Improved upper bounds on the stabilizer rank of magic states

arXiv: 2106.07740

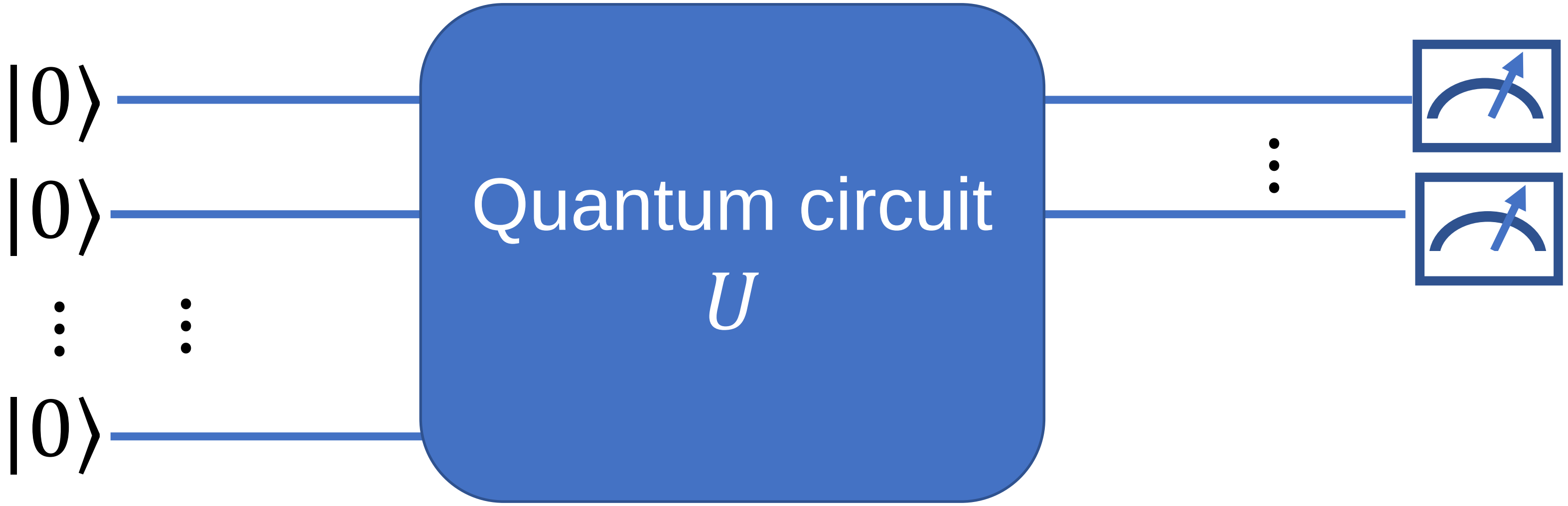
Hammam Qassim, Hakop Pashayan, **David Gosset**

Builds on: Garcia-Ramirez Ph.D thesis 2014
1506.01396 [Bravyi Smith Smolin 15]
1601.07601 [Bravyi DG 16]
….

