Mismatched Monte Carlo for the Planted clique problem

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Max-Clique Problem

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Any polinomial algorithm stops at $K = \log_2(N)$

The Jerrum Metropolis MC

- Start from $x_i = 0 \quad \forall i$
- At each time n choose i u.a.r. and flip x_i with probabilities:

$$P_{\text{Jerrum}}(x_i^n = 0 \rightarrow x_i^{n+1} = 1) = \begin{cases} 0 & \text{if } \exists j : x_j^n = 1 \text{ and } A_{ij} = 0 \\ 1 & \text{otherwise} \end{cases}$$

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State space: collection Ω of cliques $\mathcal{C}$ of any size in G.

Stationary distribution on Ω: $\pi(\mathcal{C}) = \frac{w(\mathcal{C})}{\sum_{\mathcal{C} \in \Omega} w(\mathcal{C})}$

$w(\mathcal{C}) = \lambda^{|\mathcal{C}|}$: weight asssigned to each clique $\mathcal{C} \in \Omega$

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Theorem (Jerrum '92)

Suppose $\epsilon > 0$. For a.e. $G \in \mathcal{G}(N, \frac{1}{2})$ and every $\lambda \geq 1$, the expected time for MC to reach a clique of size at least $(1 + \epsilon) \log N$ exceeds $N^{\Omega(\log N)}$

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Given $G \in \mathcal{G}(N, 1/2)$, select u.a.r. a subset $\mathcal{C}$ of size $|\mathcal{C}| \equiv K$.
Add to G all the edges between two nodes in $\mathcal{C}$.
(These operations define the new ensemble $\mathcal{G}(N, 1/2, K)$)
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Possible for $K > 2\log_2 N$

BUT many known algorithms are proved to fail in the regime $K/\sqrt{N} \to 0$:

- Spectral algorithms Alon et al., Random Structures & Algorithms (1998)
- Message Passing Deshpande, Montanari, Found. of Comp. Math. (2015)
- Sum of Squares Barak, SIAM Journal on Computing (2019)

The Jerrum MC for the planted clique problem

Same algorithm as in the random case

Theorem (Jerrum '92)

Suppose $\epsilon > 0$ and $0 < \beta < \frac{1}{2}$. For a.e. $G \in \mathcal{G}(N, \frac{1}{2}, \lceil N^{\beta} \rceil)$ and every $\lambda \geq 1$ the expected time for the MC process to reach a clique of size at least $(1+\epsilon)\log N$ exceeds $N^{\Omega(\log N)}$

But is the MC linear (polynomial) for $\beta \geq \frac{1}{2}$?

A different Metropolis MC for the planted clique

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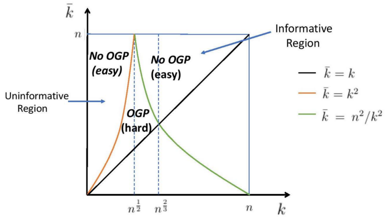


It becomes polynomial down to $K = \sqrt{N}$, working with a mismatched fixed magnetization $\overline{K} > K$.

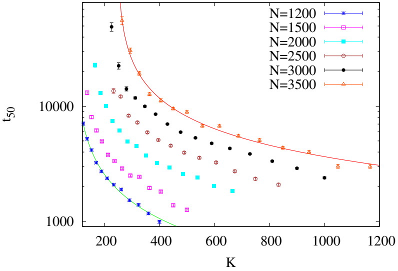
Two questions:

MCA, deFeo, Fachin, arXiv:2106.05720

- Is the Jerrum algorithm suboptimal? (super-polynomial for $K \leq N^{\beta}$ with $\beta > 1/2$)

- If yes, can we introduce a mismatched parameter to enhance its performances?

