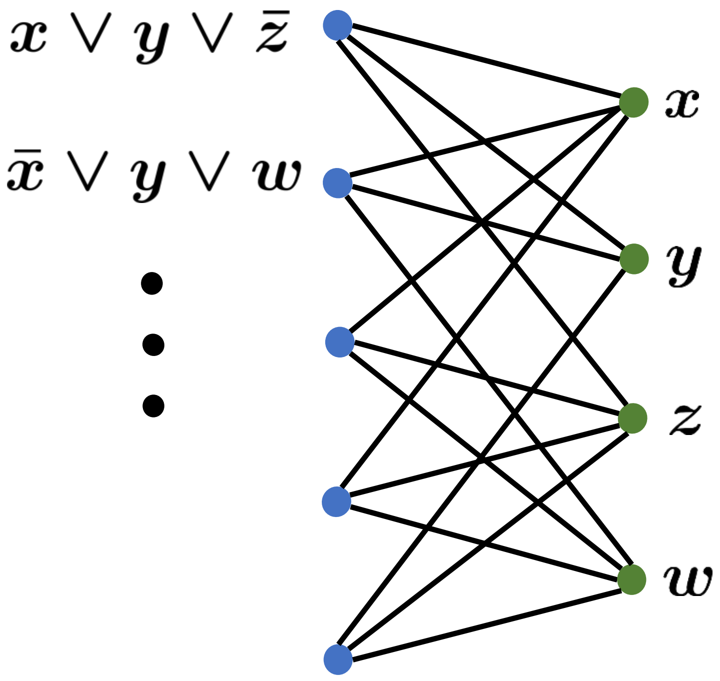

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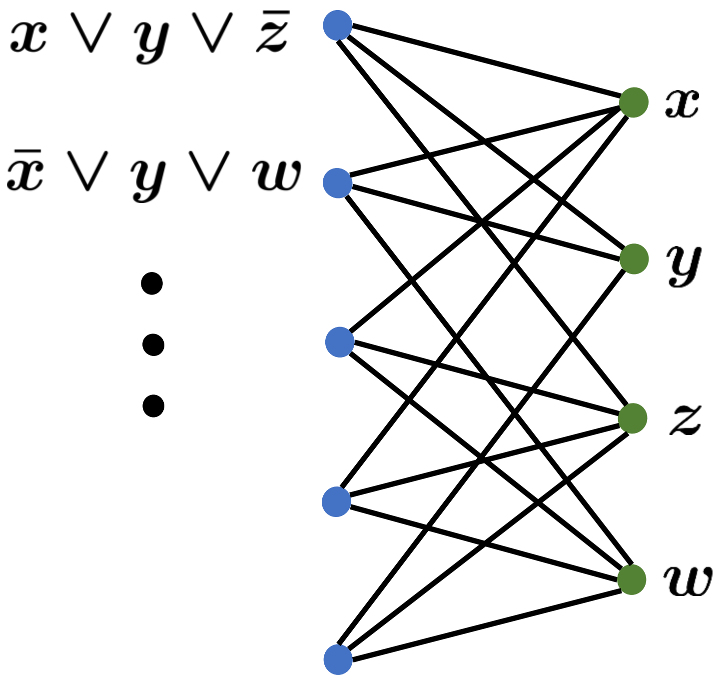

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- Rewrite constraint in CNF
  - $x \oplus y \oplus z = 0$ becomes 4 clauses:

$$(\bar{x} \vee \bar{y} \vee \bar{z}) \wedge (x \vee y \vee \bar{z}) \wedge (x \vee \bar{y} \vee z) \wedge (\bar{x} \vee y \vee z)$$

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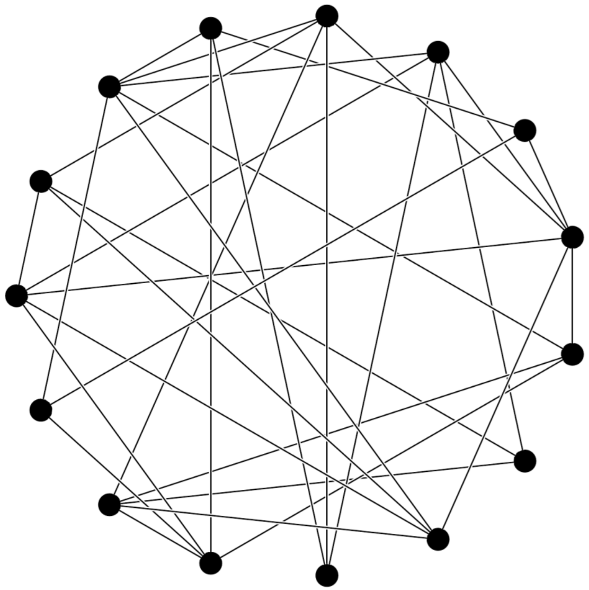

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  - This talk: focus on average-case complexity of three NP-hard problems
- Many combinatorial formulas are of independent interest

Proof Systems and Lower Bounds

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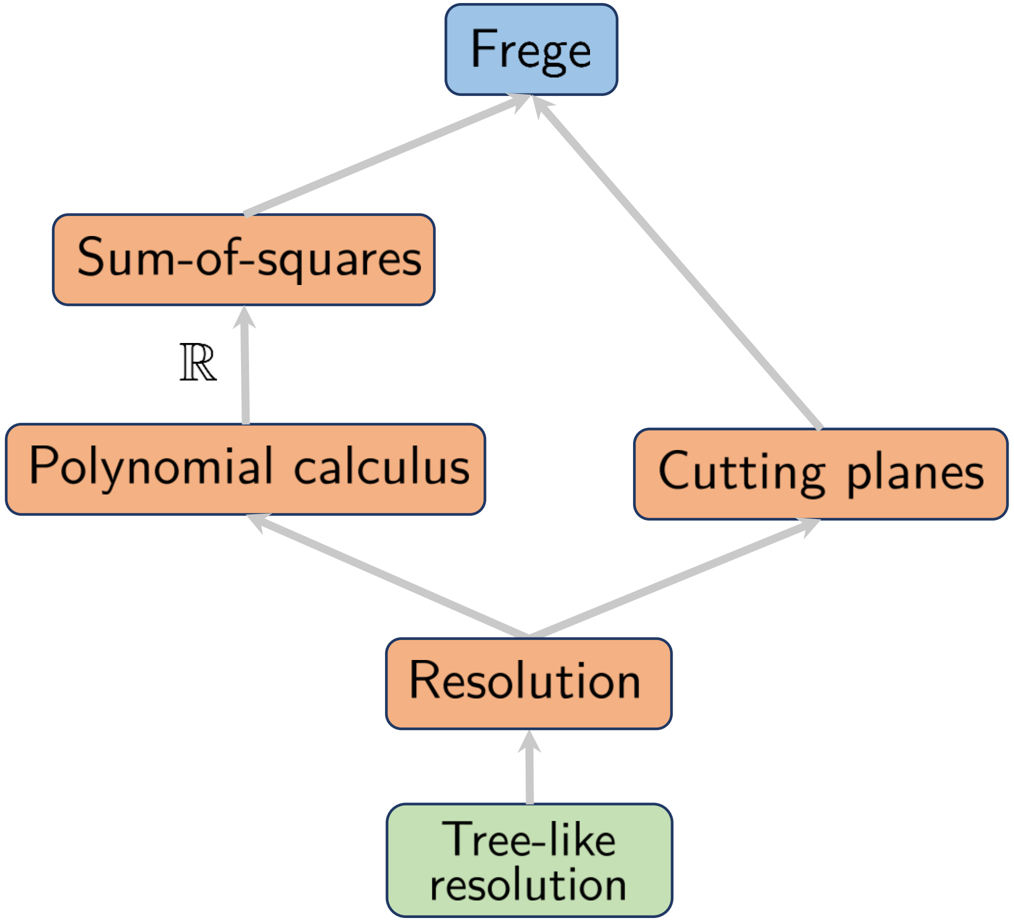

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UNSAT k-CNF formula F: $(\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z)$

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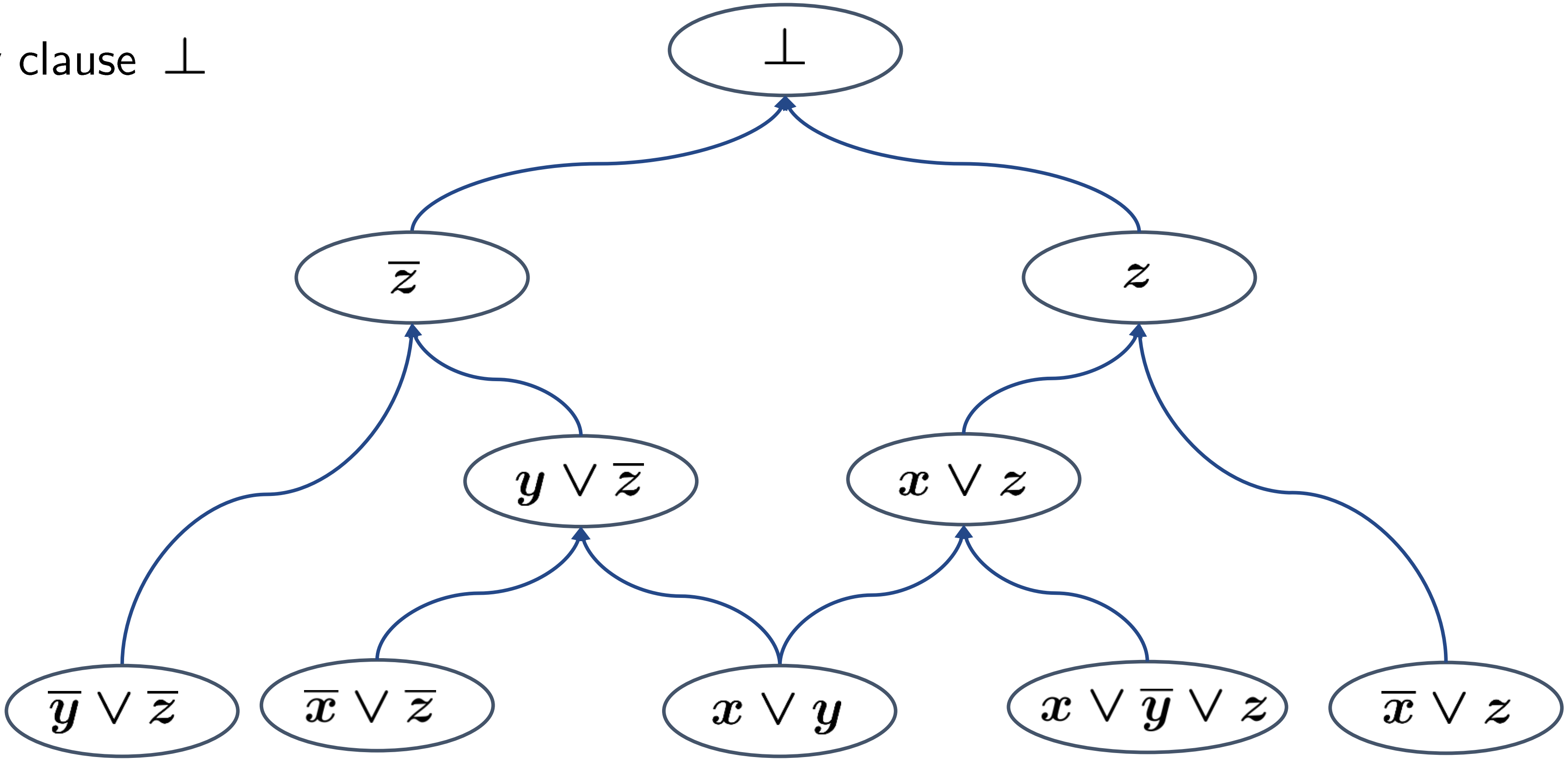

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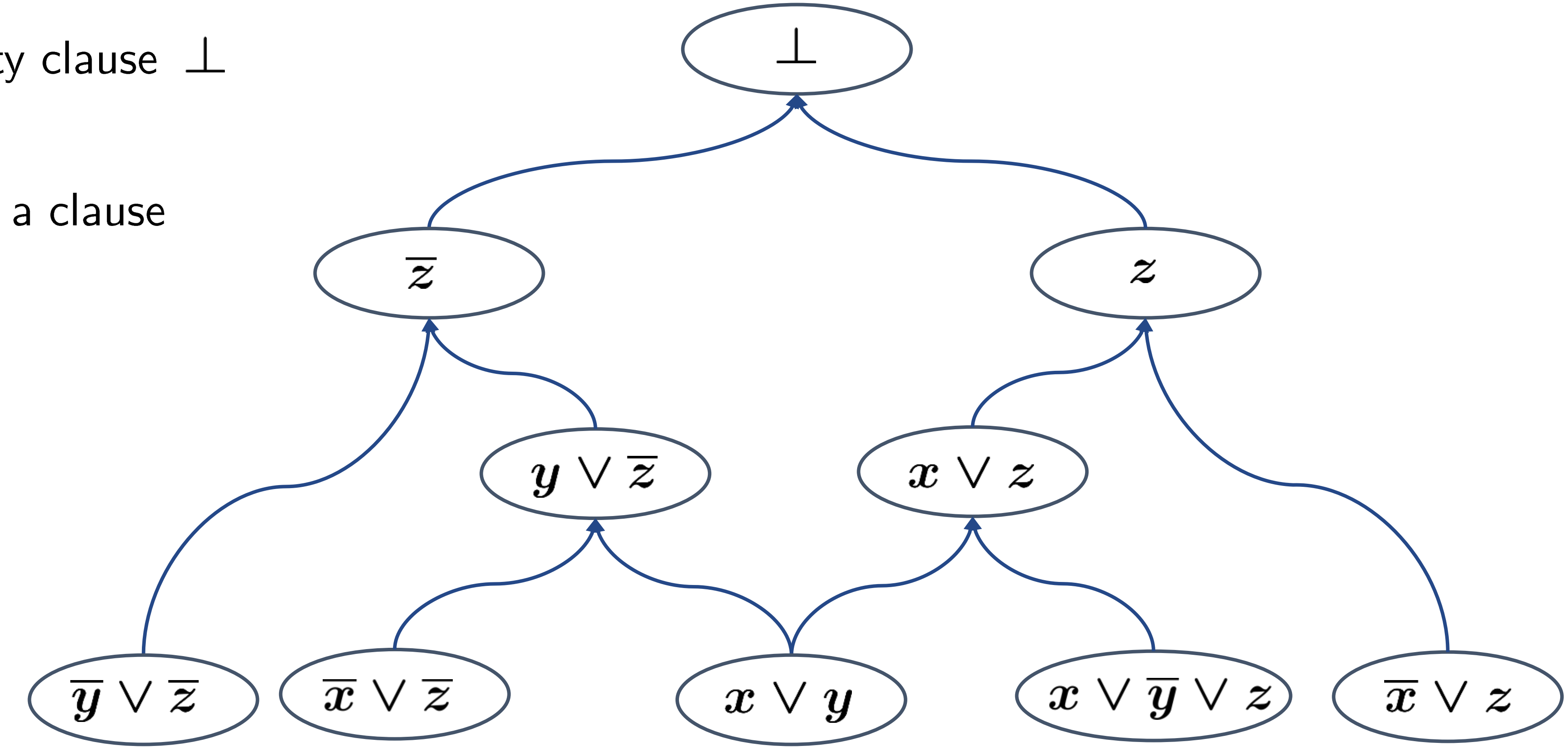
Proof size: # clauses in proof
Proof width: max # literals in a clause



Resolution

UNSAT k-CNF formula F: $(\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z)$

Resolution rule: $\dfrac{C \vee x \quad D \vee \overline{x}}{C \vee D}$

Refutation: Derivation of empty clause $\perp$

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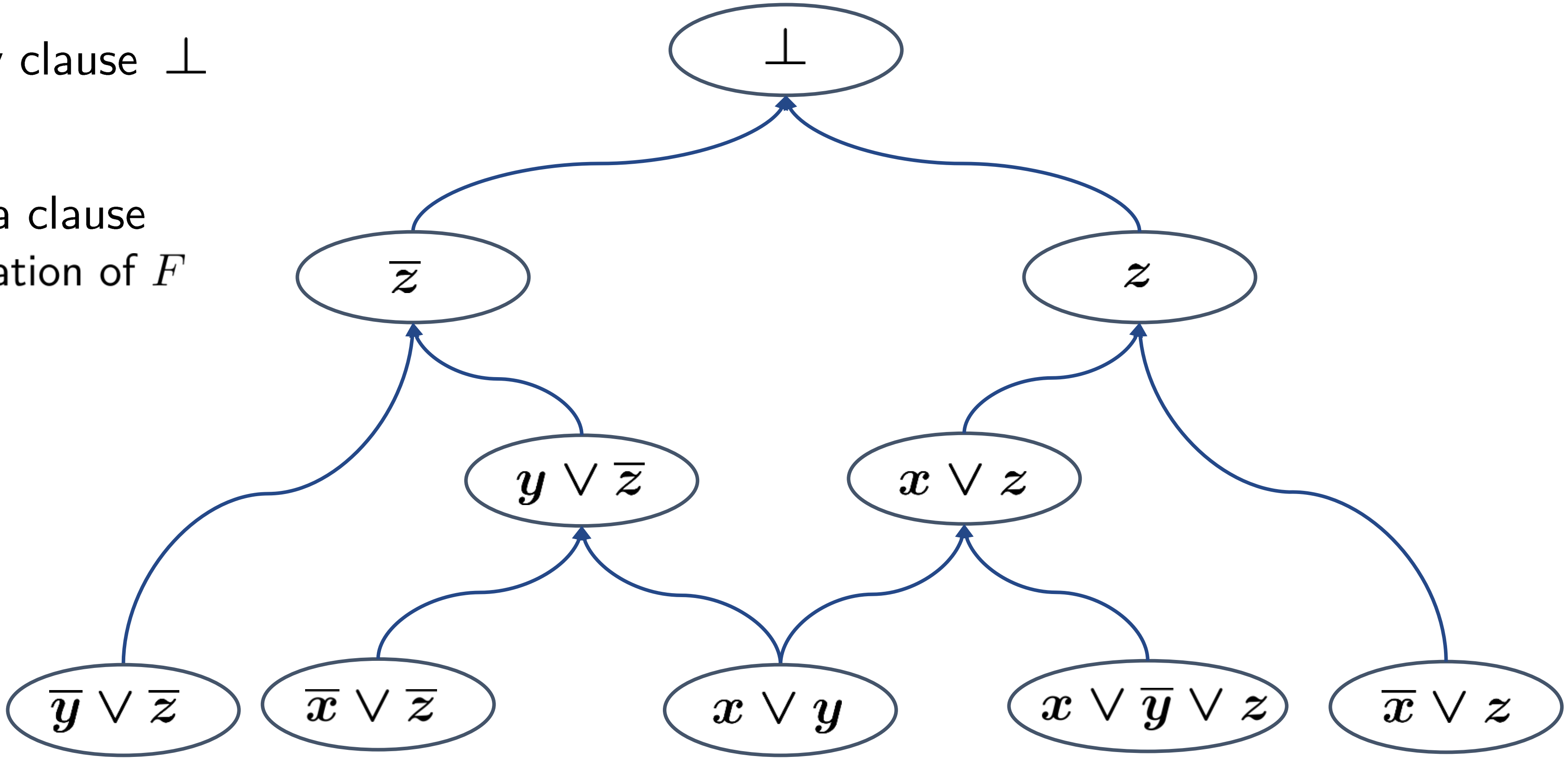
$w =$ smallest width of any refutation of F

Algorithm in time $\approx n^w$



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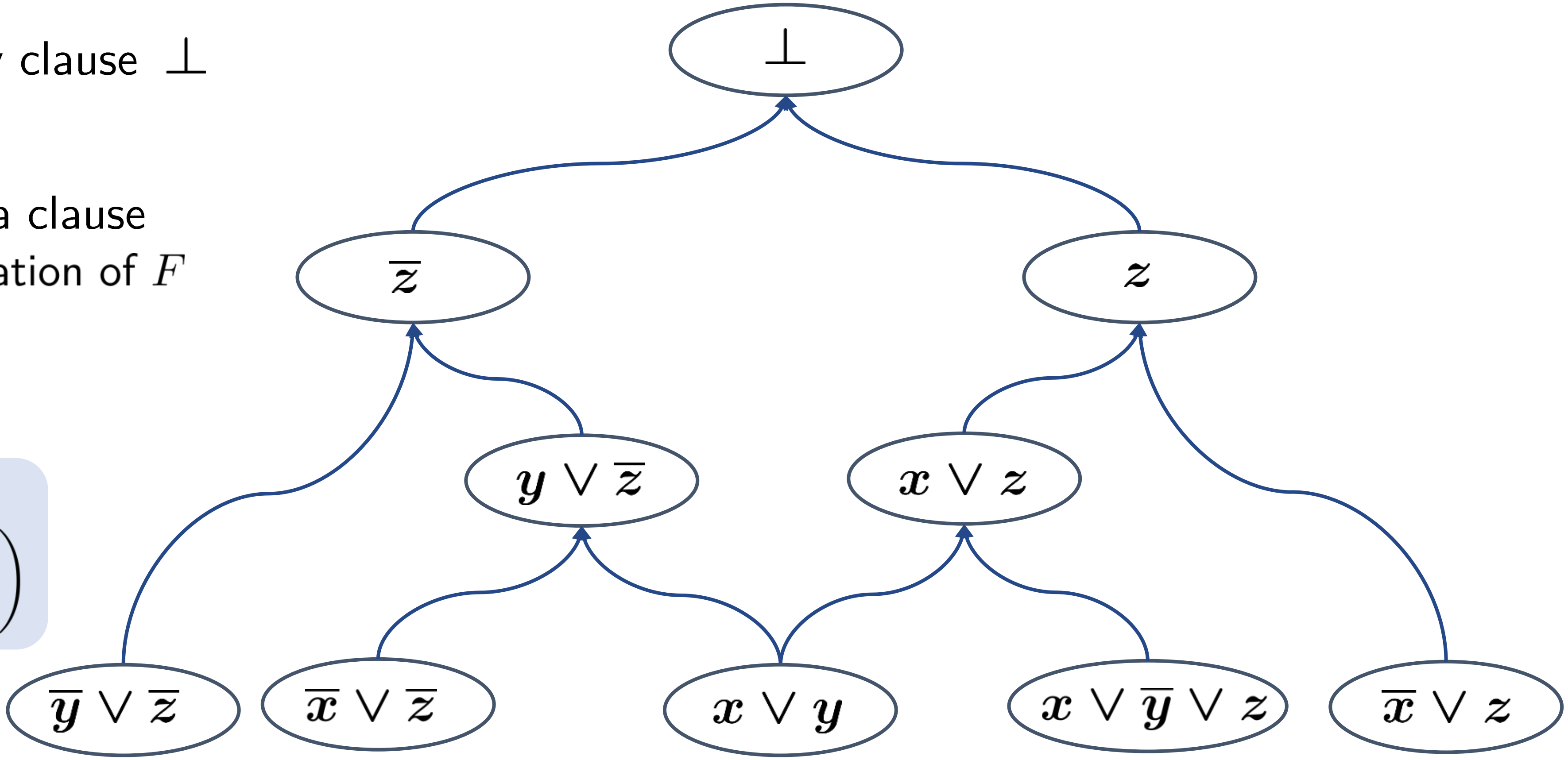