Institute for Quantum Computing

UNIVERSITY OF WATERLOO

Classical simulation of quantum circuits



$$z \in \{0,1\}^w$$

$$p(z) = \| |z\rangle\langle z|_{out} U |0^n\rangle \|^2$$

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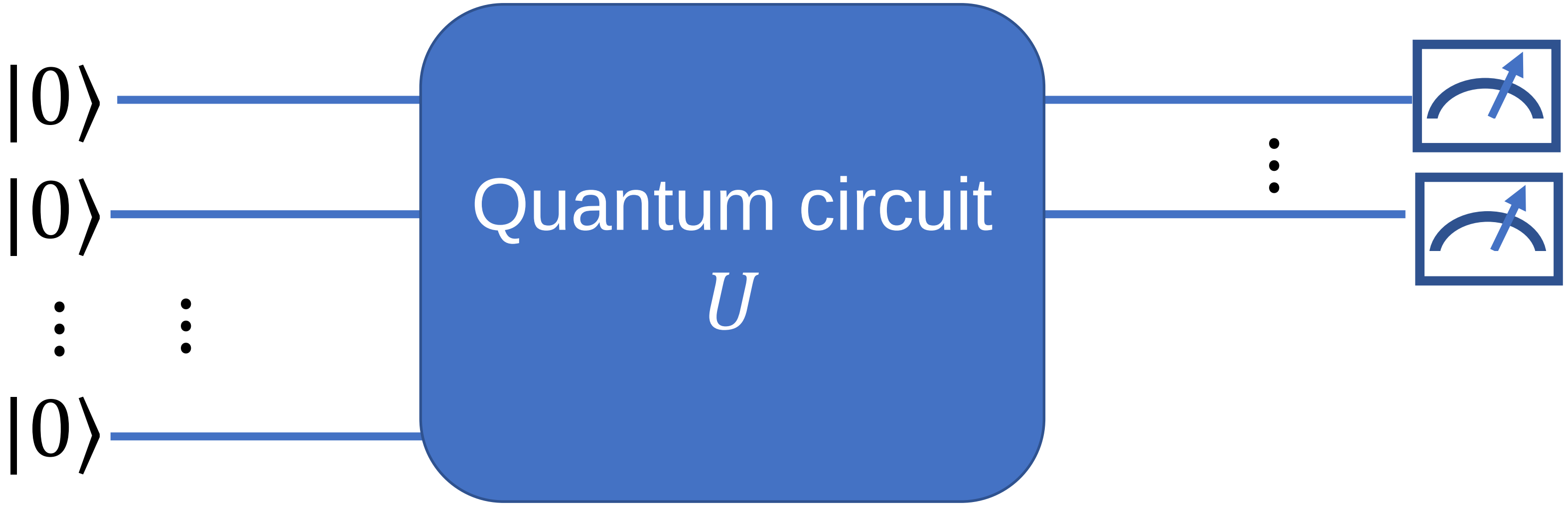

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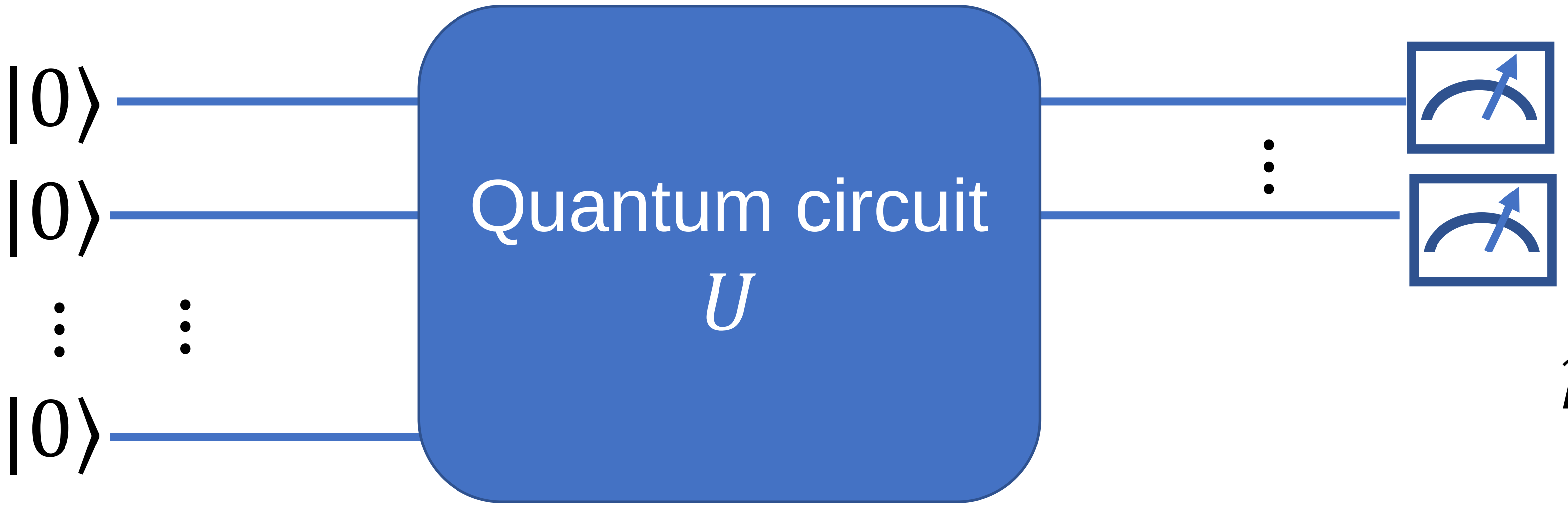