Numerical simulation of Jerrum algorithm



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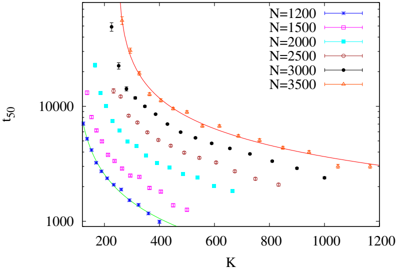

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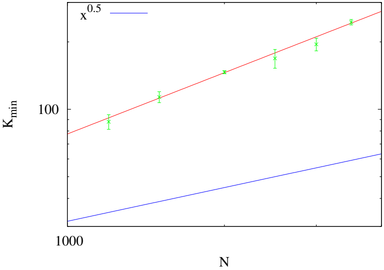

$$K_{min}(N) = bN^{\alpha}$$

$$\alpha = 0.91$$

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  To answer this question we introduce a slightly different MC

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As statistical physicists, we love Gibbs-Boltzmann weights:

$$P_\beta(\{x\}|\{A\}) \equiv P^\beta(\{x\}|\{A\}) \equiv \frac{1}{\mathcal{N}} e^{-\beta H(\{x\})}, \qquad \beta_{\text{Bayes}} = 1$$

introducing the Hamiltonian:

$$H(\{x\}) = -\sum_i \log(P(x_i)) + -\sum_{ij} \left[(1 - A_{ij}) \log \frac{(1 - x_i x_j)}{2} + A_{ij} \log \frac{(1 + x_i x_j)}{2} \right].$$

Metropolis algorithm:

$$P(x_i^n = 0 \to x_i^{n+1} = 1) = \begin{cases} 0 & \text{if } \exists j : x_j^n = 1 \text{ and } A_{ij} = 0 \\ \min\left(e^{-\beta \Delta E}, 1\right) = \min\left(e^{-\beta\left[\log(1-\frac{K}{N}) - \log(\frac{K}{N}) + m\log(\frac{1}{2})\right]}, 1\right) & \text{o.w.} \end{cases}$$

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Working only on perfect-clique configurations of different sizes m.

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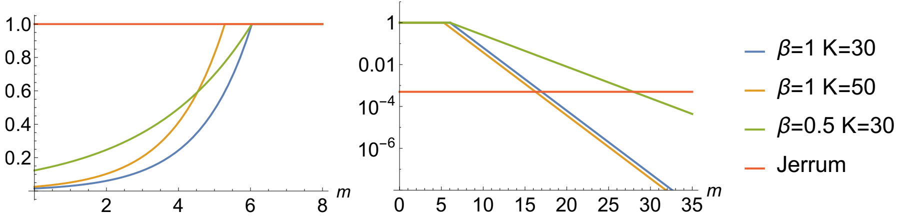
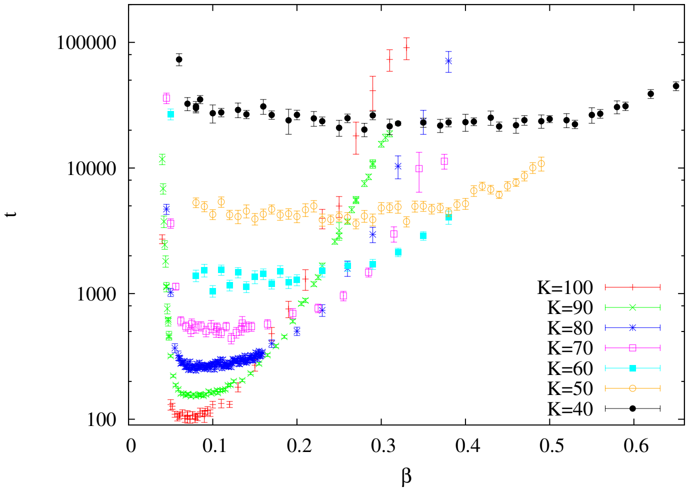


$P(x_i^n = 0 \to x_i^{n+1} = 1)$ $\quad$ $P(x_i^n = 1 \to x_i^{n+1} = 0)$ $\quad$ $N = 2000$