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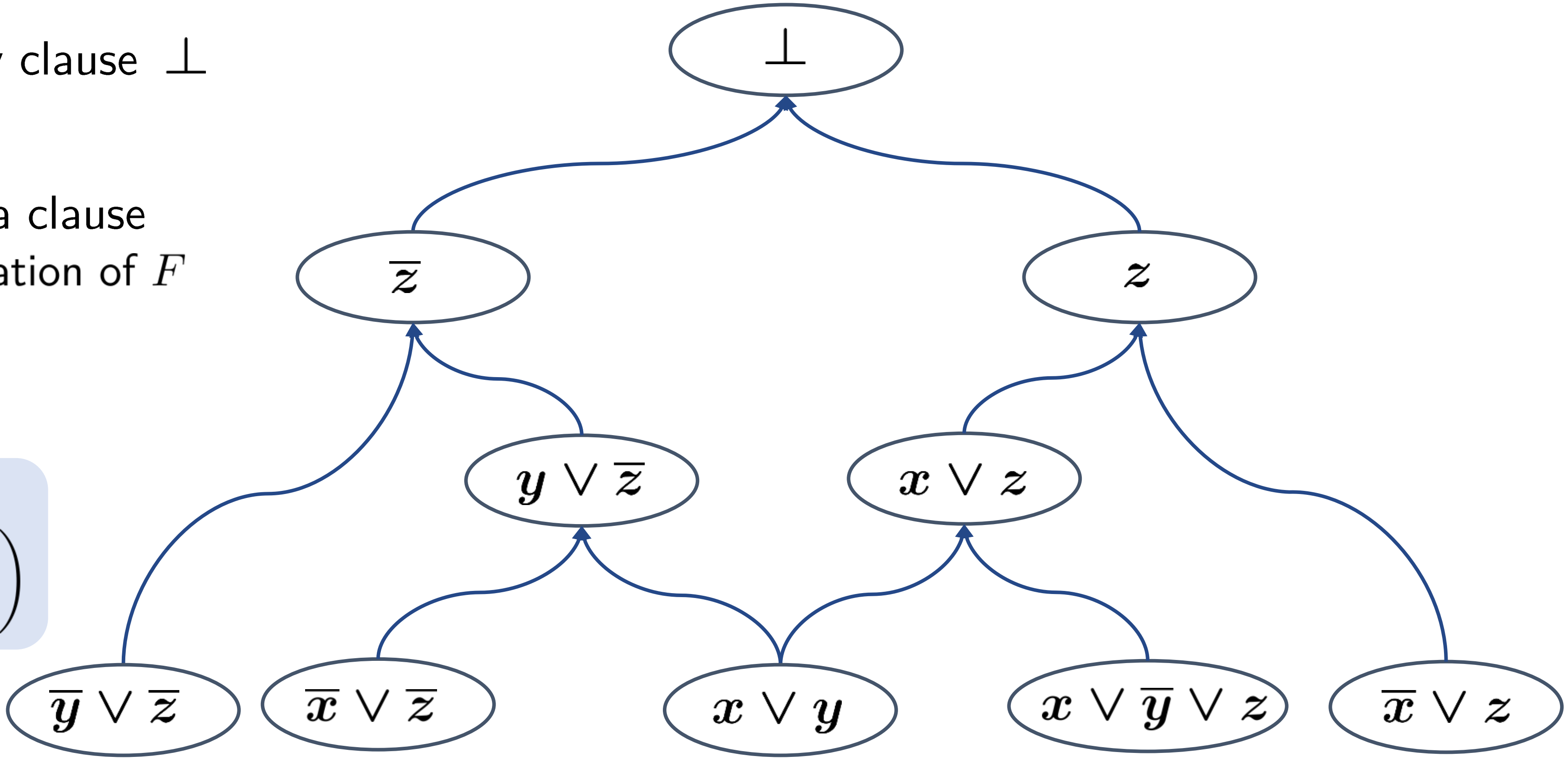

Tree-like: proof DAG is a tree

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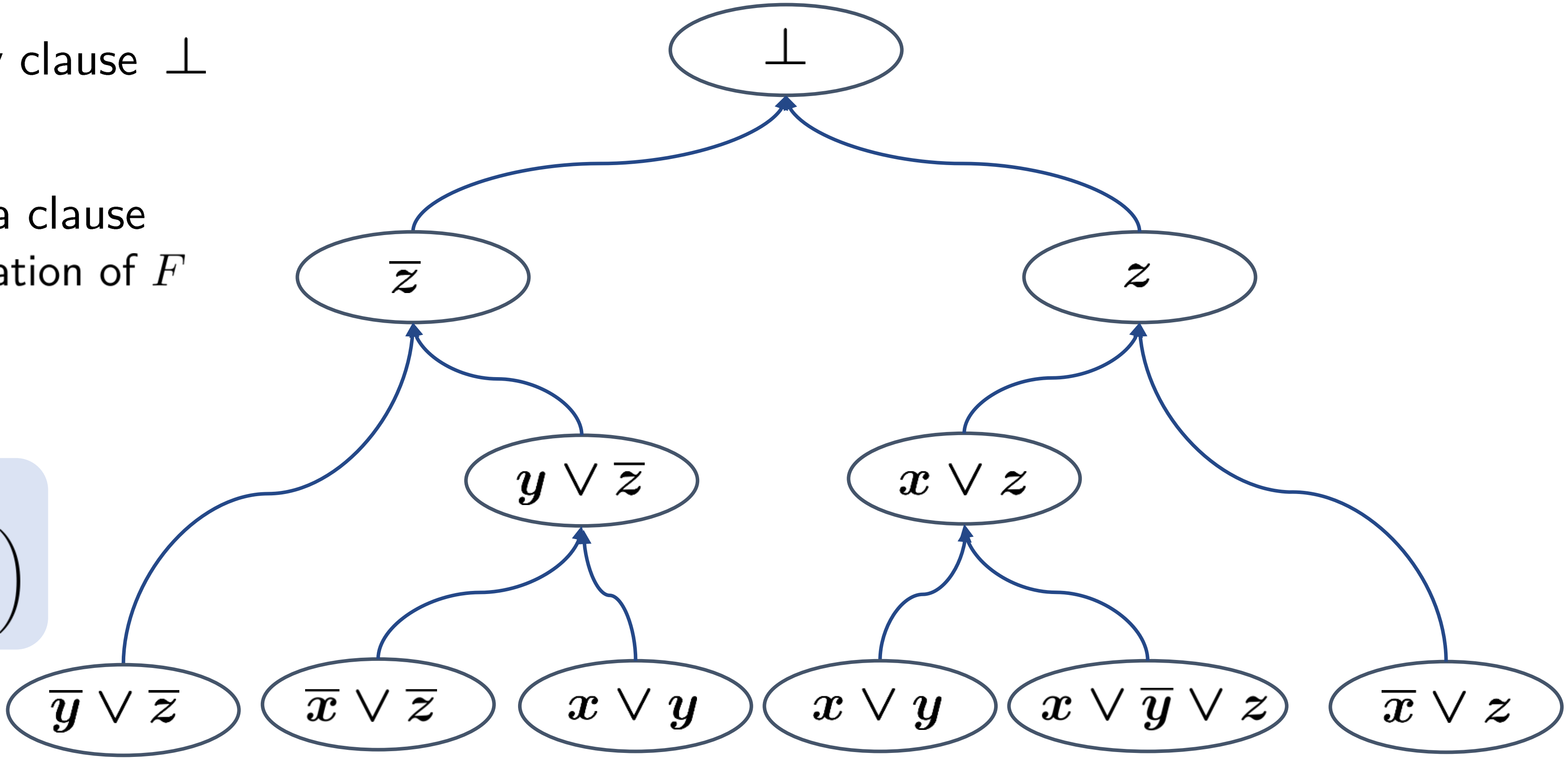

Polynomial calculus

Cutting planes

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Multiply by variable: $\dfrac{p = 0}{xp = 0}$

Refutation: Derivation of $1 = 0$

Polynomial calculus

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Refutation: Derivation of $-1 \geq 0$

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Proof size: # inequalities in proof

Sum of squares

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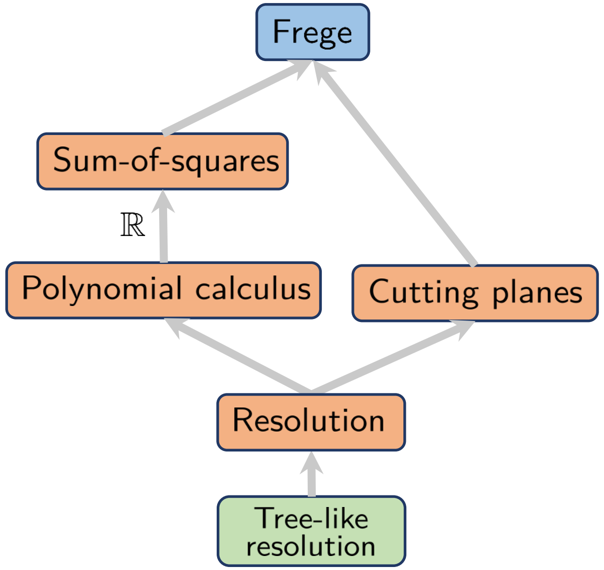

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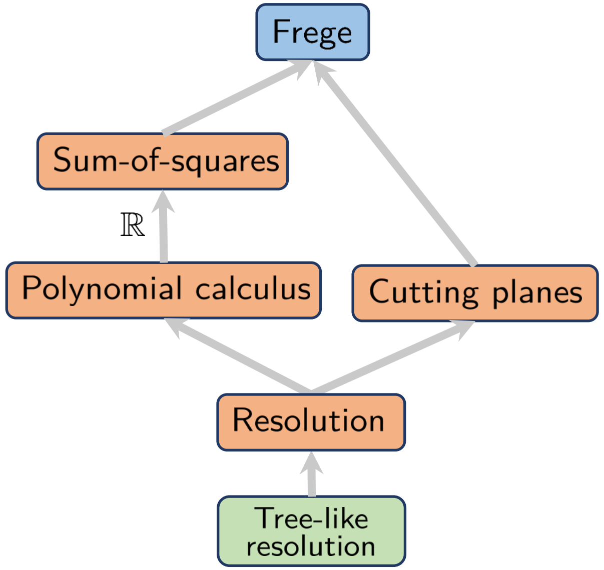
Polynomials $\mathcal{P} = \{P_1 = 0, P_2 = 0, \ldots, P_m = 0; Q_1 \geq 0, Q_2 \geq 0, \ldots, Q_\ell \geq 0\}$

SoS refutation of $\mathcal{P}$: $R_1, R_2, \ldots, R_m; S_1, S_2, \ldots, S_\ell$ s.t.

$$\sum_{i \in [m]} R_i P_i + \sum_{i \in [\ell]} S_i Q_i = -1$$

where each S_i is a sum of squares



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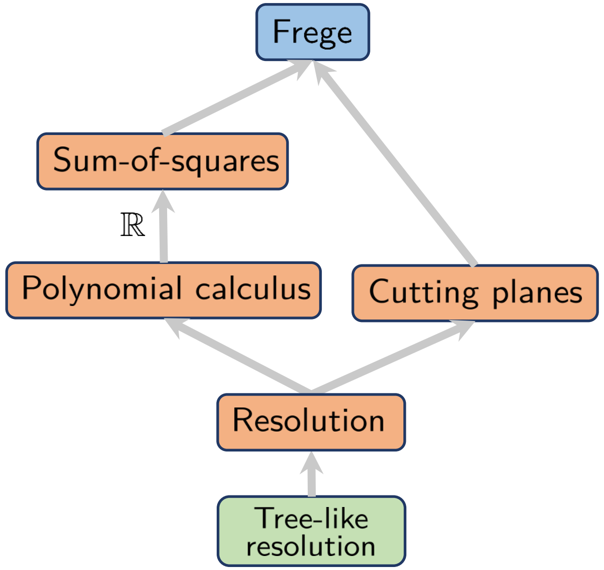

Proof size: # monomials when we expand proof
Proof degree: max degree of any polynomial

Average-case hardness results

	k-clique	k-coloring	3-SAT	3-XOR
Tree-like Resolution			HARD [Chvátal, Szemerédi '88] Improved [Ben-Sasson, Galesi '01] (size $\exp(n/\Delta^{1+\epsilon})$) $\Delta = m/n$	
Resolution			HARD [Chvátal, Szemerédi '88] $\exp(n/\Delta^{2+\epsilon})$ Improved [Beame, Karp, Pitassi, Saks '98], [Ben-Sasson '01]	
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			$\mathbb{F} = 2$: HARD [Alekhnovich, Razborov '01]	EASY
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Average-case hardness results

<table>
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<tr><td>Tree-like Resolution</td><td rowspan="8"></td><td rowspan="2">HARD
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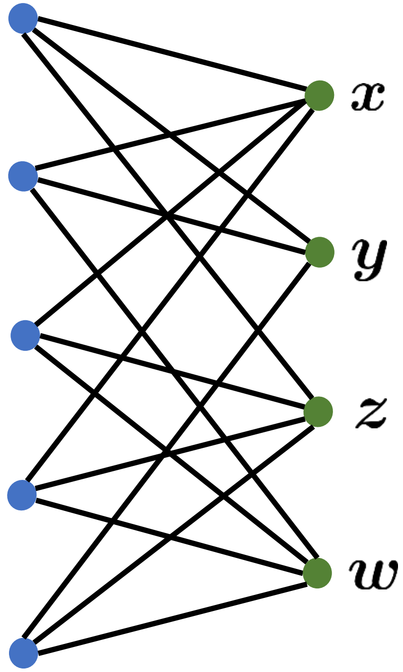
* [Beame, Impagliazzo, Sabharwal ‘01], [Pang ’21], [Atserias, Bonacina, **dR**, Lauria, Nordström, Razborov ‘18], [Lauria, Pudlák, Rödl, Thapen ‘13]
** [Meka, Potechin and Wigderson ‘15], ..., [Barak, Hopkins, Kelner, Kothari, Moitra, Potechin ’16]

Finding Structure in Randomness

Random k-SAT (and k-XOR)

- Expansion G is (s, ϵ)-bipartite expander if $\forall U \subseteq V$: $|U| \leq s \Rightarrow |N(U)| \geq (1+\epsilon)|U|$
- Constraint-variable graph



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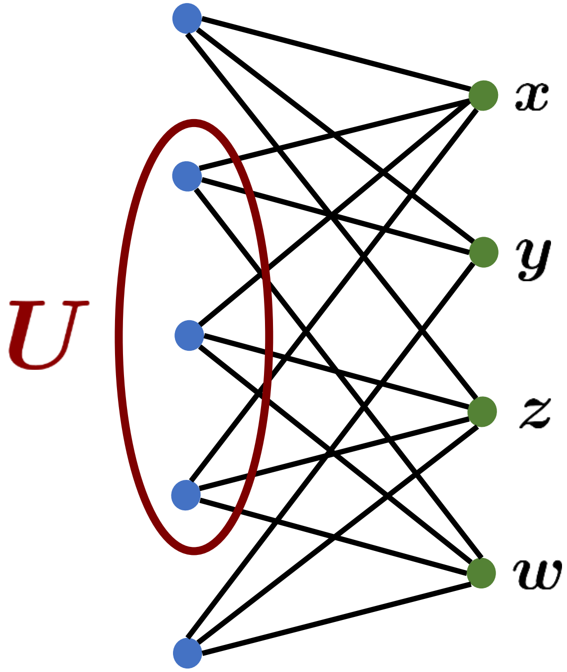

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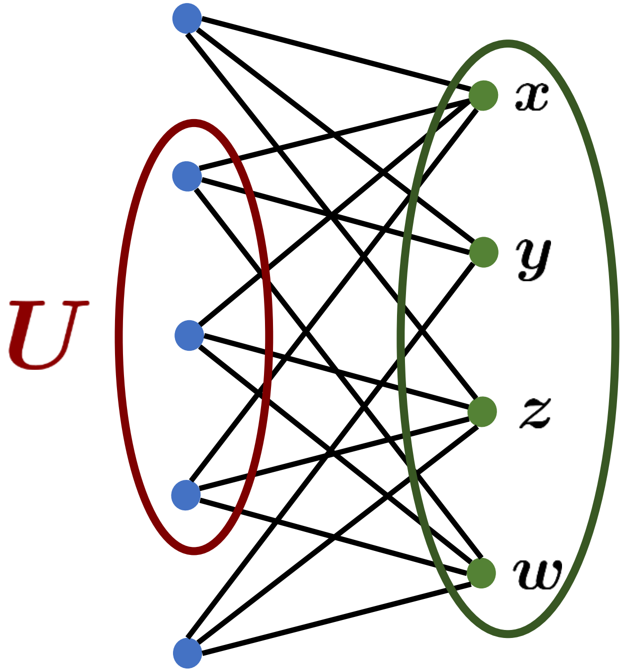

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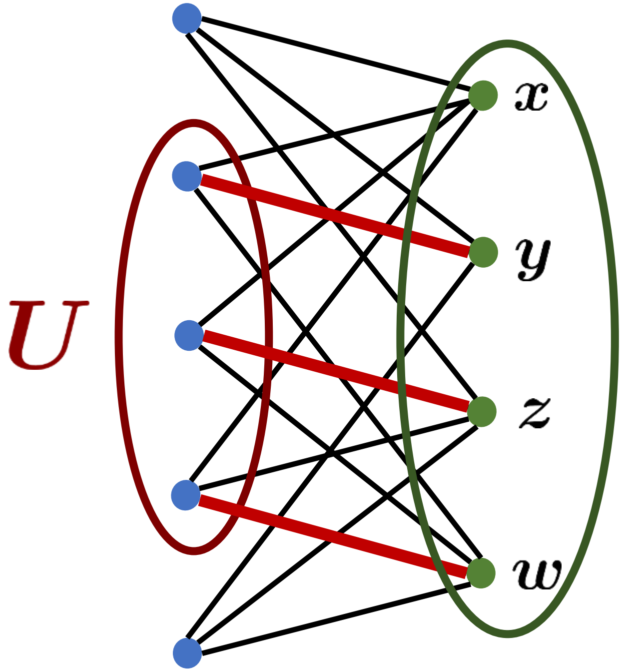

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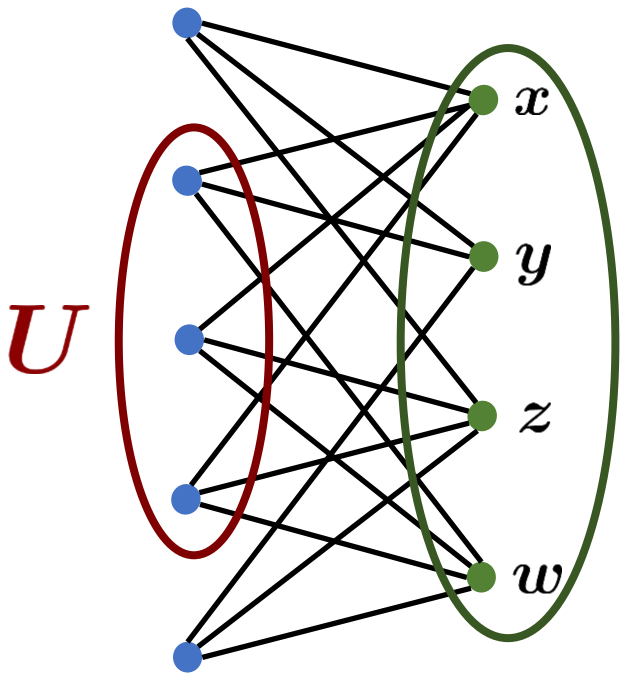

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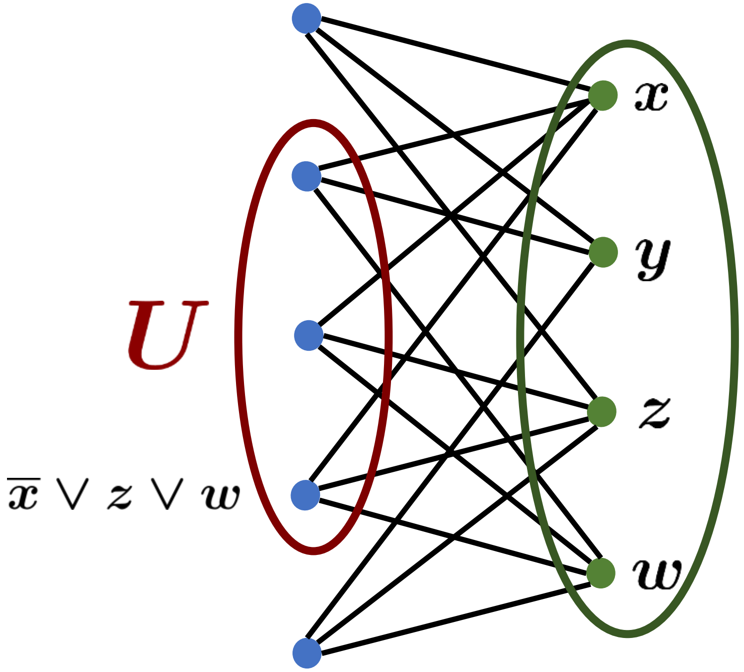

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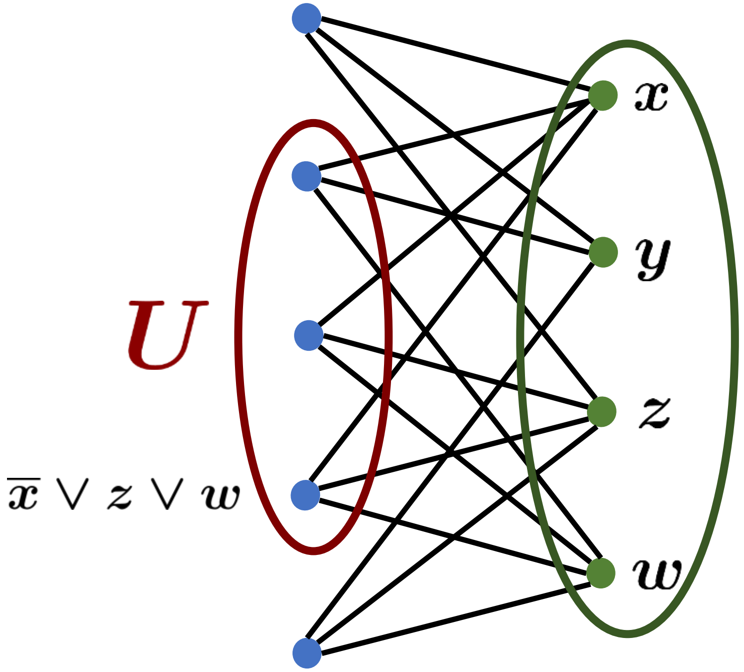

Prover

Delayer

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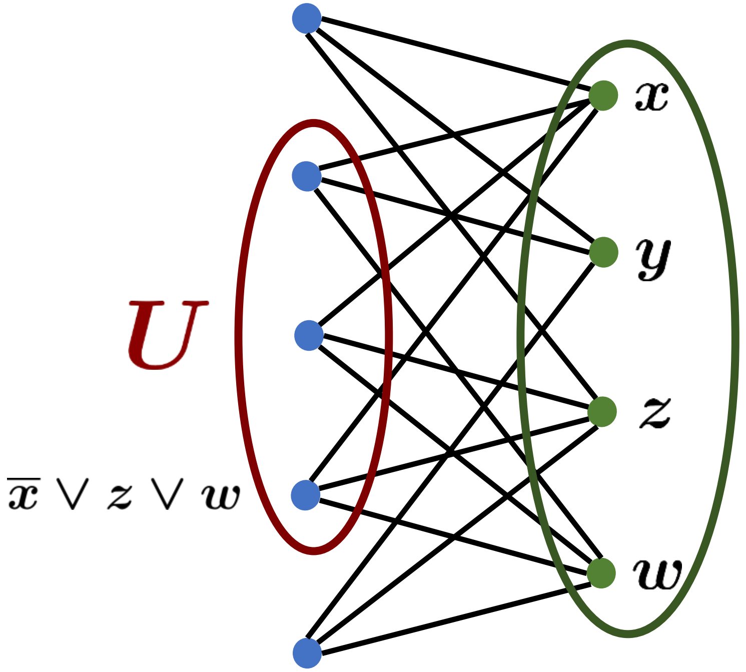