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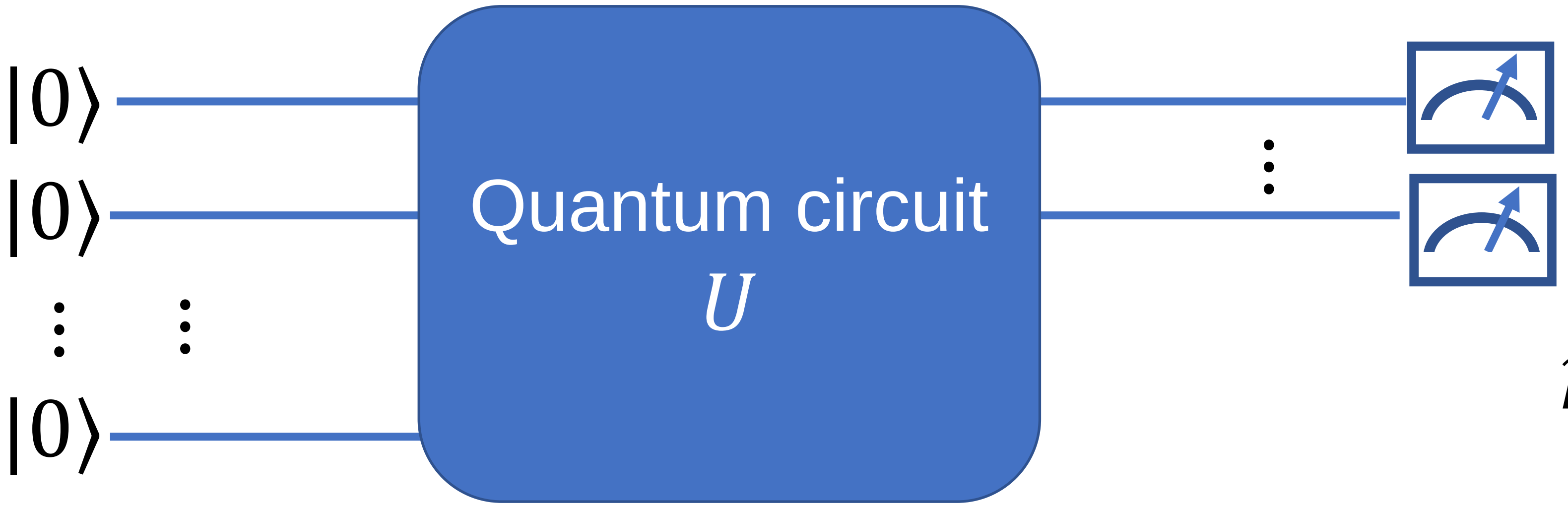

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Strong simulation: Given x, compute $p(x)$.
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ϵ-Strong simulation: Given x, compute $\tilde{p}$ such that $(1-\epsilon)p(x) \leq \tilde{p} \leq (1+\epsilon)p(x)$.
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Weak simulation: Sample a bit string from the distribution p.
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Classical Simulation

Most algorithms have exponential scaling in the number of qubits or number of gates