Finding the optimal β



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($\beta = 1$: the planted clique is not recovered in $t \leq 10^7$)

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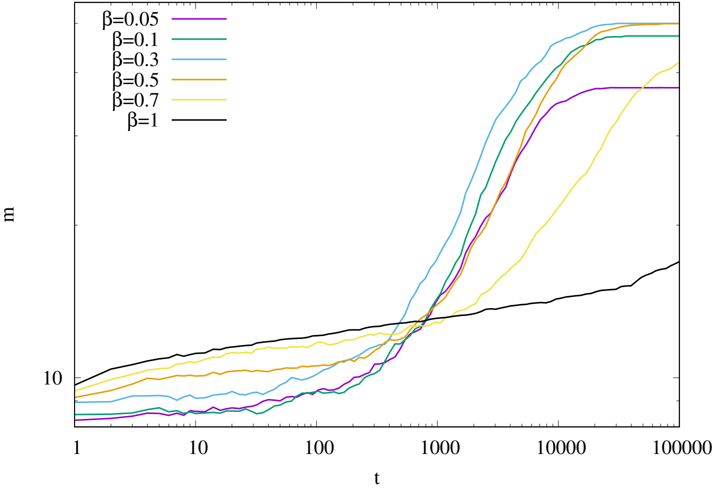
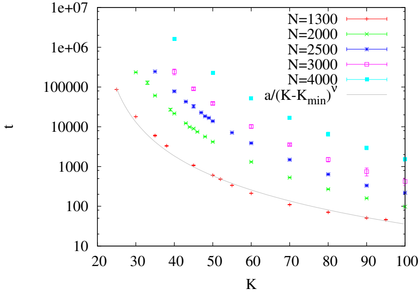


$N = 2000$, $K = 50$

Finding the MC threshold



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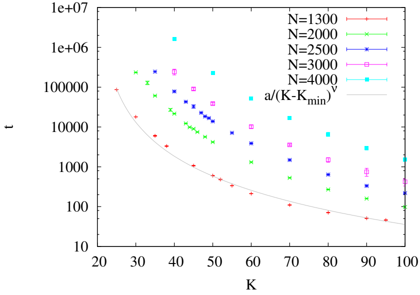

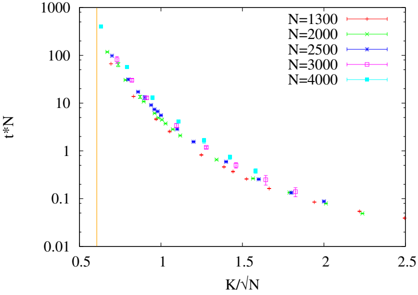

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- If yes, can we introduce a mismatched parameter to enhance its performances?

  Yes, we introduce a “temperature”. MC seems to reach the threshold for linear algorithms $K = \sqrt{\frac{N}{e}}$ at “mismatched” temperature $T > 1$.

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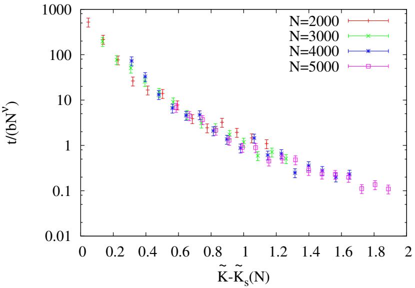

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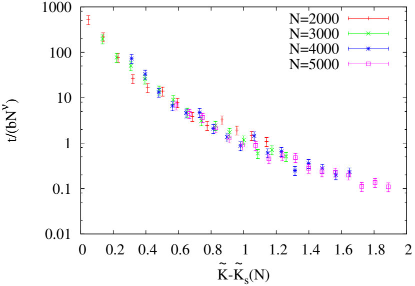

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- Overparametrization is essential in Deep NN. Simple cases can be useful in understanding complex ones.