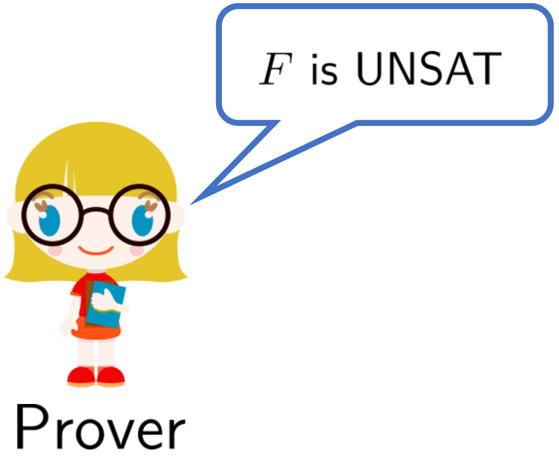

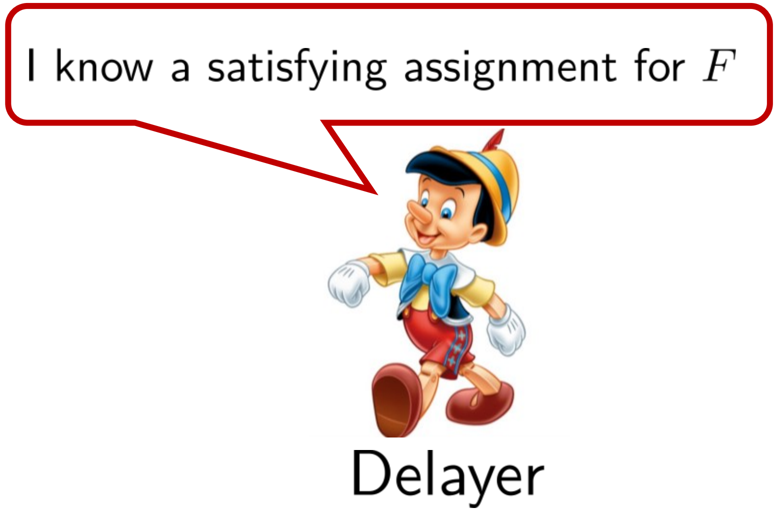

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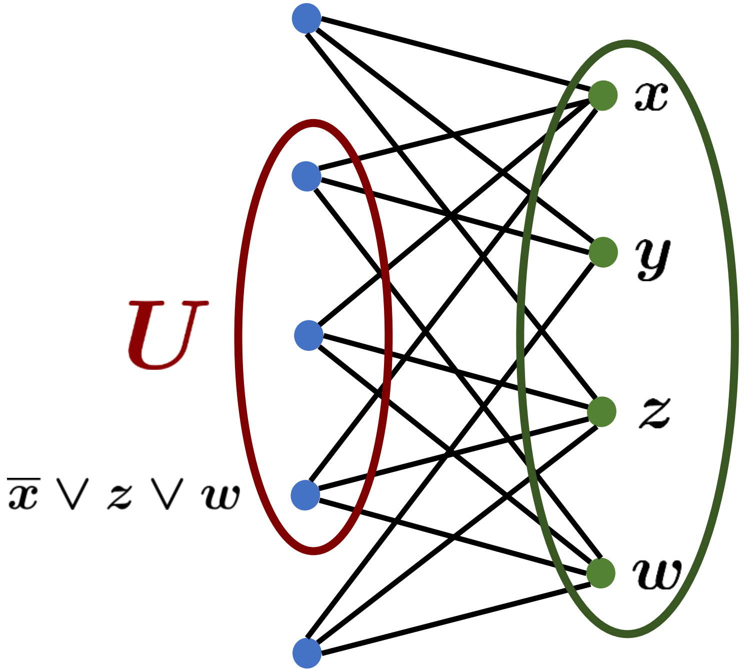

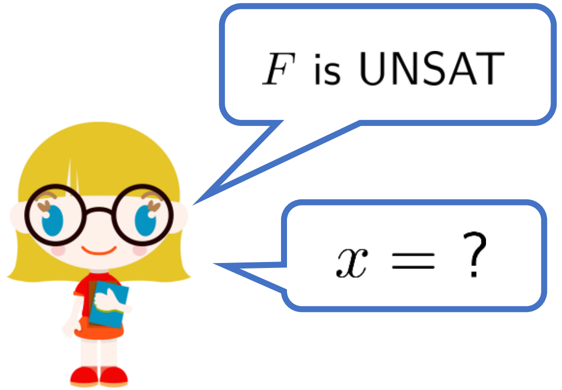

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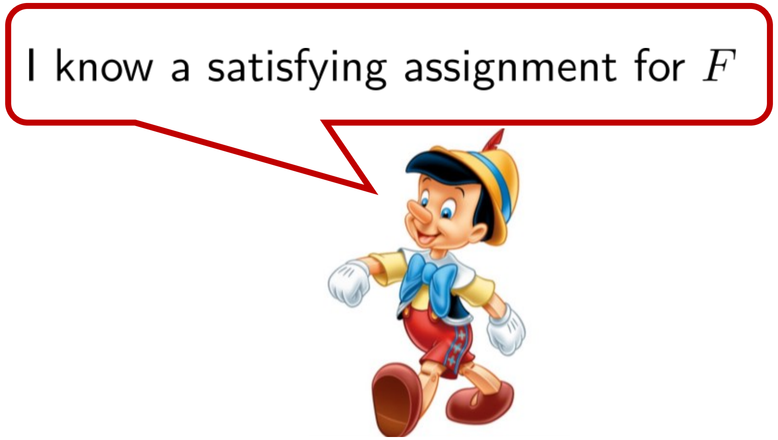

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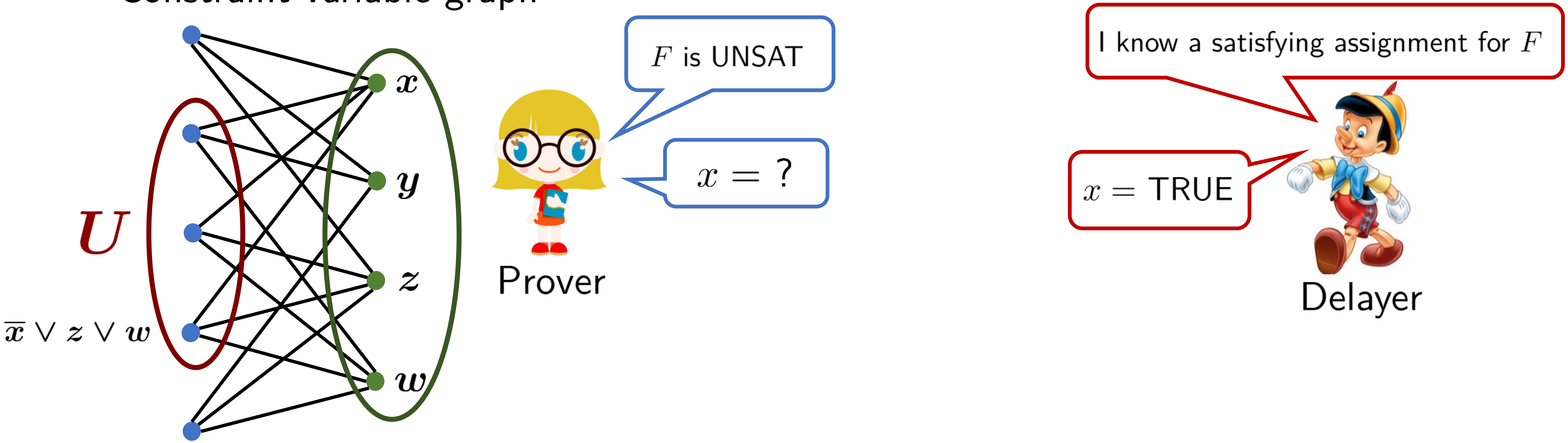

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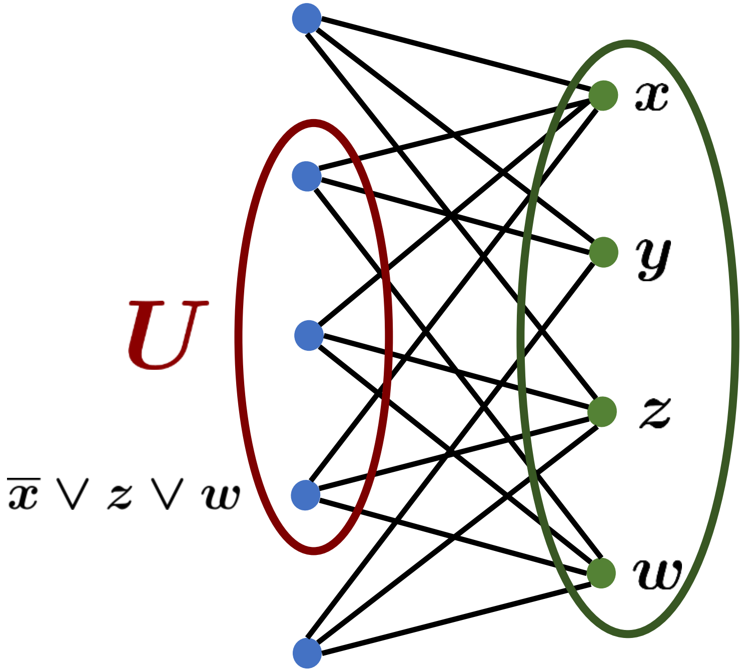

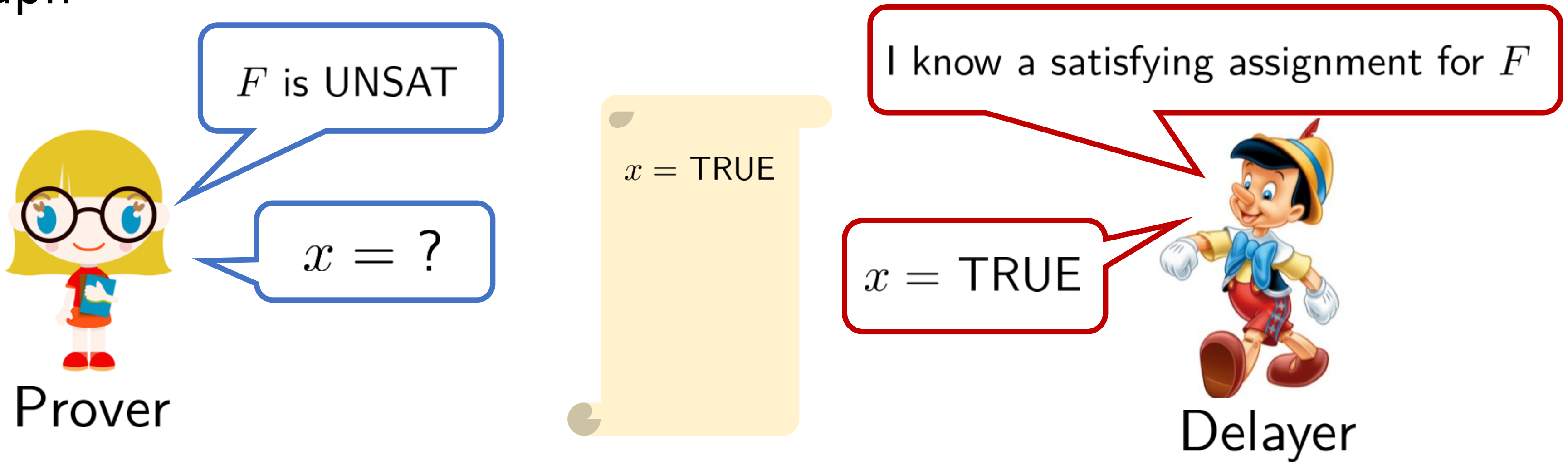

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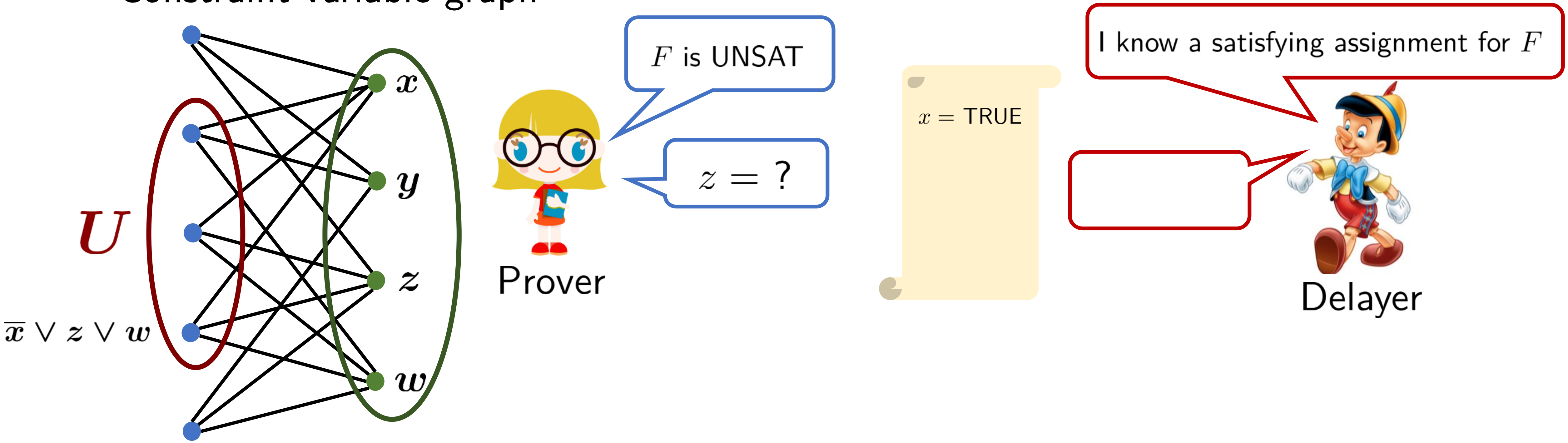

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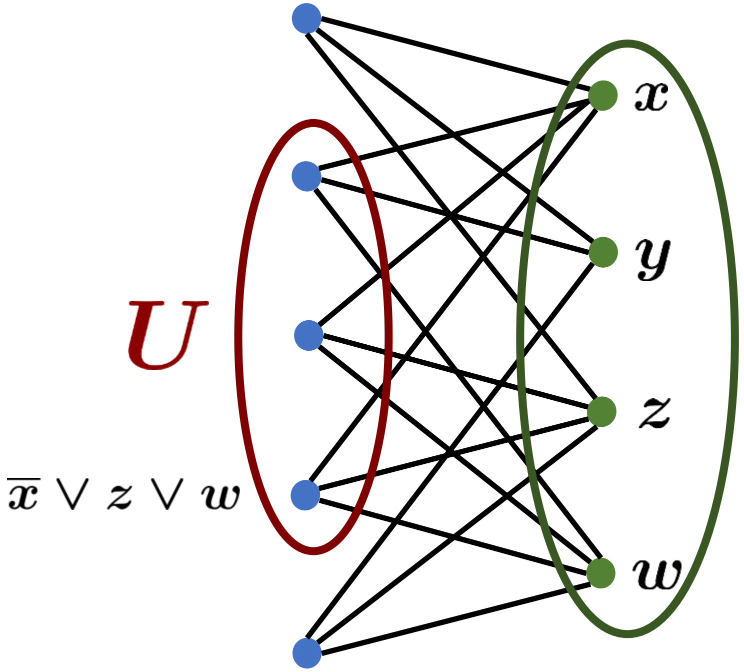

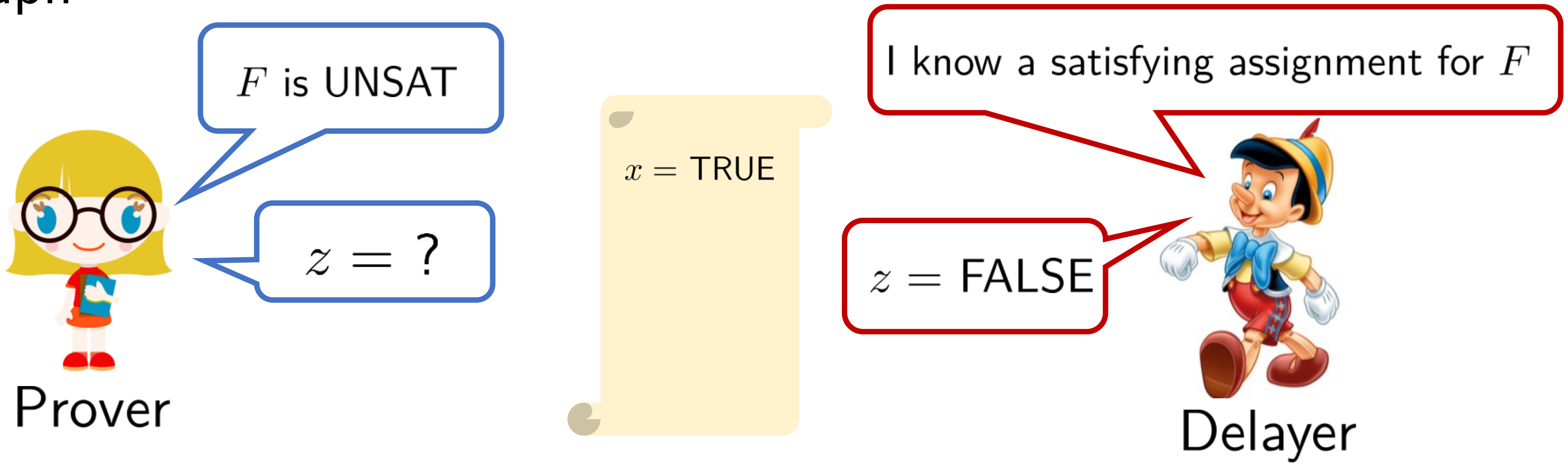

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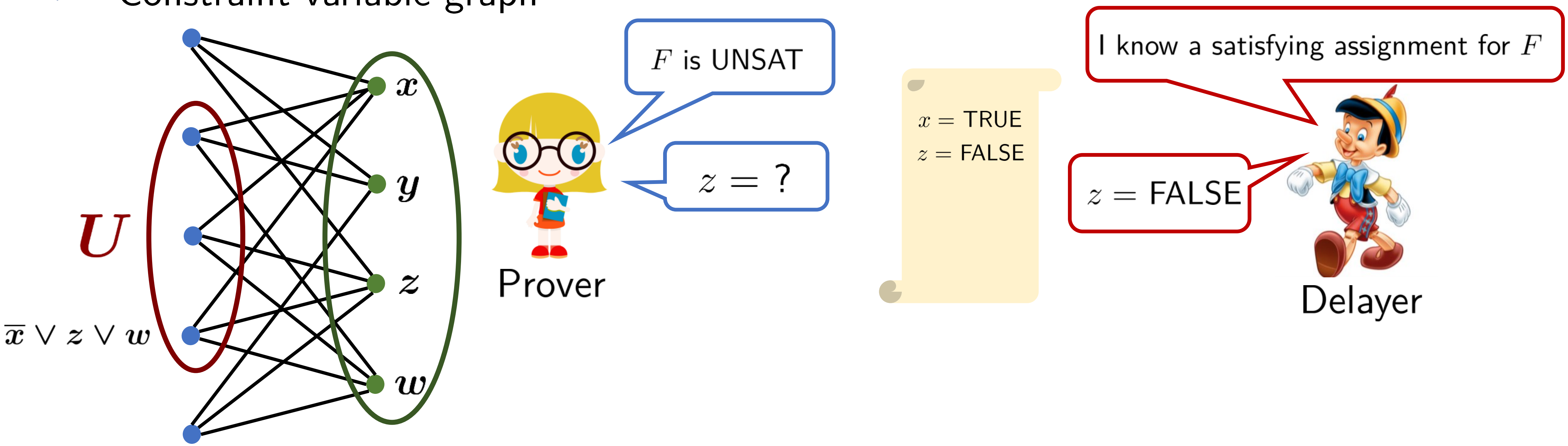

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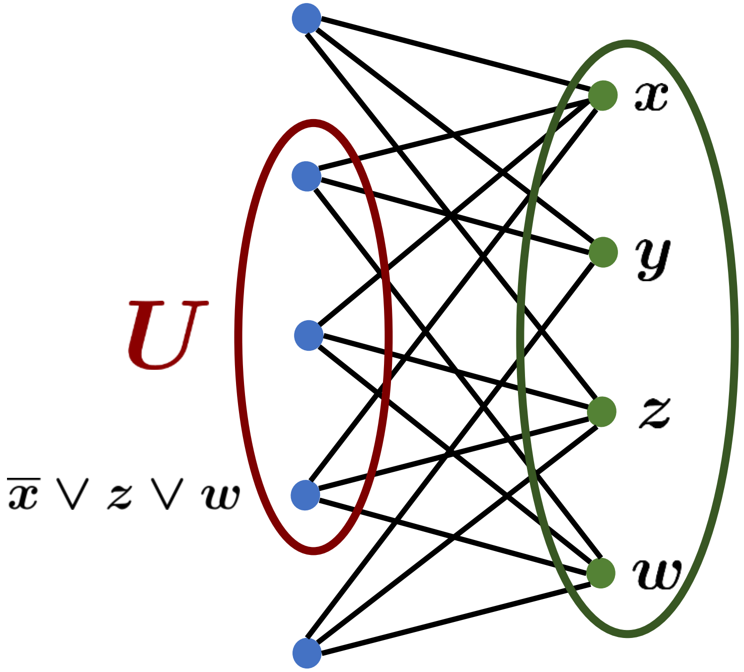

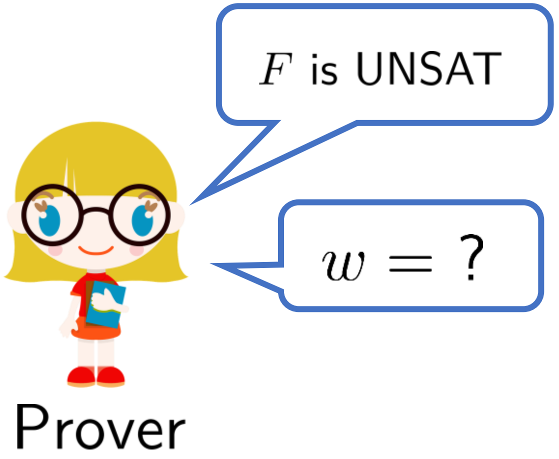

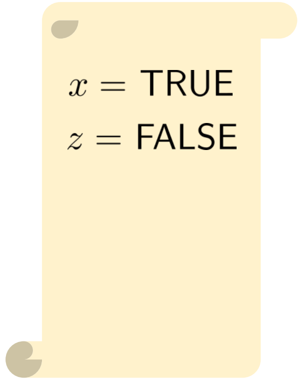

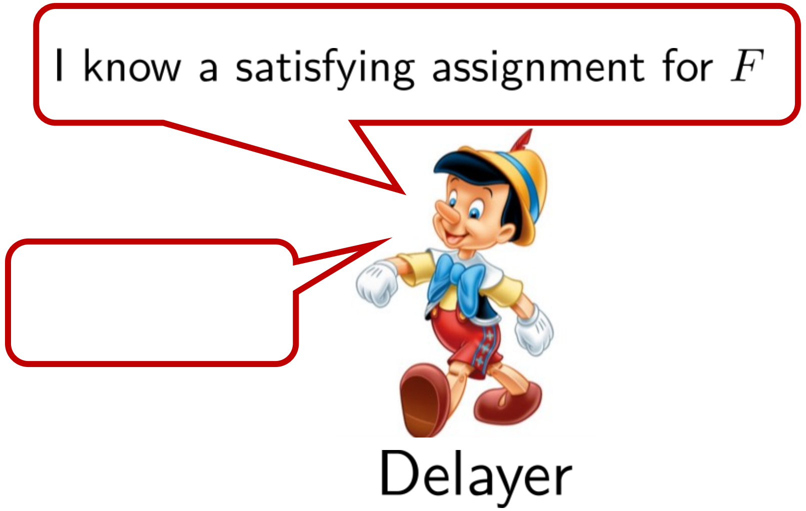

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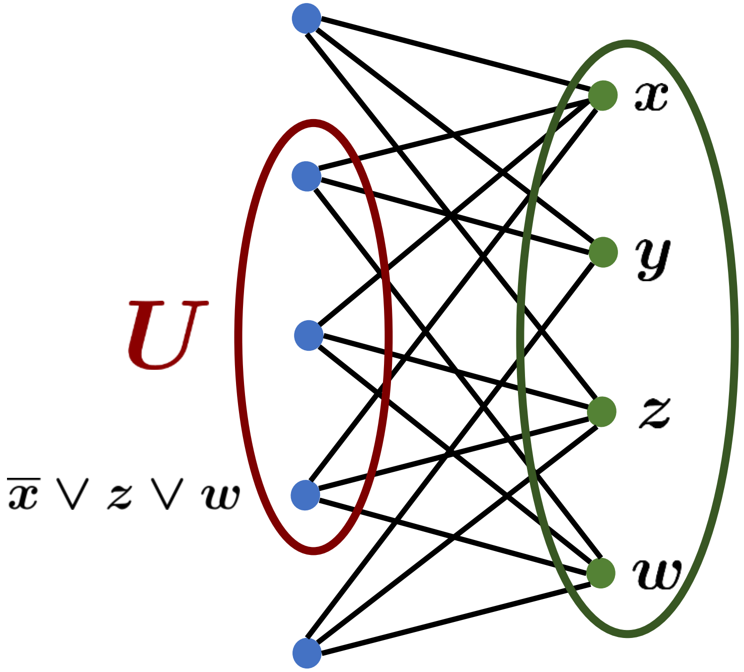