Multiply by sparse matrix

$$|0^{\otimes n}\rangle \longrightarrow U_1|0^{\otimes n}\rangle \longrightarrow U_2U_1|0^{\otimes n}\rangle \;\cdots\; \longrightarrow U_m \ldots U_2U_1|0^{\otimes n}\rangle$$

Complex vector
2^n entries

Runtime: $2^n m$ **Memory: 2^n**

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Most algorithms have exponential scaling in the number of qubits or number of gates

$$\langle x|U_m \ldots U_2 U_1 \big|0^{\otimes n}\big\rangle = \sum_{z_1, z_2 \ldots z_{m-1}} \langle x|U_m|z_{m-1}\rangle \ldots \langle z_2|U_2|z_1\rangle\langle z_1|U_1|0\rangle$$

Runtime: 4^m **Memory: $m + n$**

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Most algorithms have exponential scaling in the number of qubits or number of gates

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Recursive variant
[Aaronson Chen 2016]

Runtime: $\boldsymbol{n(2d)^{n+1}}$ **Memory:** $\boldsymbol{n \log(d)}$

Circuit depth

Classical Simulation

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$$\langle x|U_m \ldots U_2 U_1\left|0^{\otimes n}\right\rangle = \sum_{z_1, z_2 \ldots z_{m-1}} \langle x|U_m|z_{m-1}\rangle \ldots \langle z_2|U_2|z_1\rangle\langle z_1|U_1|0\rangle$$

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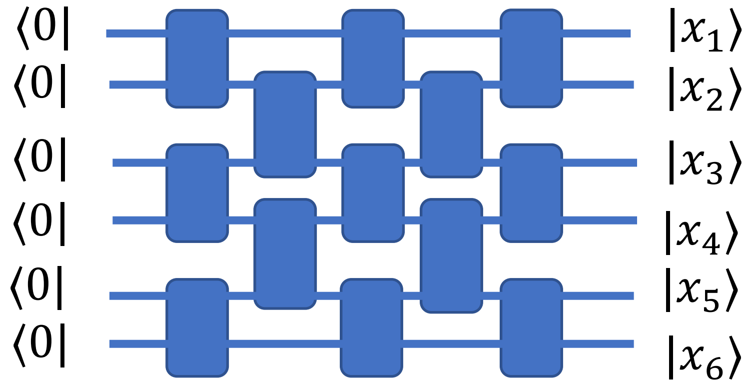
Tensor network contraction methods:
[Markov Shi 2005][Pednault et al. 2017][Boixo et al. 2017][Li et al. 2018][Chen et al. 2018]

Tensor network contraction methods

Simulation algorithm idea: view circuit as a tensor network and choose contraction ordering to minimize memory/runtime.



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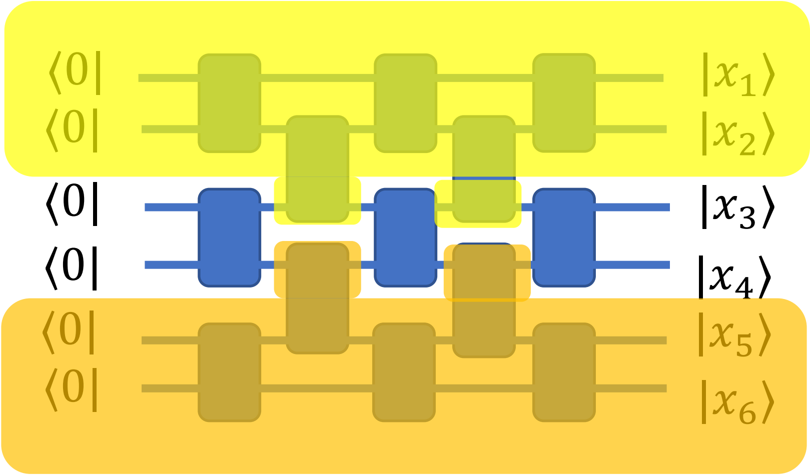

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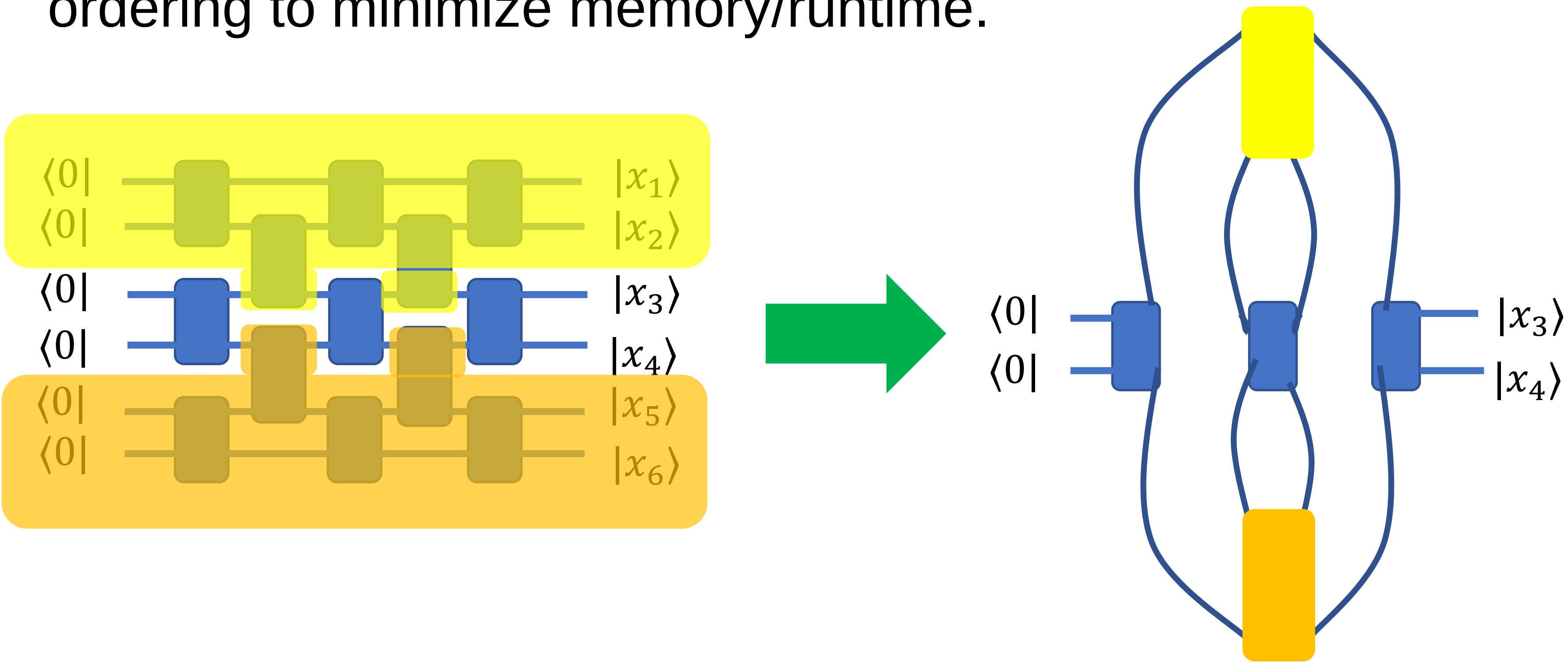