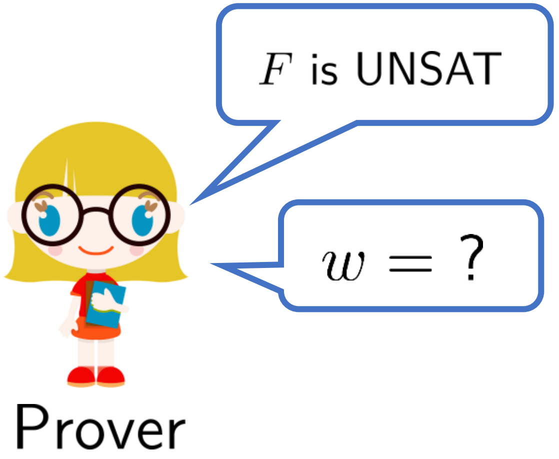

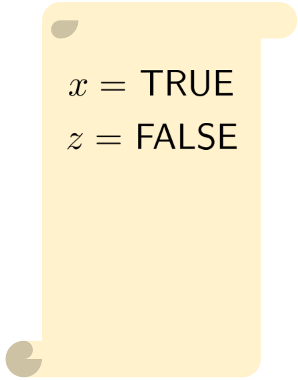

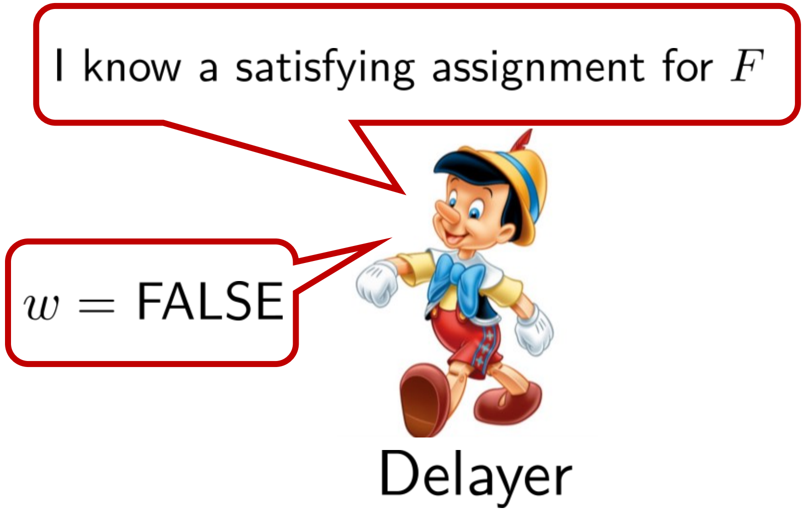

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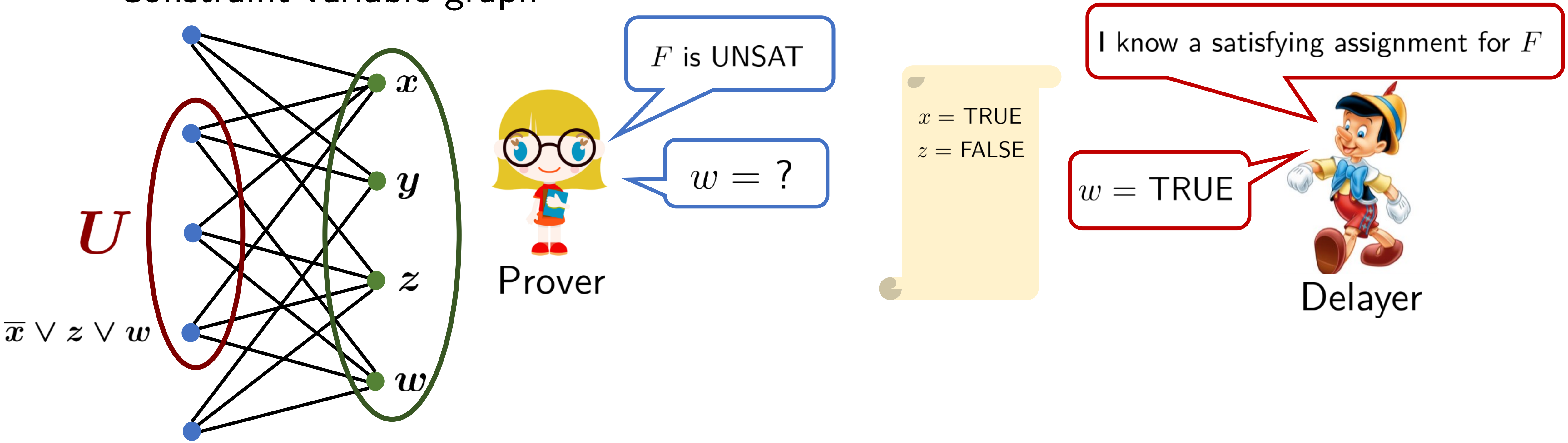

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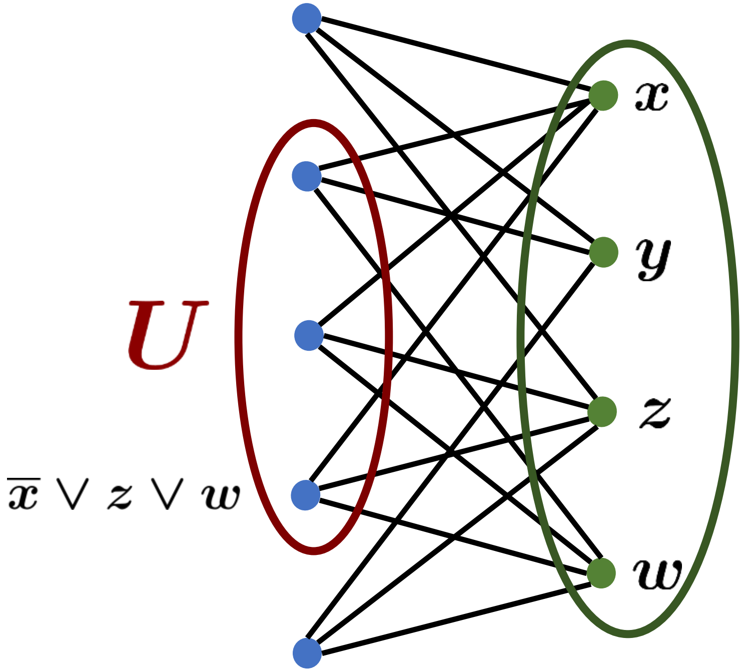

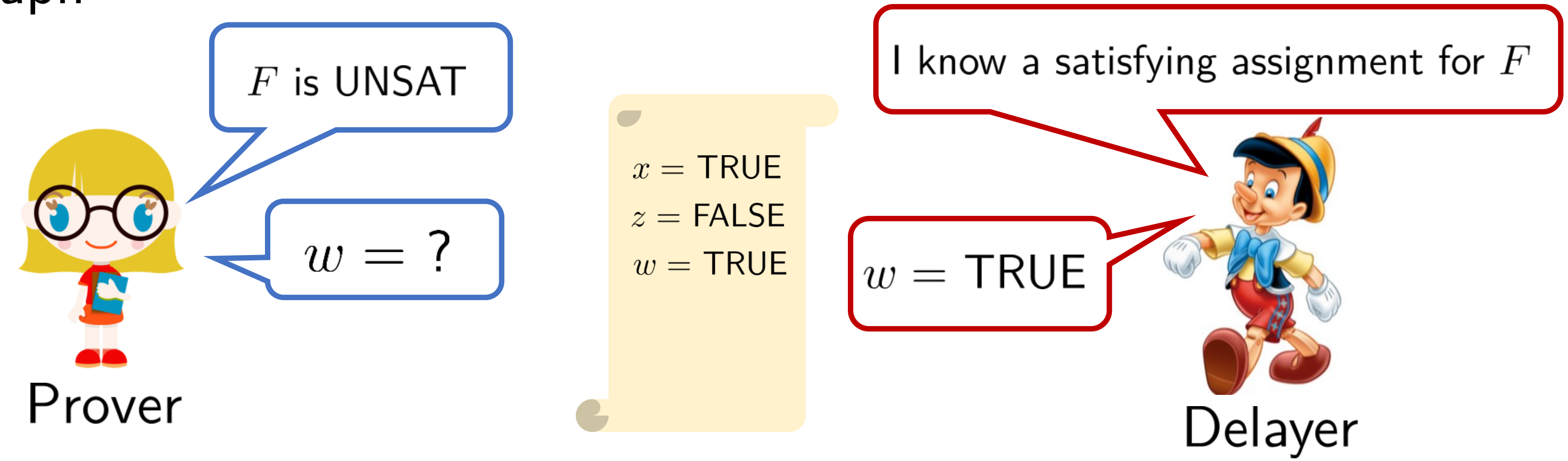

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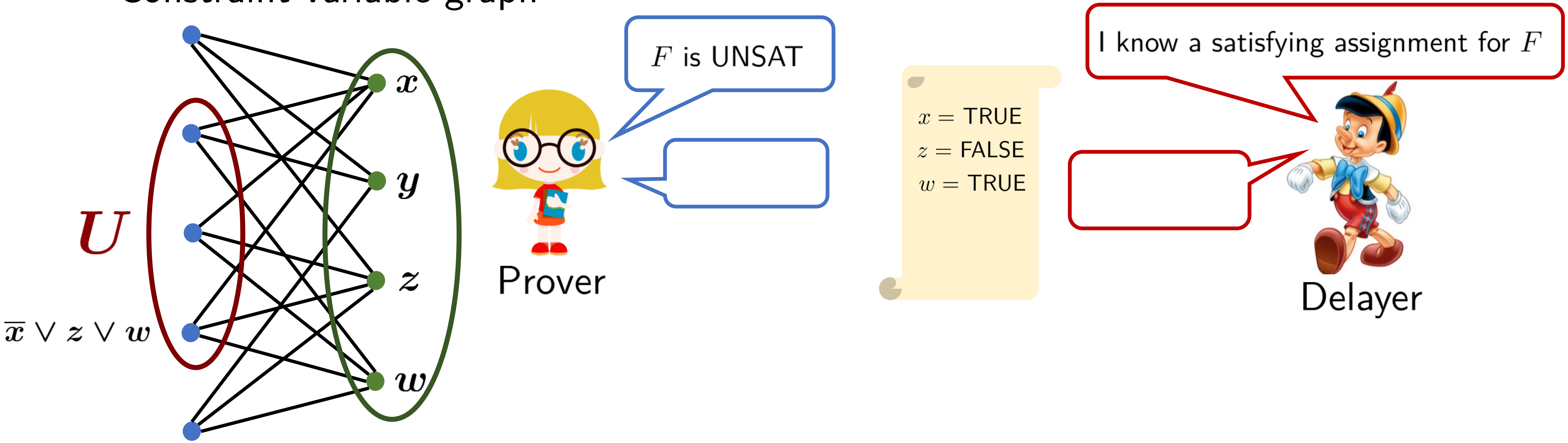

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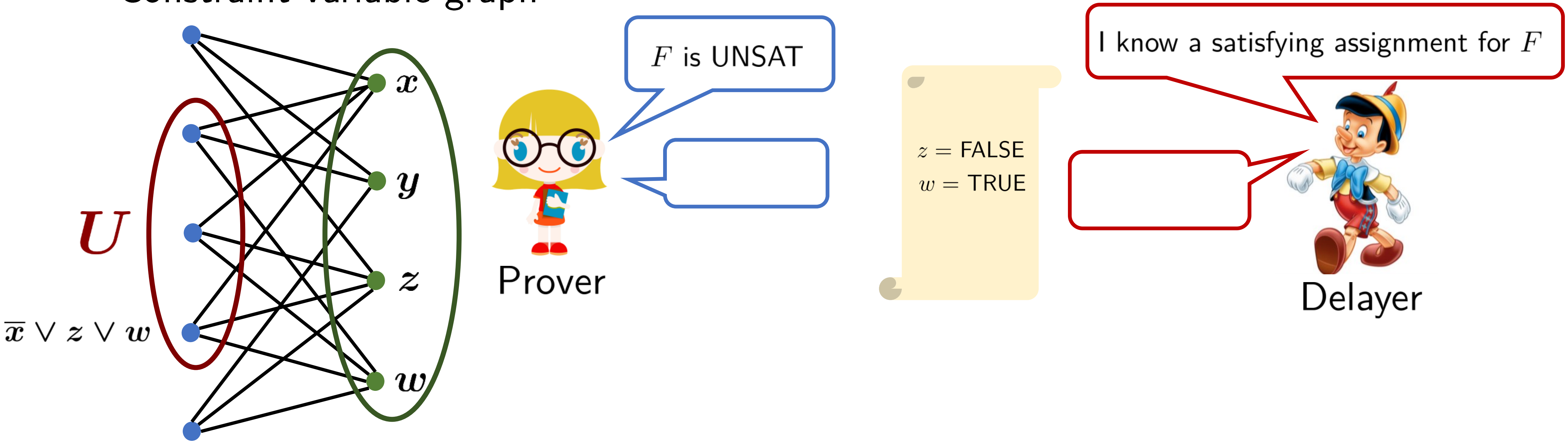

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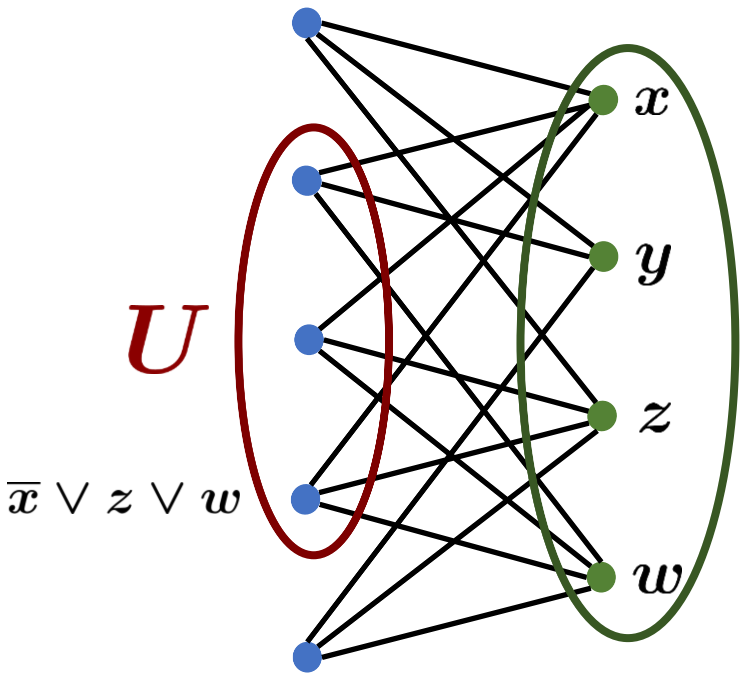

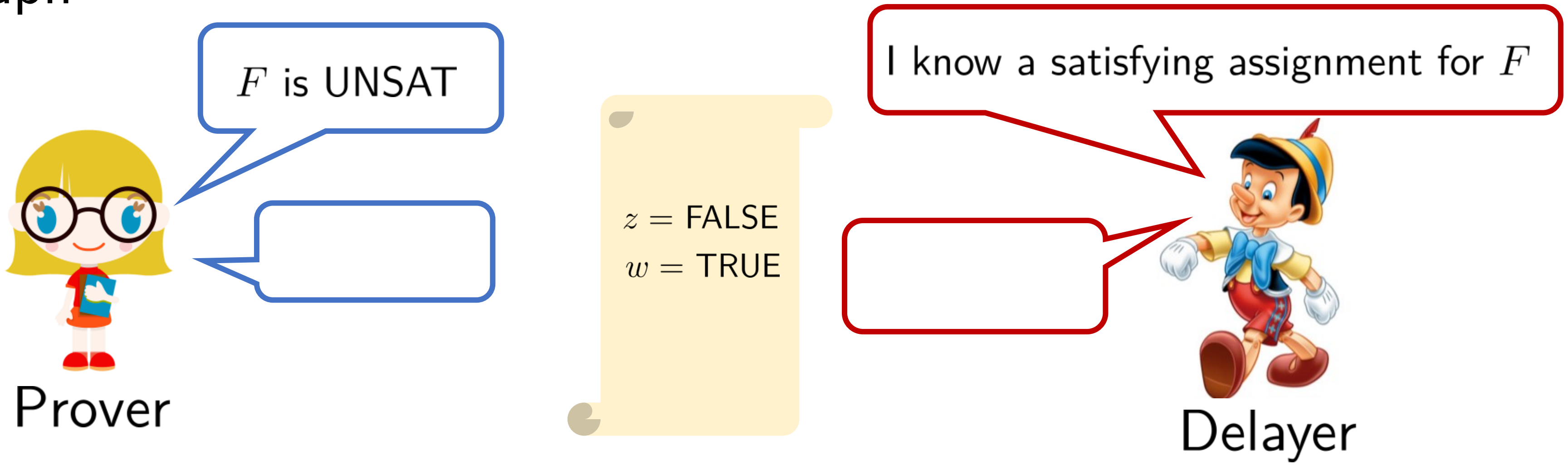

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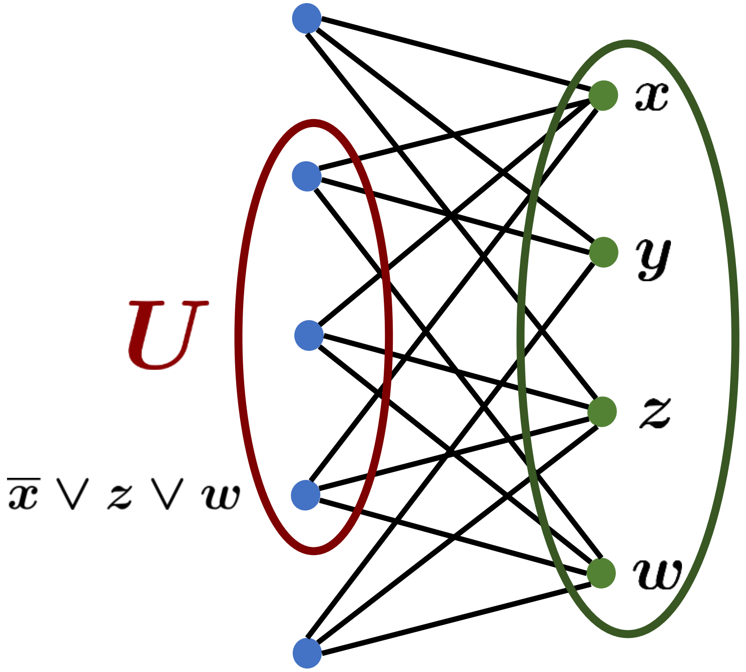

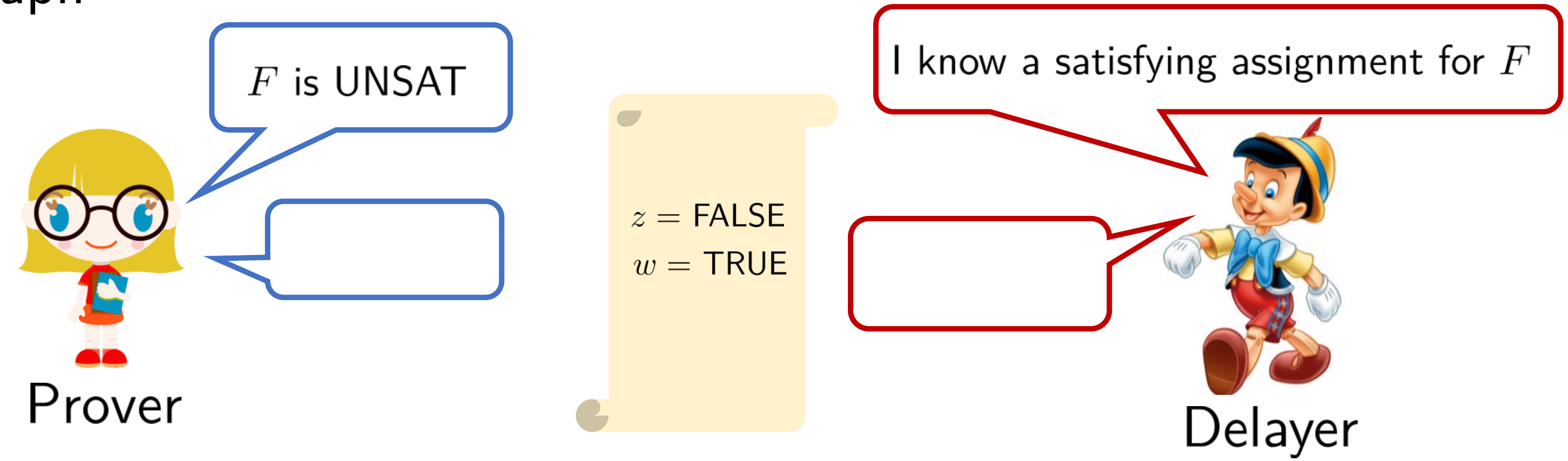

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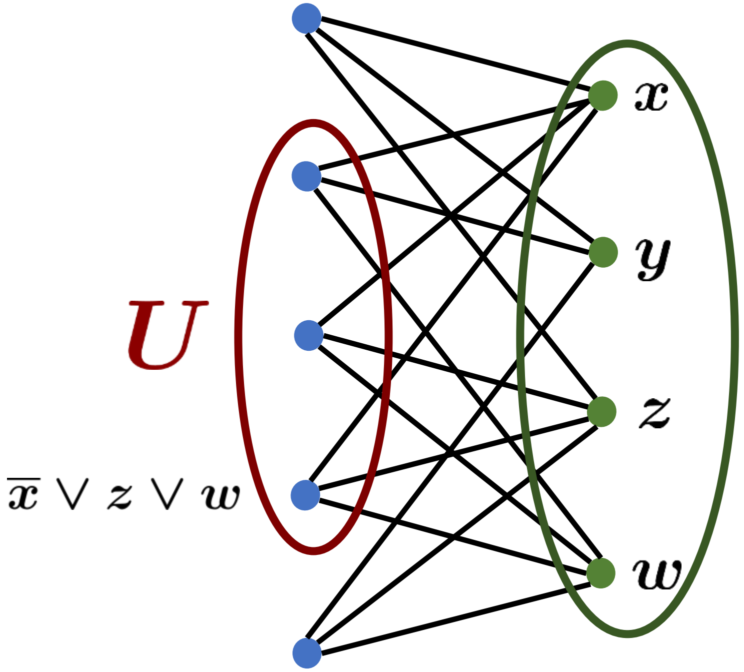