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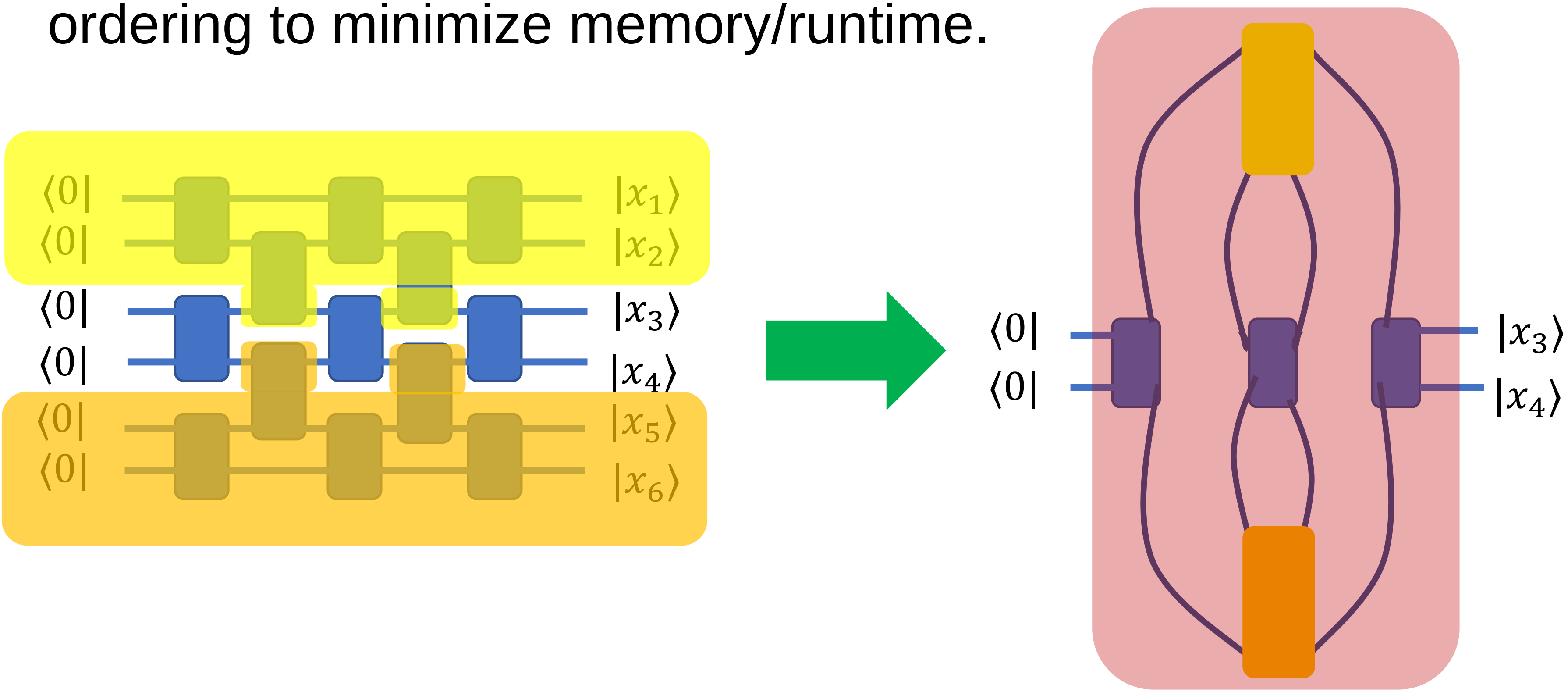