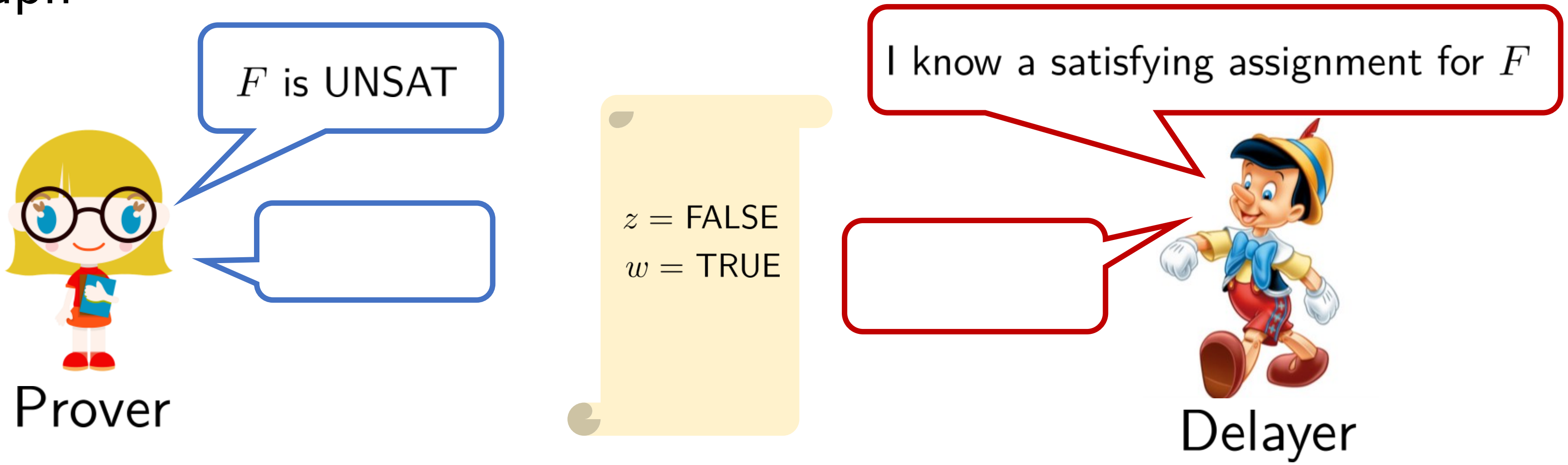

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Lemma 1. If G is (s,ϵ)-bipartite expander Delayer wins if $r \leq \epsilon s/(d+\epsilon)$

Lemma 2. W.h.p. constraint-variable graph of random 3-CNF is a good expander

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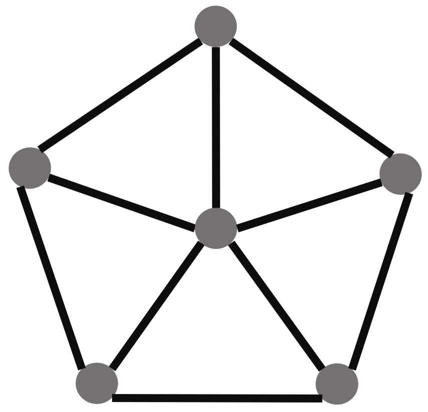

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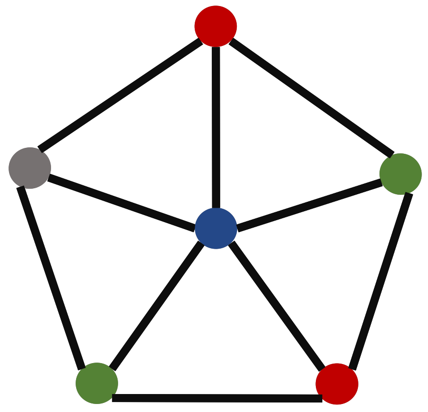

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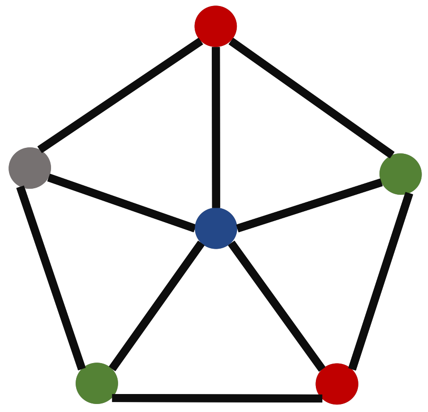

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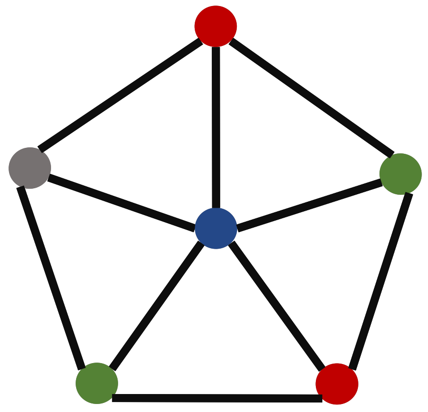

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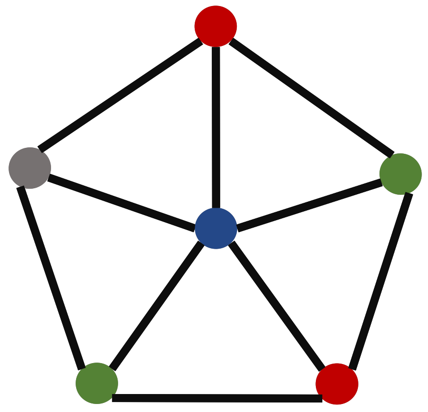

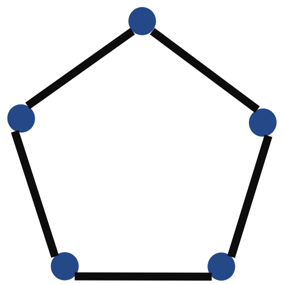

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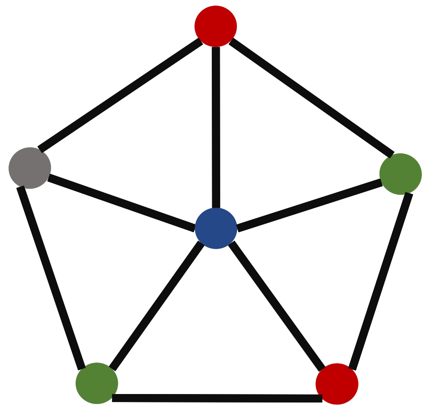

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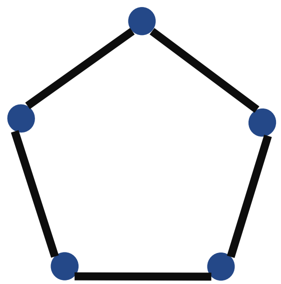

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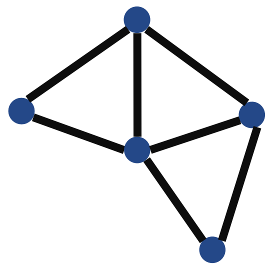

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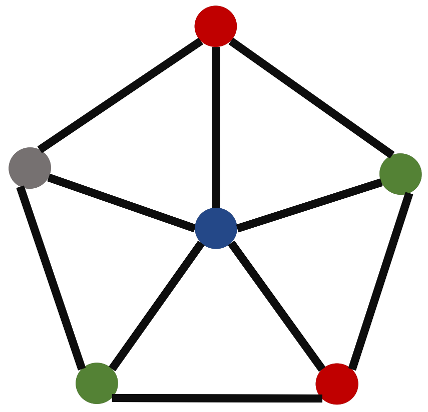

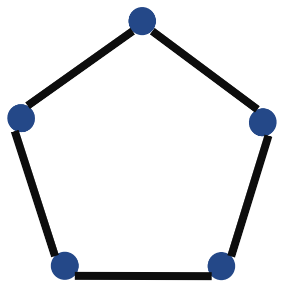

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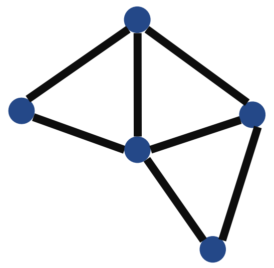

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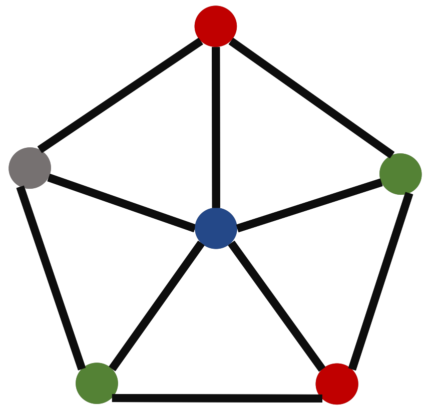

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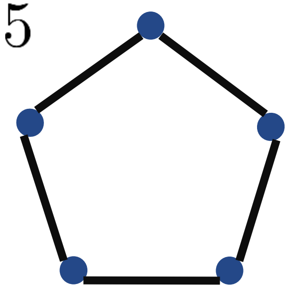

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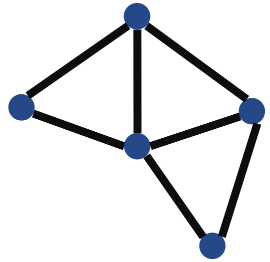

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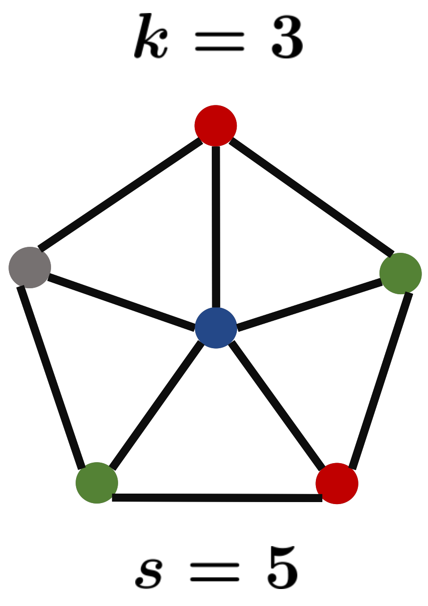

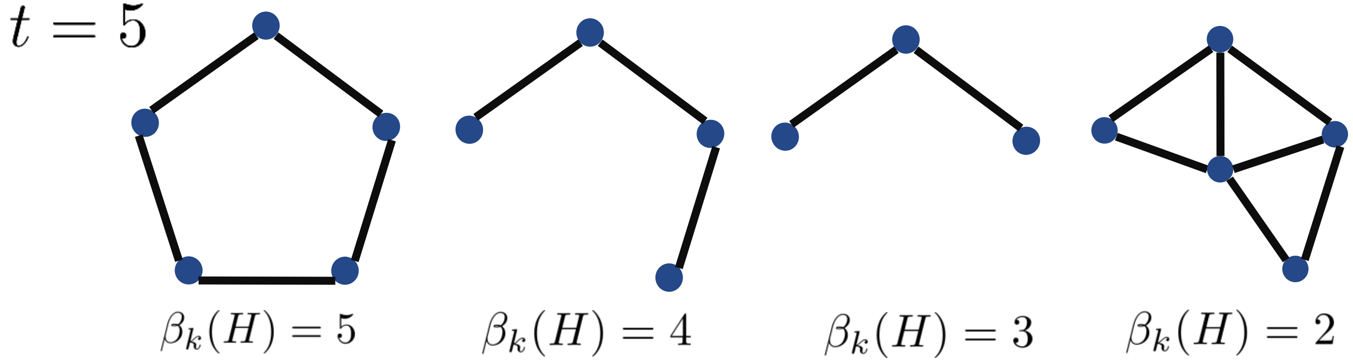

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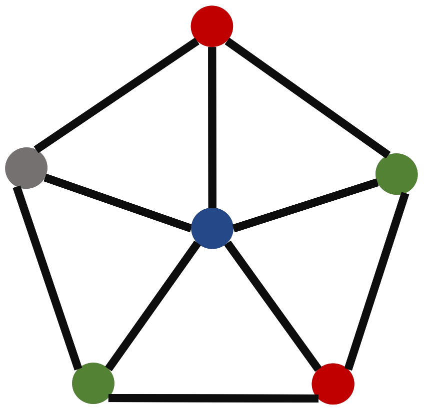

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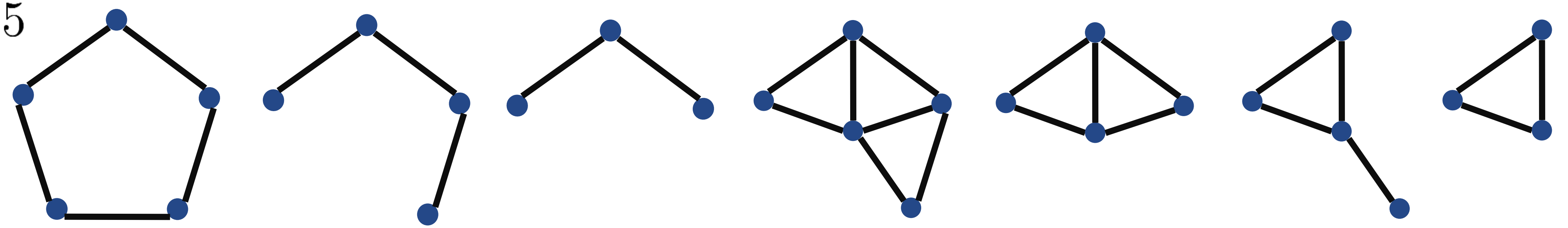

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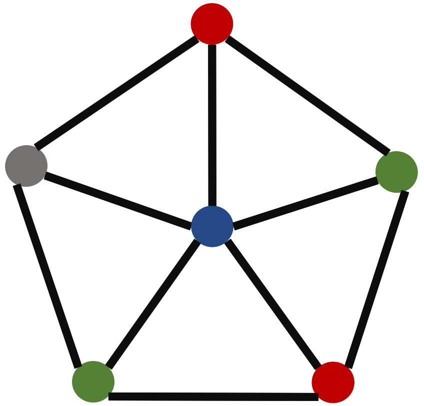

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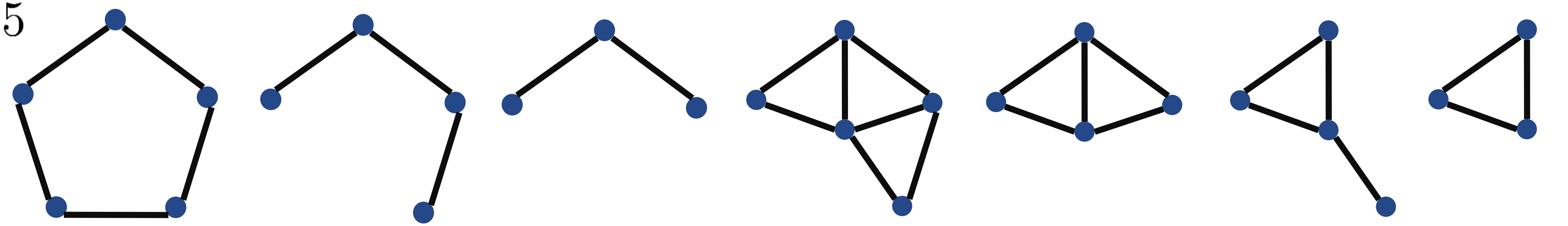

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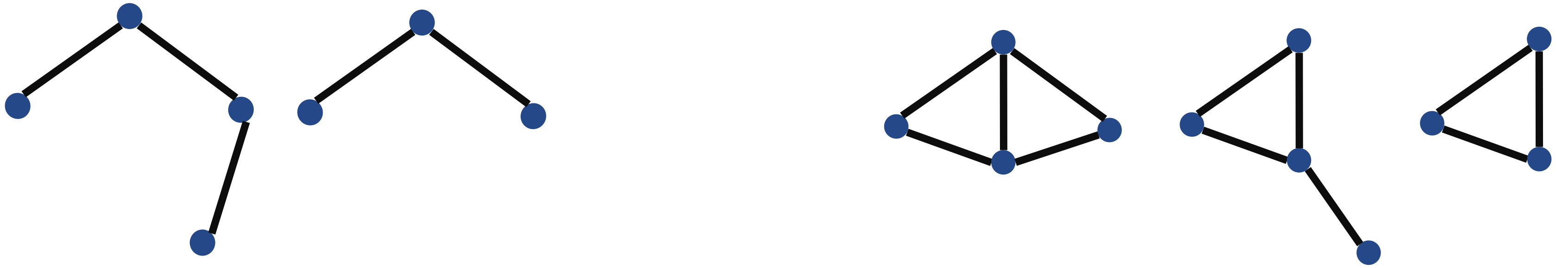

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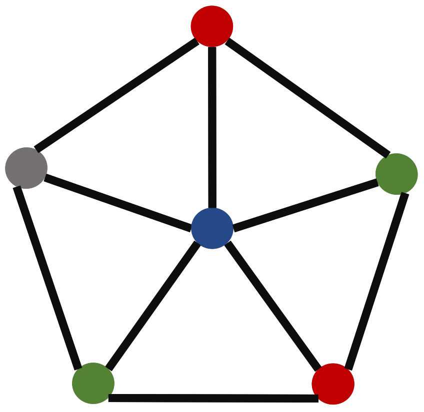

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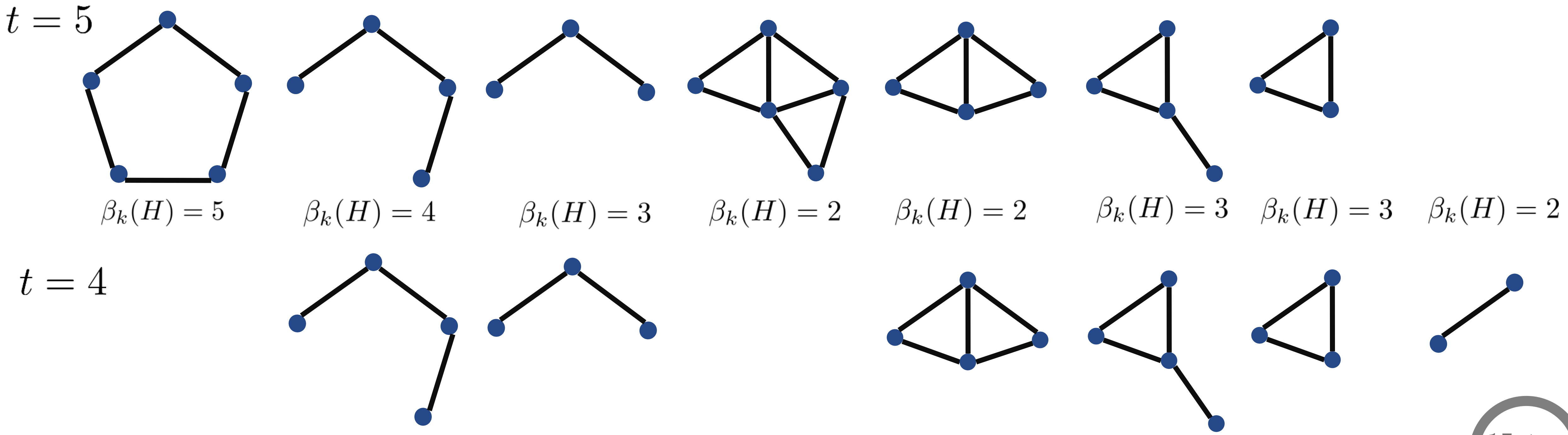

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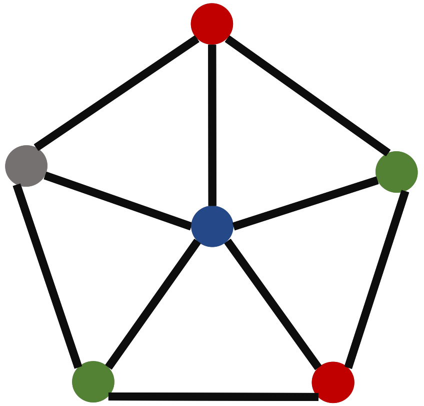

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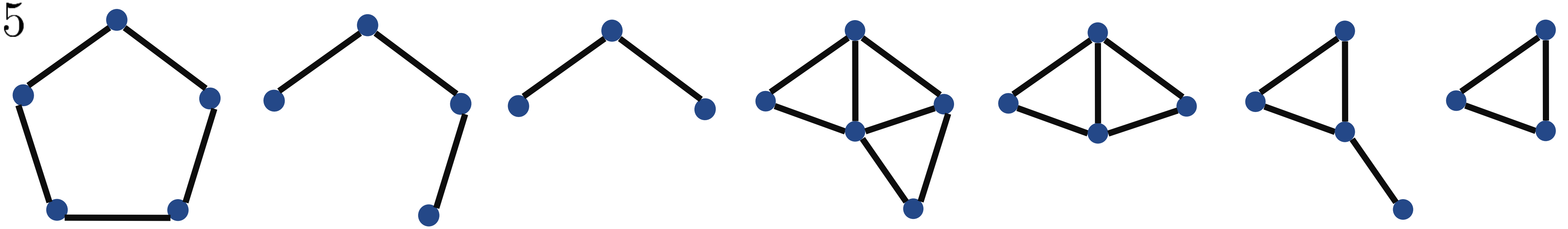
Subcritical k-expansion $\displaystyle e_k(G) = \max_{2 \leq t \leq s} \min_{\substack{H \subseteq G \\ H \text{ connected} \\ t/2 \leq V(H) \leq t}} \beta_k(H)$

$t = 5$



$\beta_k(H) = 5$ $\beta_k(H) = 4$ $\beta_k(H) = 3$ $\beta_k(H) = 2$ $\beta_k(H) = 2$ $\beta_k(H) = 3$ $\beta_k(H) = 3$ $\beta_k(H) = 2$

Lemma 1. Resolution width of refuting Color$(G, k) \geq e_k(G)$

Lemma 2. Let $G \sim \mathcal{G}_{n,m}$ for $m = \Delta n$. W.h.p. $e_k(G) \geq \epsilon_k n / \Delta^{1+2/(k-2)}$

Clique formula

Clique(G, k) for $G \sim \mathcal{G}(n, p)$ and p close to k-clique threshold

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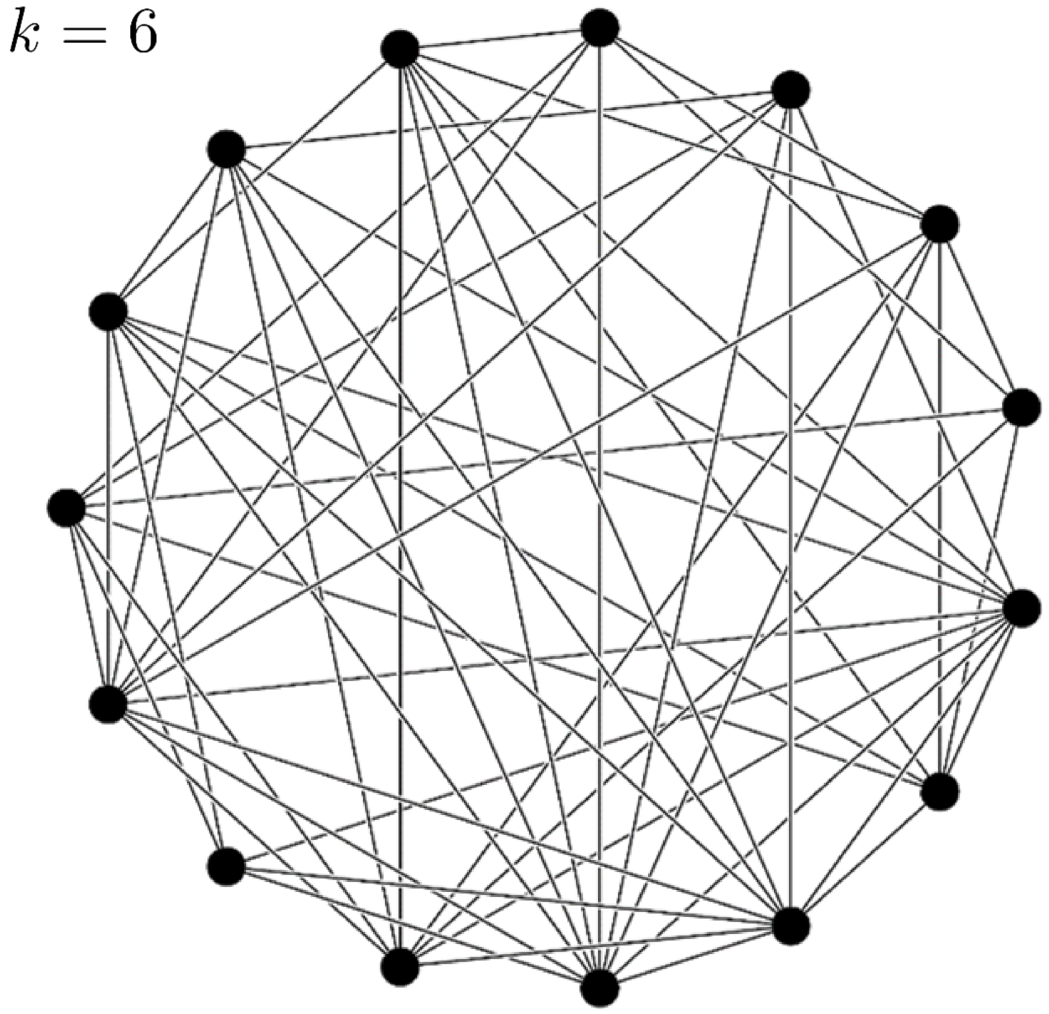
Open: Show that resolution, polynomial calculus or sum of squares requires size $n^{\Omega(k)}$ to refute Clique(G, k)

Clique formula hard for tree-like resolution

[Beyersdorff, Galesi, Lauria '11]



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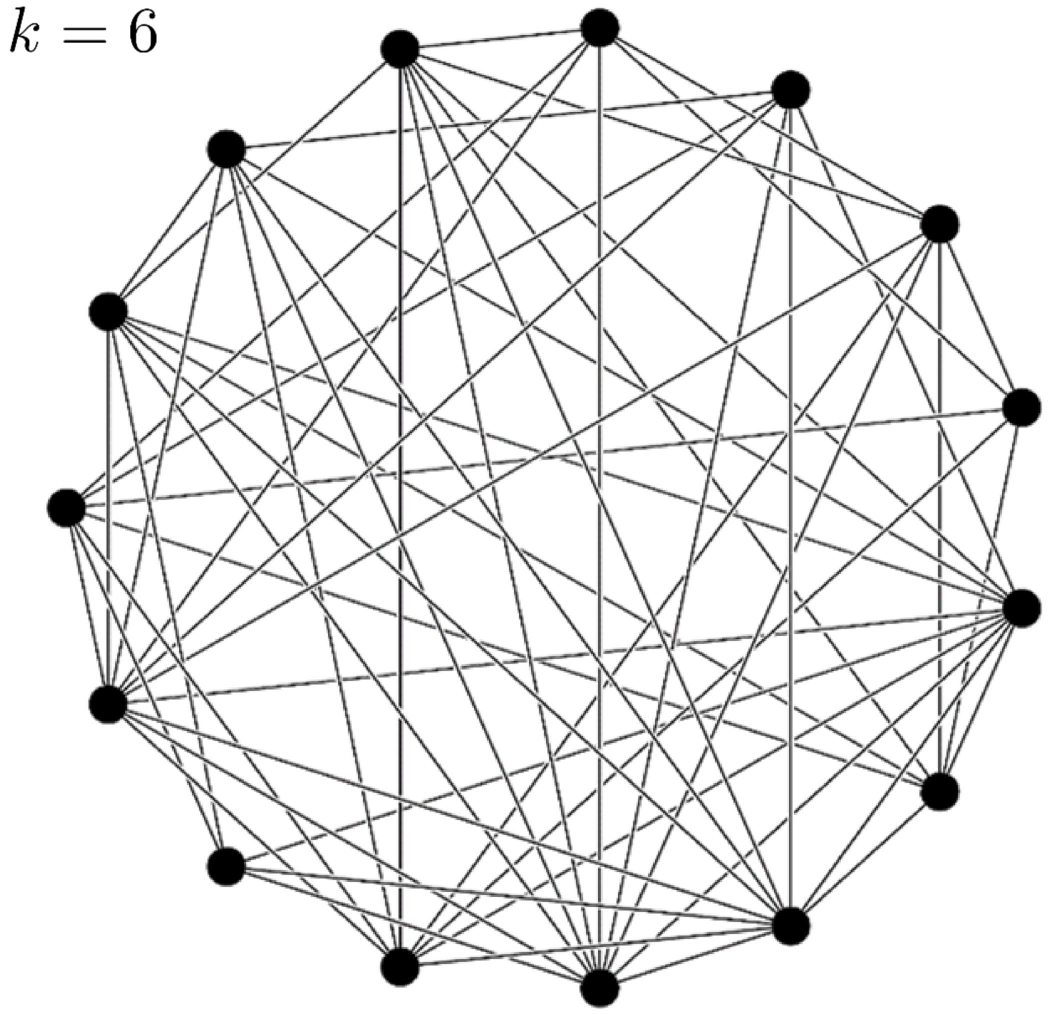

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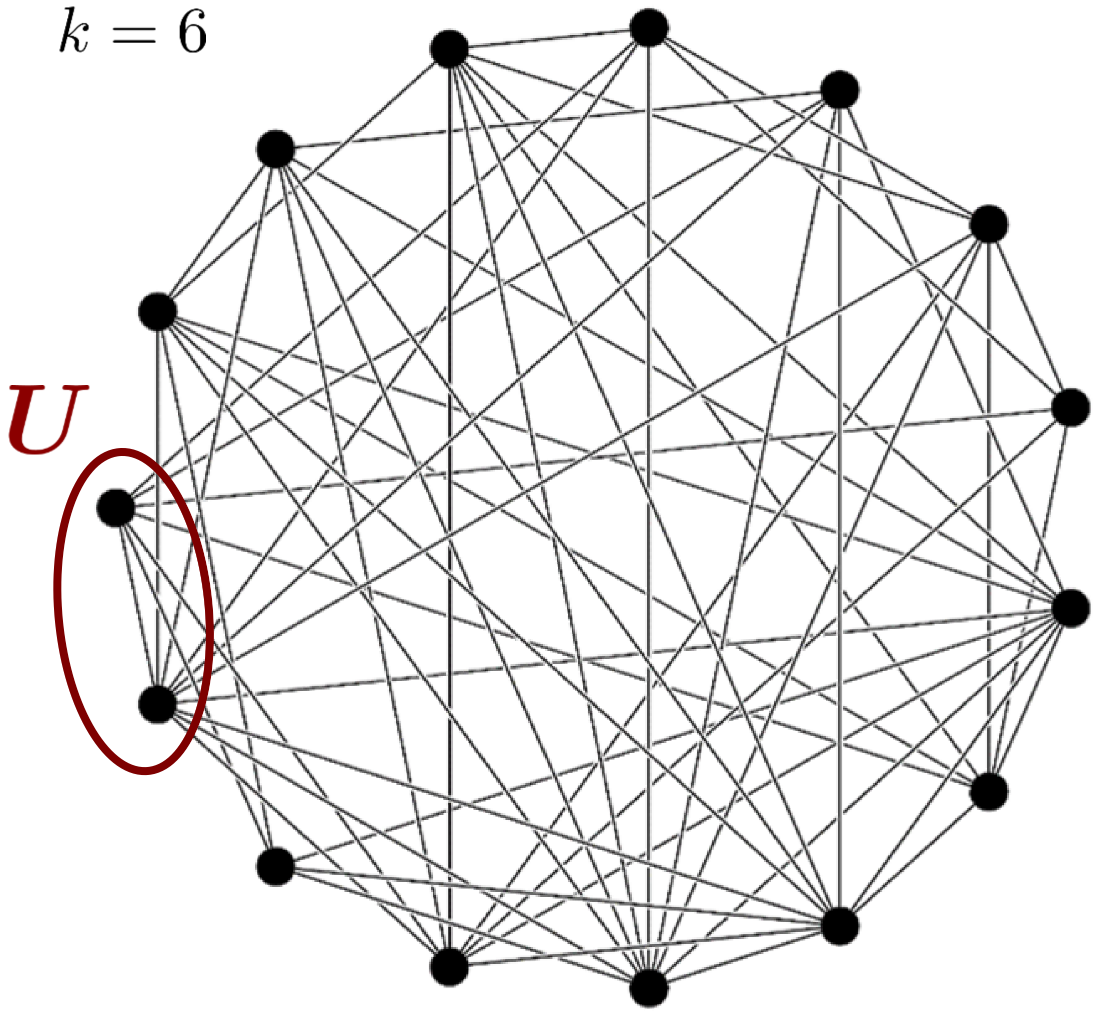

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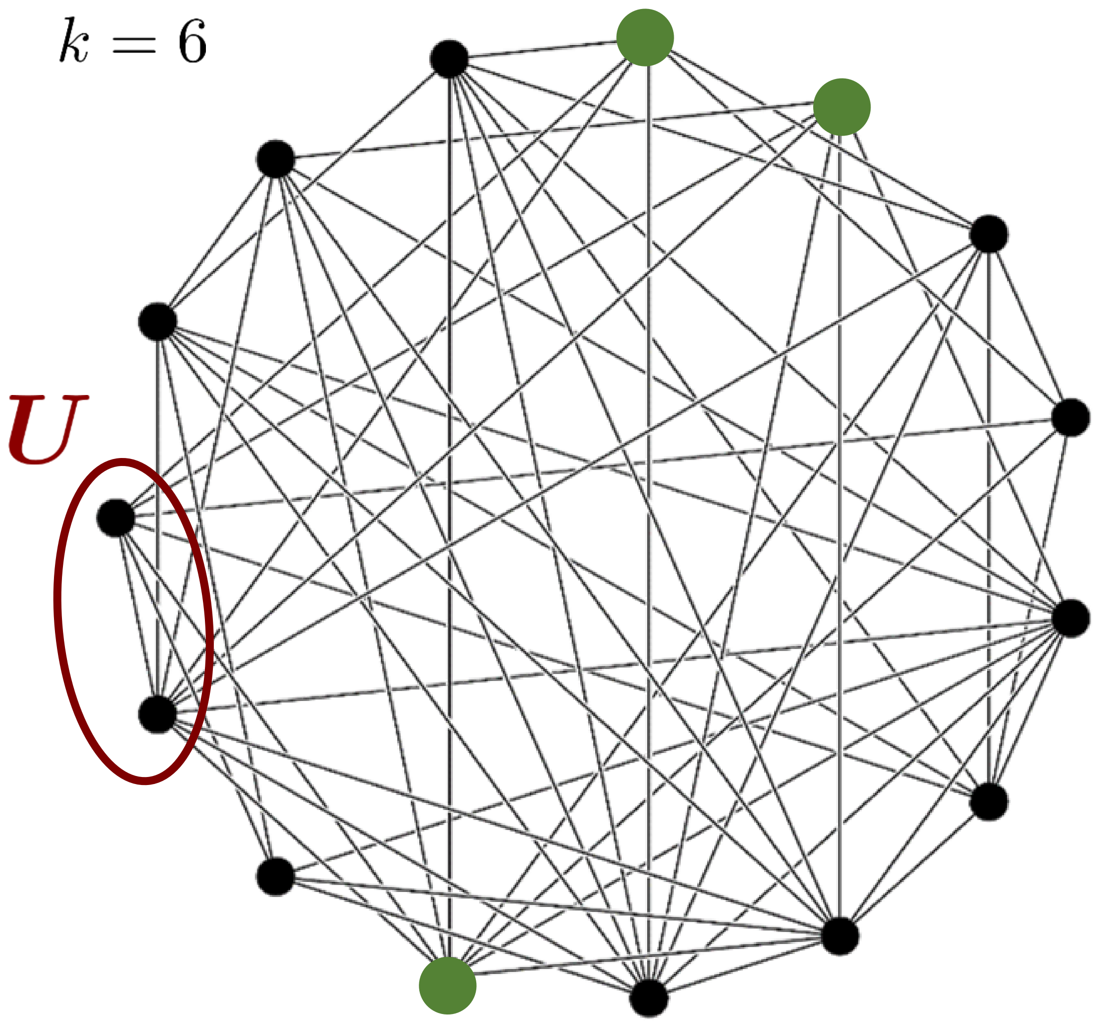

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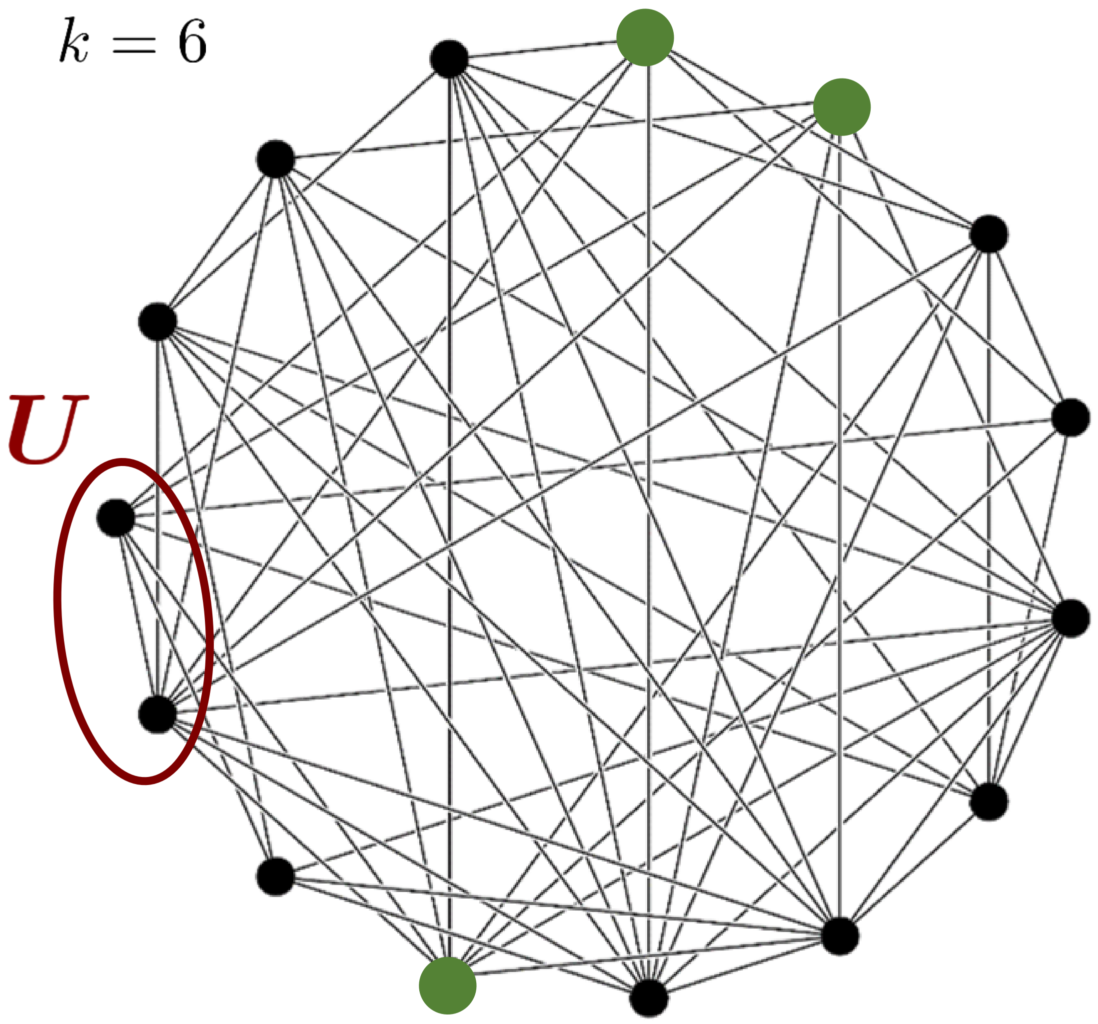

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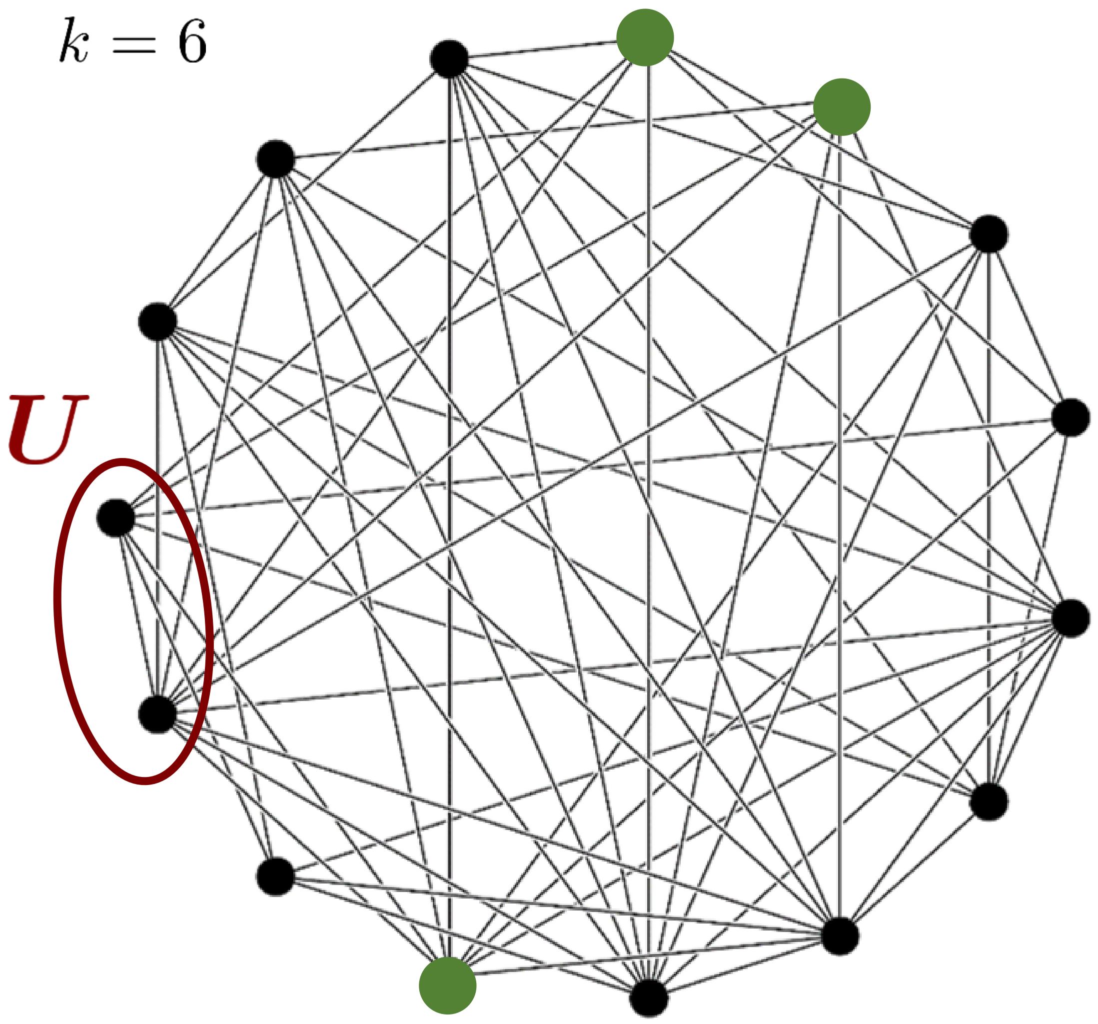

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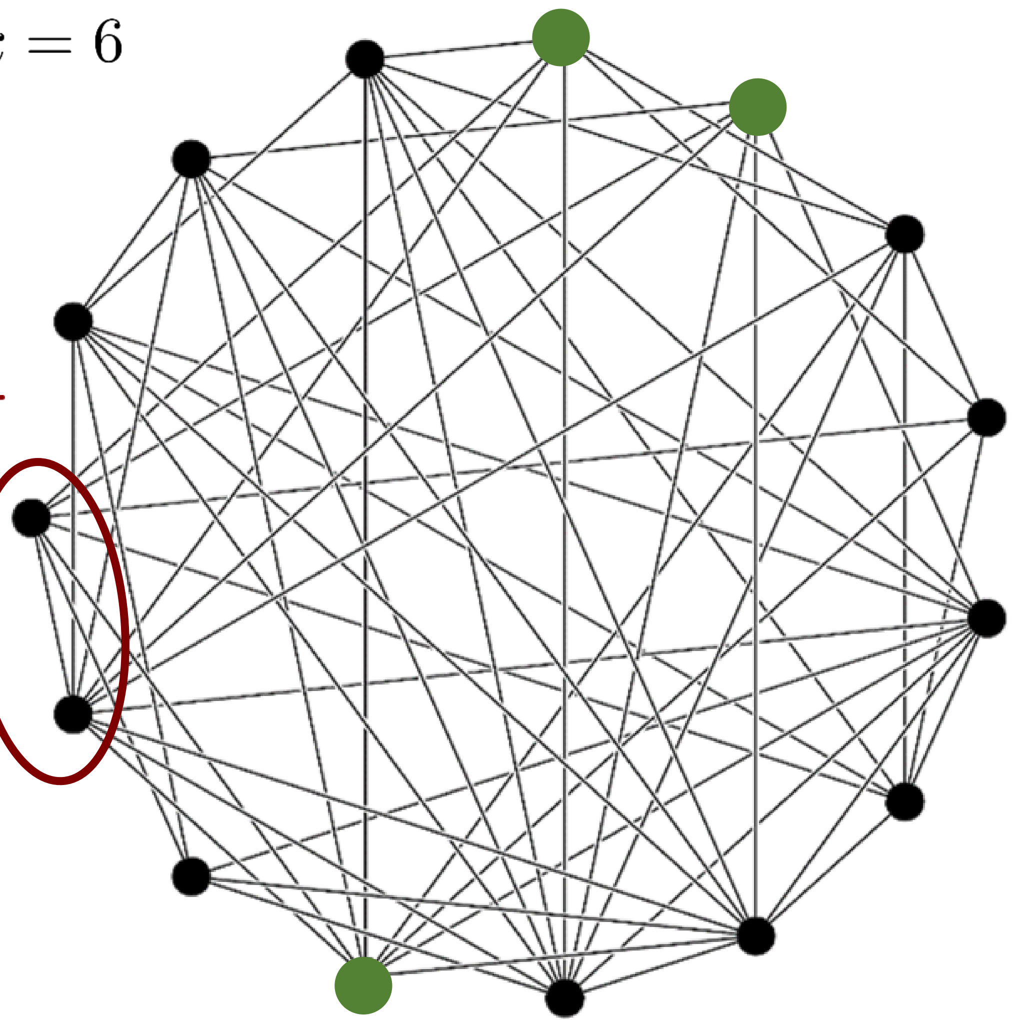

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Lemma 1. W.h.p. $G \sim \mathcal{G}(n,p)$ is neighbor dense

(for p close to k-clique threshold)

Lemma 2. Tree-like refutation of neighbor dense G

must have size $\ge n^{\Omega(k)}$

Property not enough for stronger proof systems

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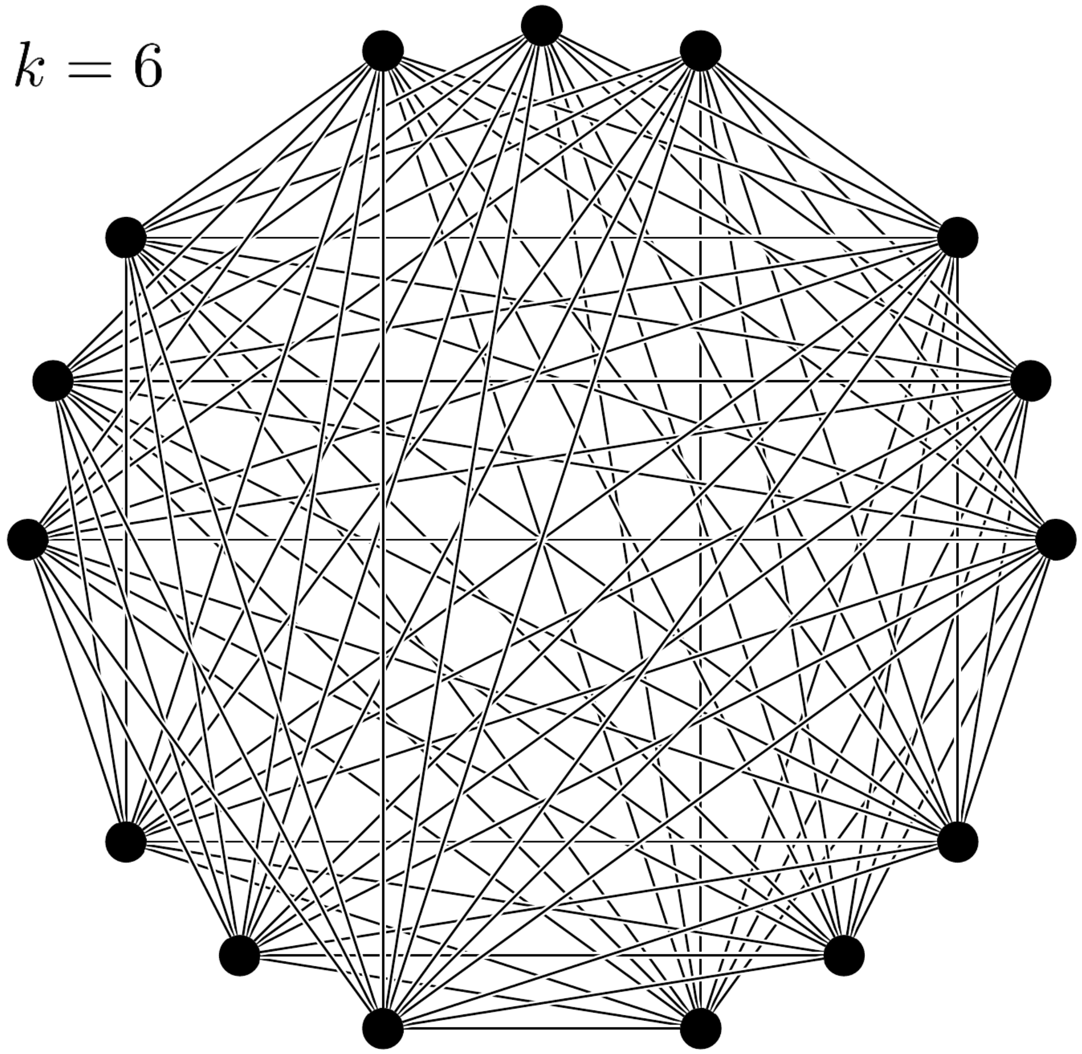

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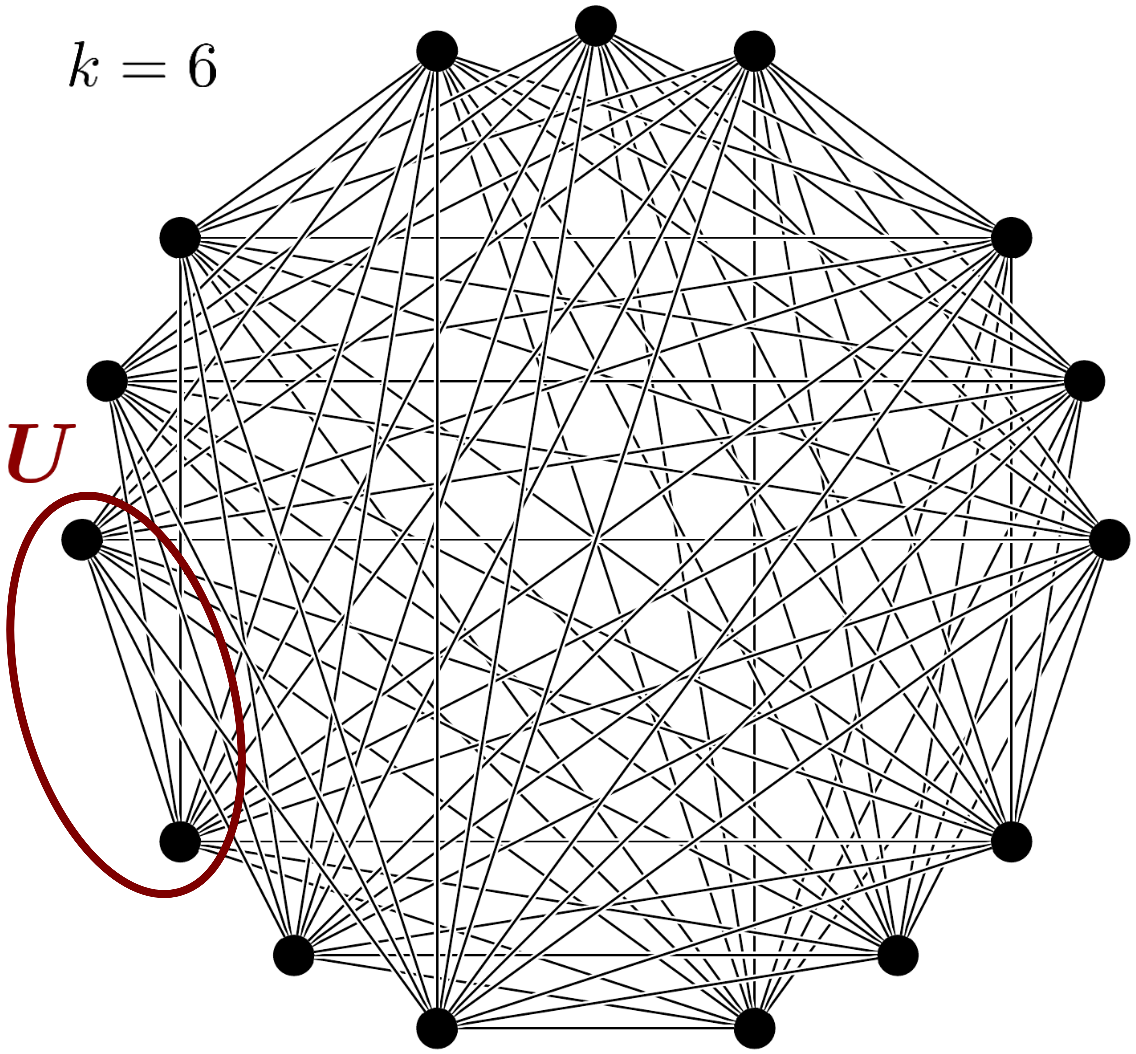

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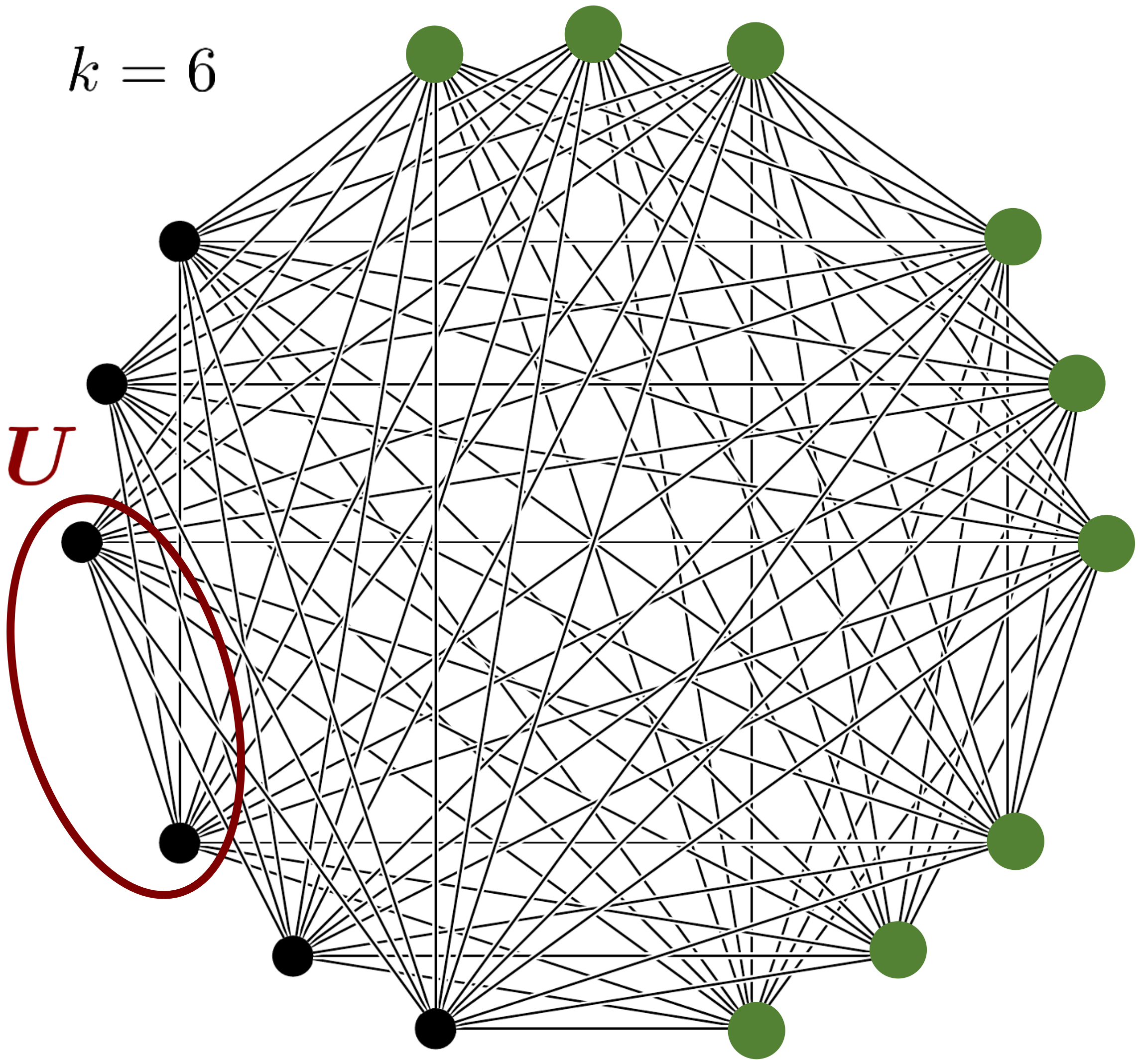

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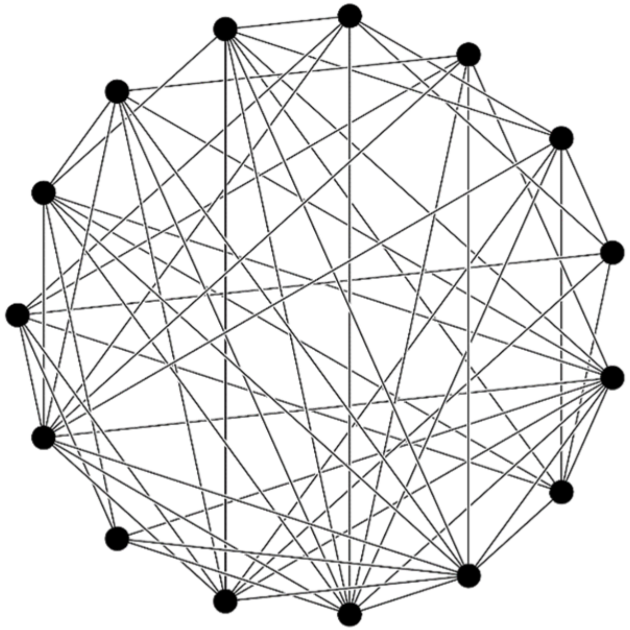

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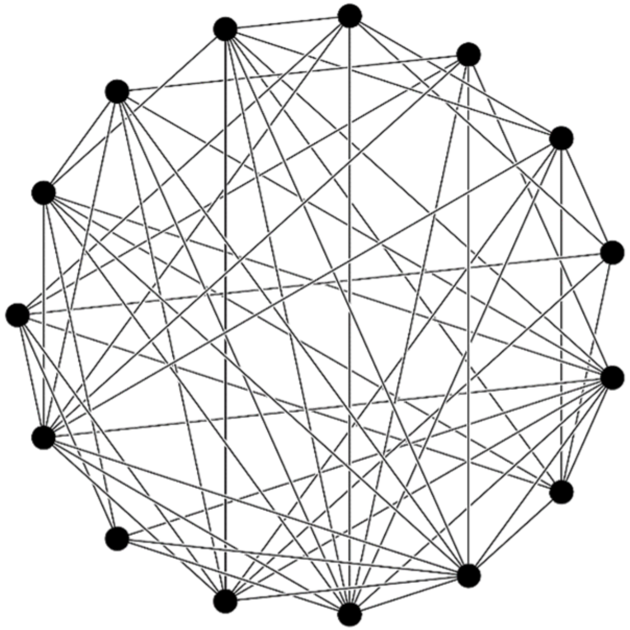

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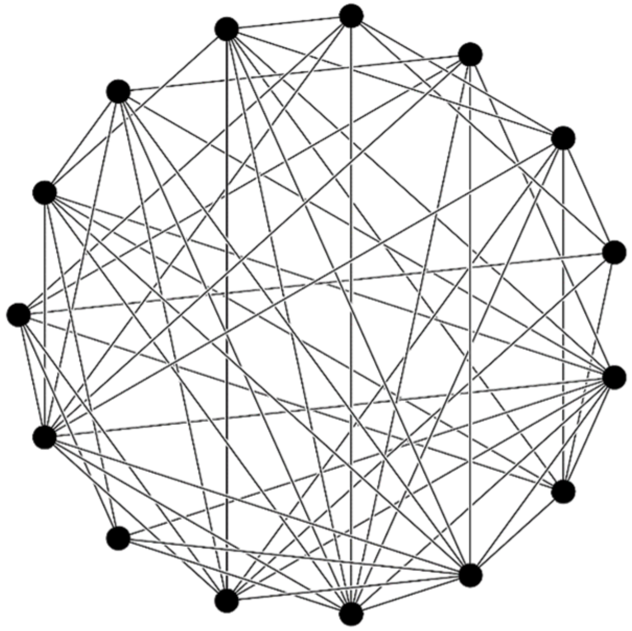

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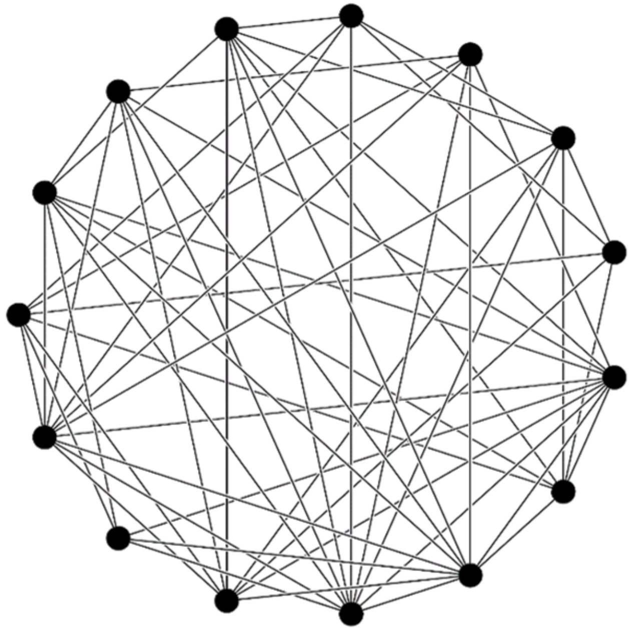

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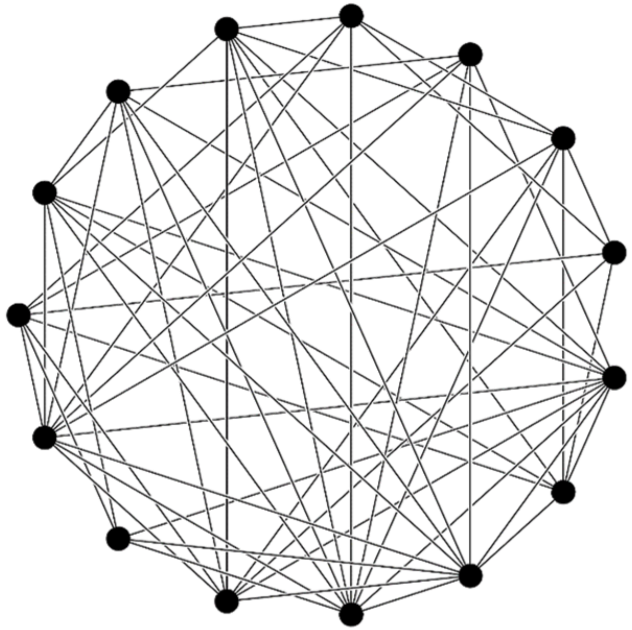

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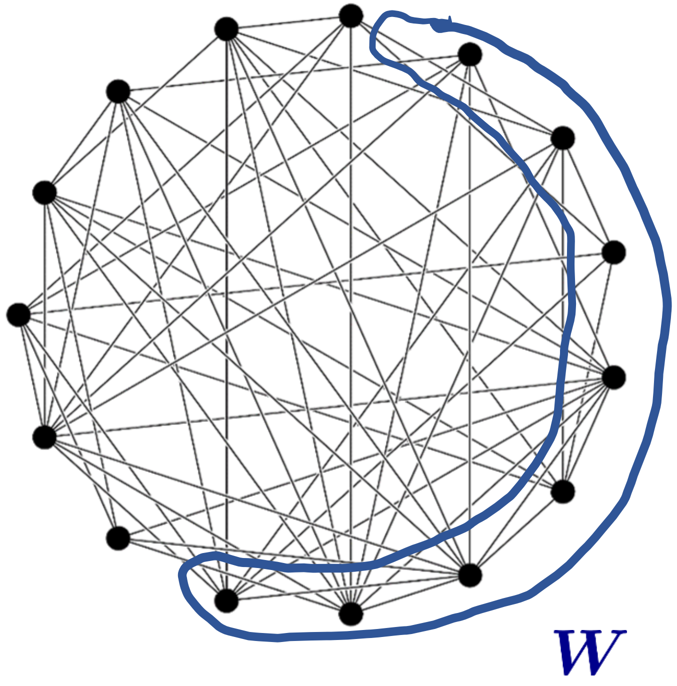

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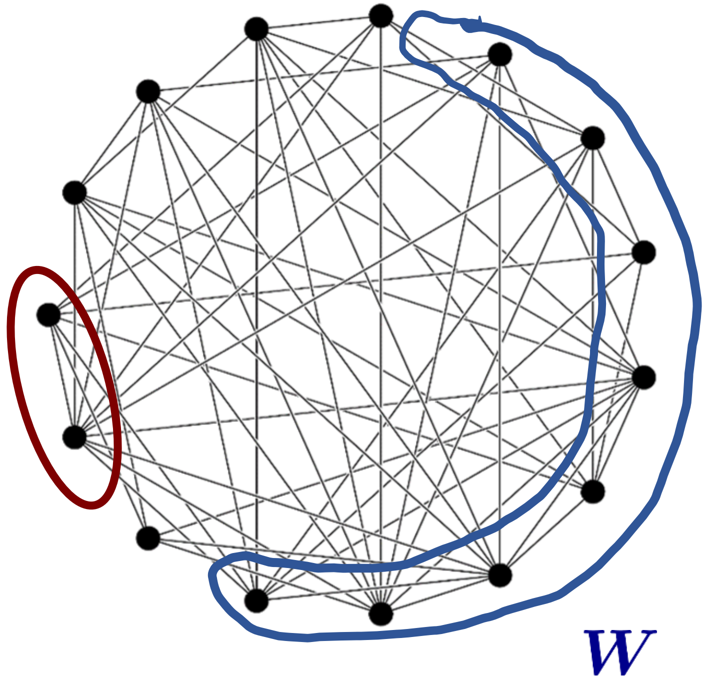

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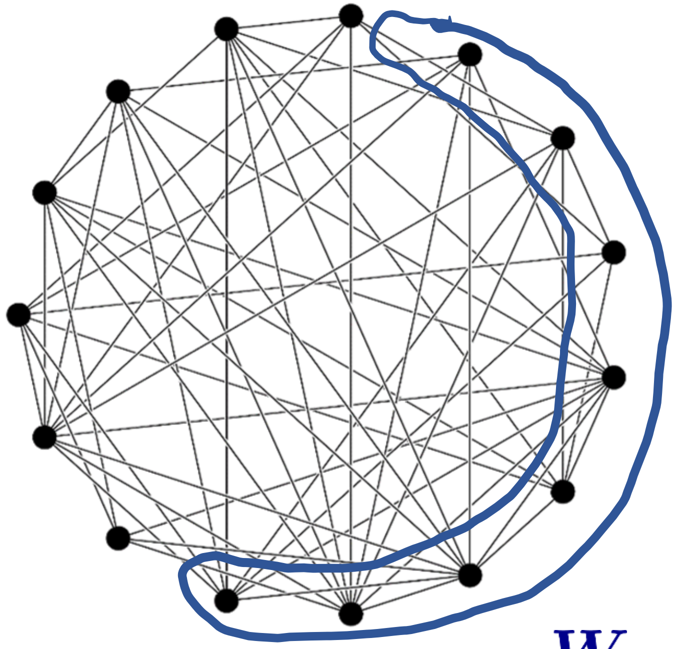

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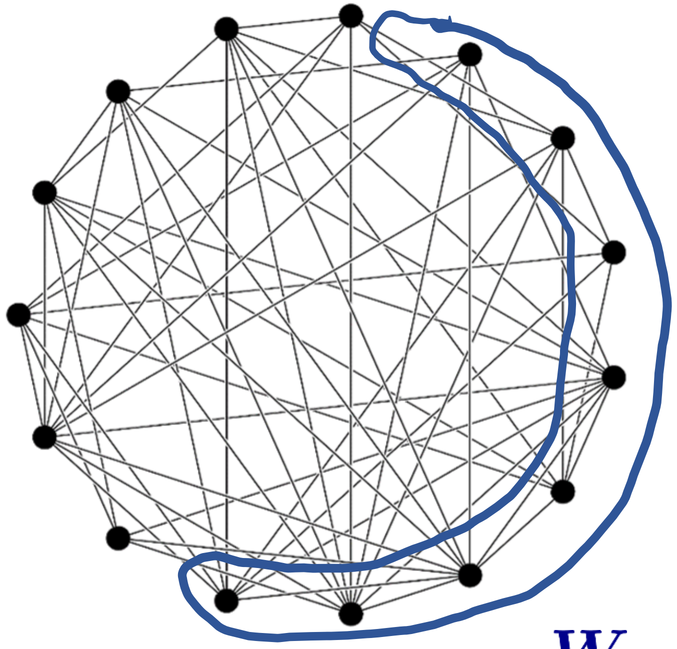

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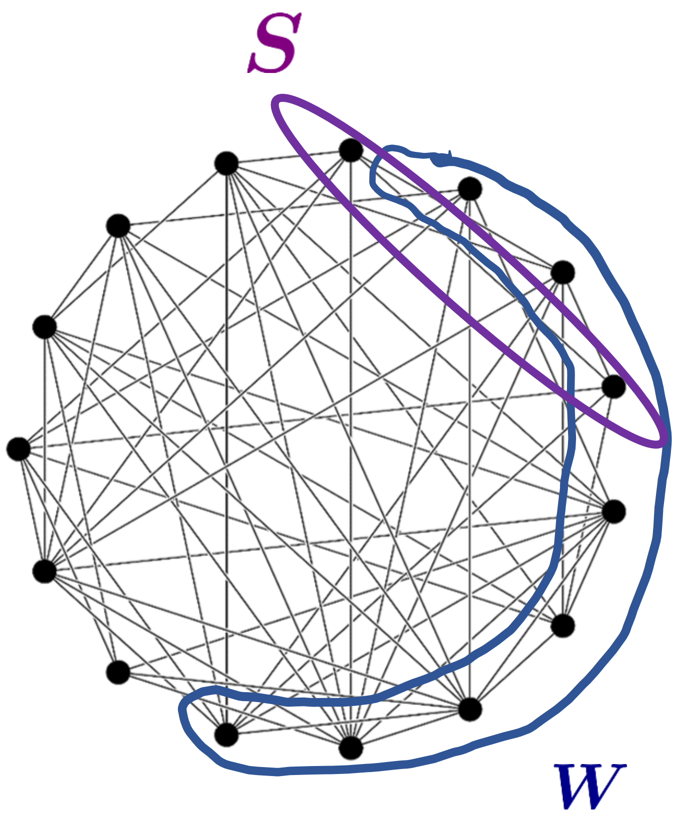

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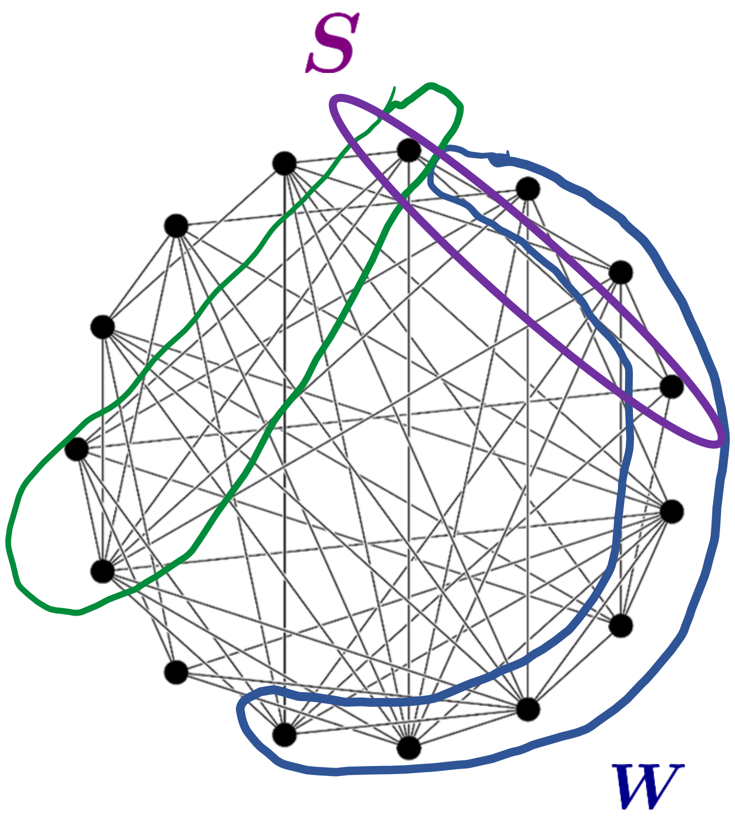

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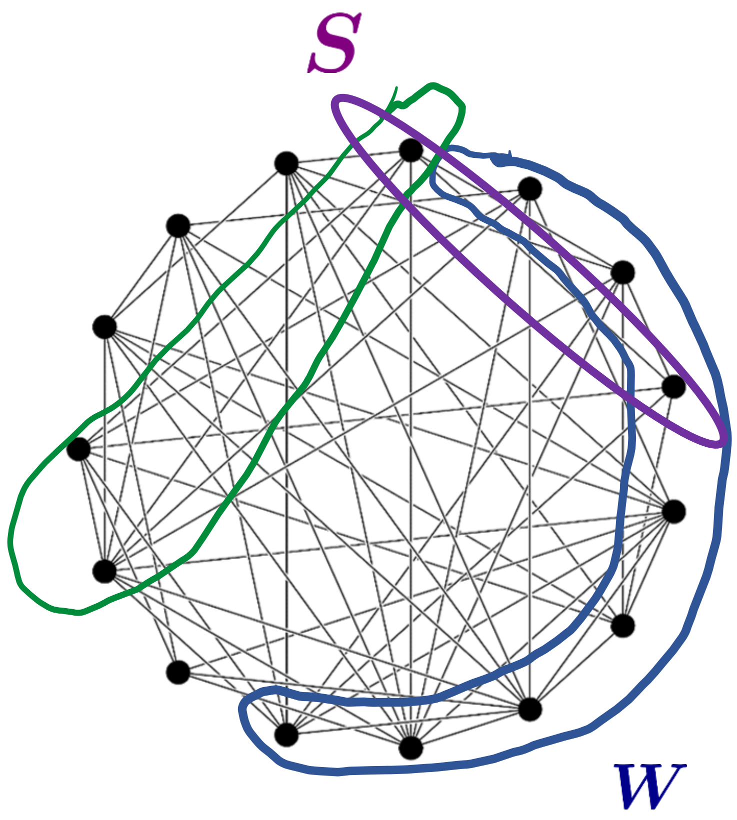
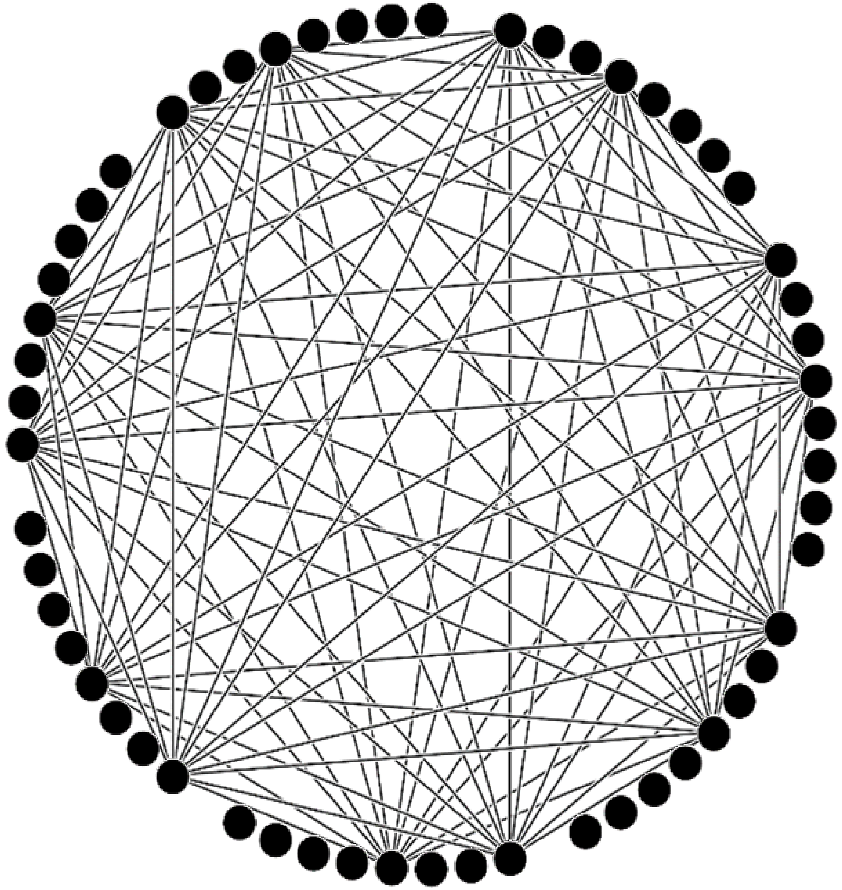


(k-1)-partite graph does not satisfy clique-denseness



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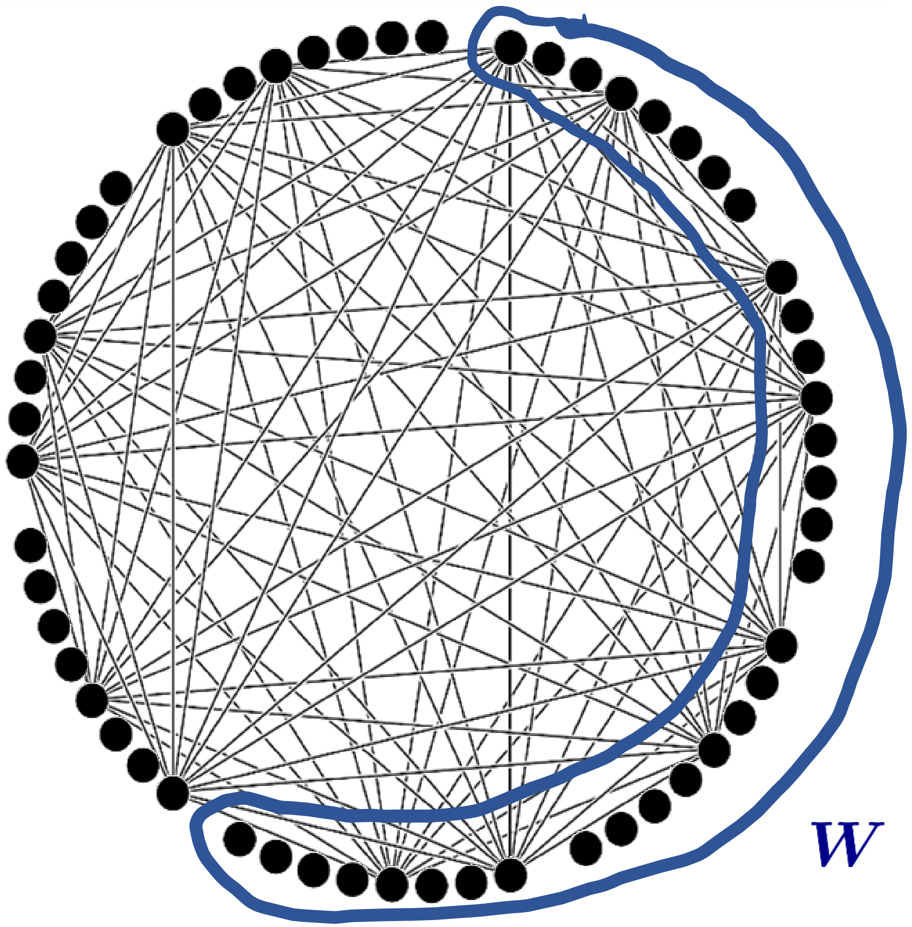

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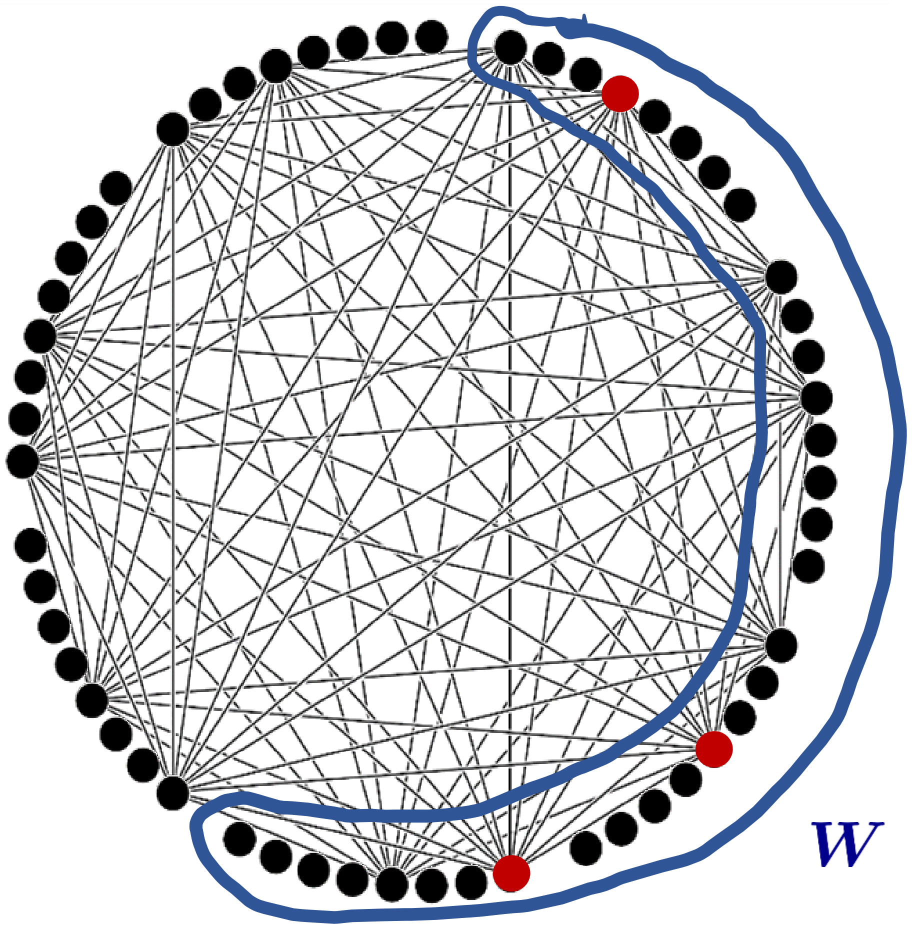

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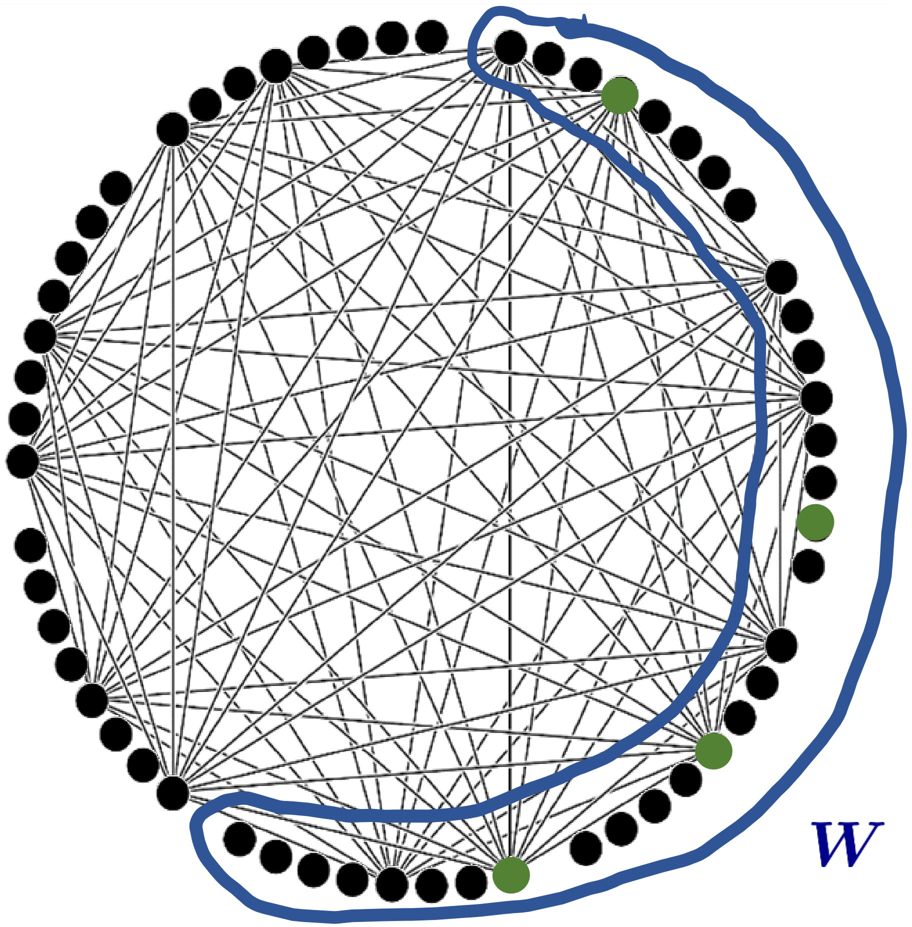

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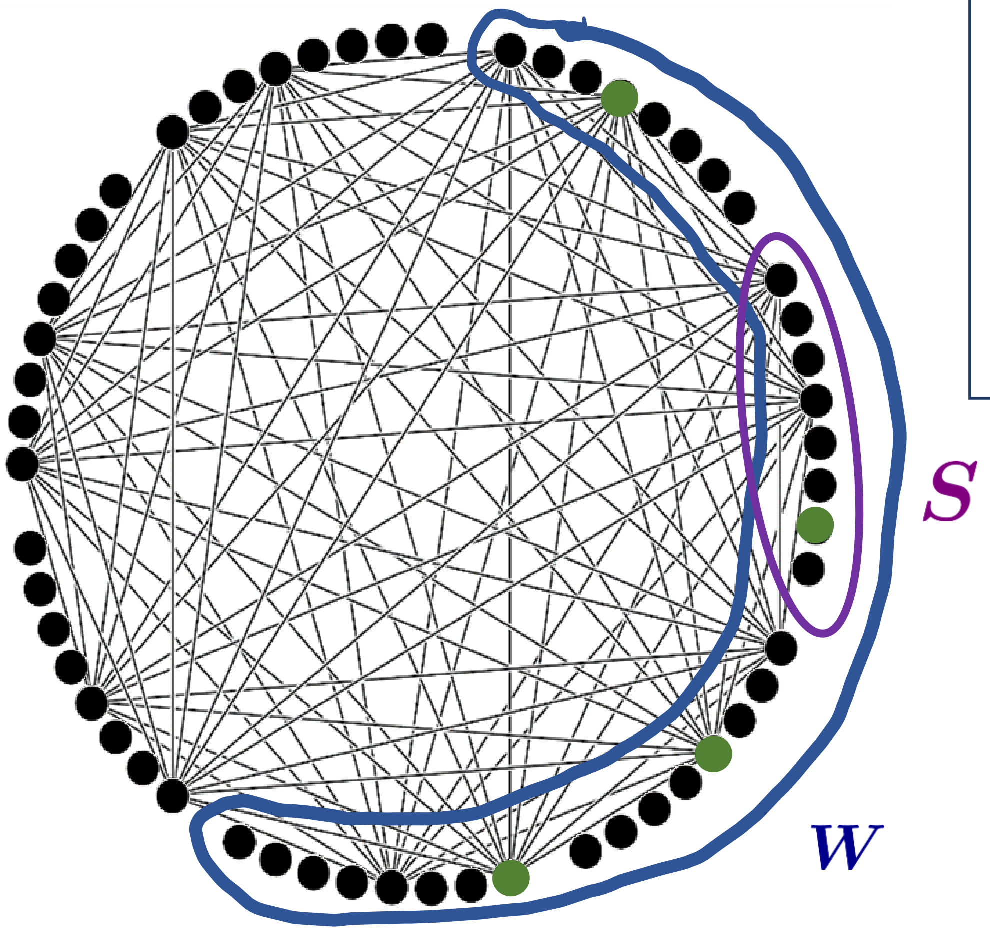

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Average-case hardness results

	k-clique	k-coloring	3-SAT	3-XOR
Tree-like Resolution	HARD (size $n^{\Omega(k)}$) [Beyersdorff, Galesi, Lauria '11]	HARD [Beame, Culberson, Mitchell, Moore '05]	HARD [Chvátal, Szemerédi '88] Improved [Ben-Sasson, Galesi '01] (size $\exp(n/\Delta^{1+\epsilon})$) $\Delta = m/n$	
Resolution	OPEN Some partial results*		HARD [Chvátal, Szemerédi '88] $\exp(n/\Delta^{2+\epsilon})$ Improved [Beame, Karp, Pitassi, Saks '98], [Ben-Sasson '01]	
Polynomial Calculus	OPEN	OPEN	$\mathbb{F} \neq 2$: HARD [Ben-Sasson, Impagliazzo '99]	
			$\mathbb{F} = 2$: HARD [Alekhnovich, Razborov '01]	EASY
Sum of Squares	OPEN Some partial results** $\mathcal{G}(n, 1/2)$: degree $= \Theta(\log n)$	OPEN [Kothari, Manohar '21] $\mathcal{G}(n, 1/2)$: $d \geq \Omega(\log n)$	HARD [Grigoriev '01, Schoenebeck '08]	
Cutting Planes	OPEN	OPEN	OPEN $\Theta(\log n)$-SAT [Fleming, Pankratov, Pitassi, Robere '17] [Hrubeš, Pudlák '17]	Quasi-poly EASY [Fleming, Göös, Impagliazzo, Pitassi, Robere, Tan, Wigderson '21] [Dadush, Tiwari '20]

* [Beame, Impagliazzo, Sabharwal '01], [Pang '21], [Atserias, Bonacina, **dR**, Lauria, Nordström, Razborov '18], [Lauria, Pudlák, Rödl, Thapen '13]
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Thank you!

	k-clique	k-coloring	3-SAT	3-XOR
Tree-like Resolution	HARD (size $n^{\Omega(k)}$) [Beyersdorff, Galesi, Lauria '11]	HARD [Beame, Culberson, Mitchell, Moore '05]	HARD [Chvátal, Szemerédi '88] Improved [Ben-Sasson, Galesi '01] (size $\exp(n/\Delta^{1+\epsilon})$) $\Delta = m/n$	
Resolution	OPEN Some partial results*		HARD [Chvátal, Szemerédi '88] $\exp(n/\Delta^{2+\epsilon})$ Improved [Beame, Karp, Pitassi, Saks '98], [Ben-Sasson '01]	
Polynomial Calculus	OPEN	OPEN	$\mathbb{F} \neq 2$: HARD [Ben-Sasson, Impagliazzo '99] $\mathbb{F} = 2$: HARD [Alekhnovich, Razborov '01]	$\mathbb{F} = 2$: EASY
Sum of Squares	OPEN Some partial results** $\mathcal{G}(n, 1/2)$: degree $= \Theta(\log n)$	OPEN [Kothari, Manohar '21] $\mathcal{G}(n, 1/2)$: $d \geq \Omega(\log n)$	HARD [Grigoriev '01, Schoenebeck '08]	
Cutting Planes	OPEN	OPEN	OPEN $\Theta(\log n)$-SAT [Fleming, Pankratov, Pitassi, Robere '17] [Hrubeš, Pudlák '17]	Quasi-poly EASY [Fleming, Göös, Impagliazzo, Pitassi, Robere, Tan, Wigderson '21] [Dadush, Tiwari '20]

* [Beame, Impagliazzo, Sabharwal '01], [Pang '21], [Atserias, Bonacina, **dR**, Lauria, Nordström, Razborov '18], [Lauria, Pudlák, Rödl, Thapen '13]
** [Meka, Potechin and Wigderson '15], ..., [Barak, Hopkins, Kelner, Kothari, Moitra, Potechin '16]