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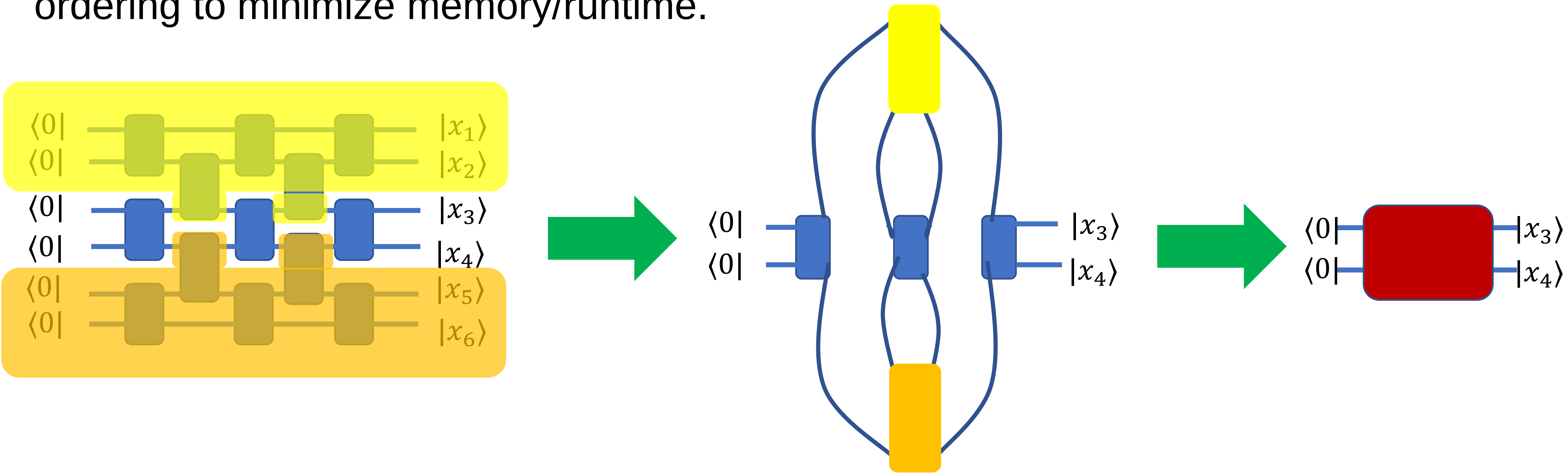

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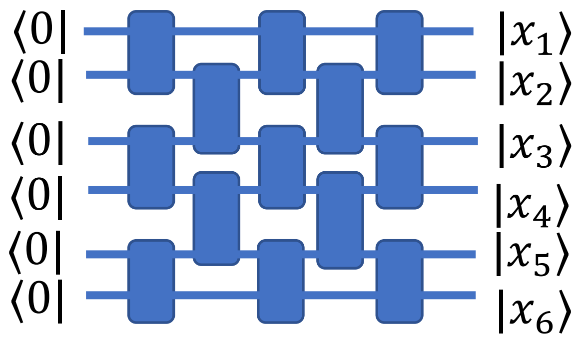

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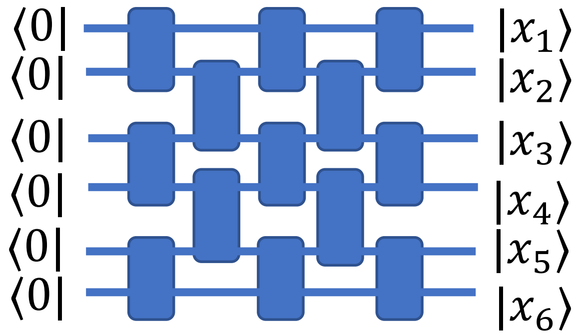

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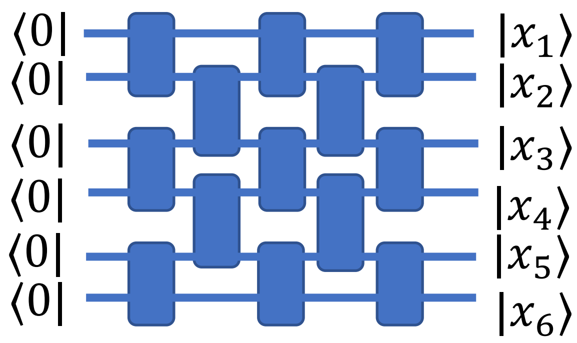

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This talk is about a different kind of simulation algorithm…

Stabilizer rank simulators for Clifford+T circuits

Clifford circuits

The **Clifford group** is generated by gates

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Gottesman-Knill Theorem [Gottesman 1997]
Quantum circuits composed only of Clifford gates can be efficiently (weakly/strongly) simulated on a classical computer.

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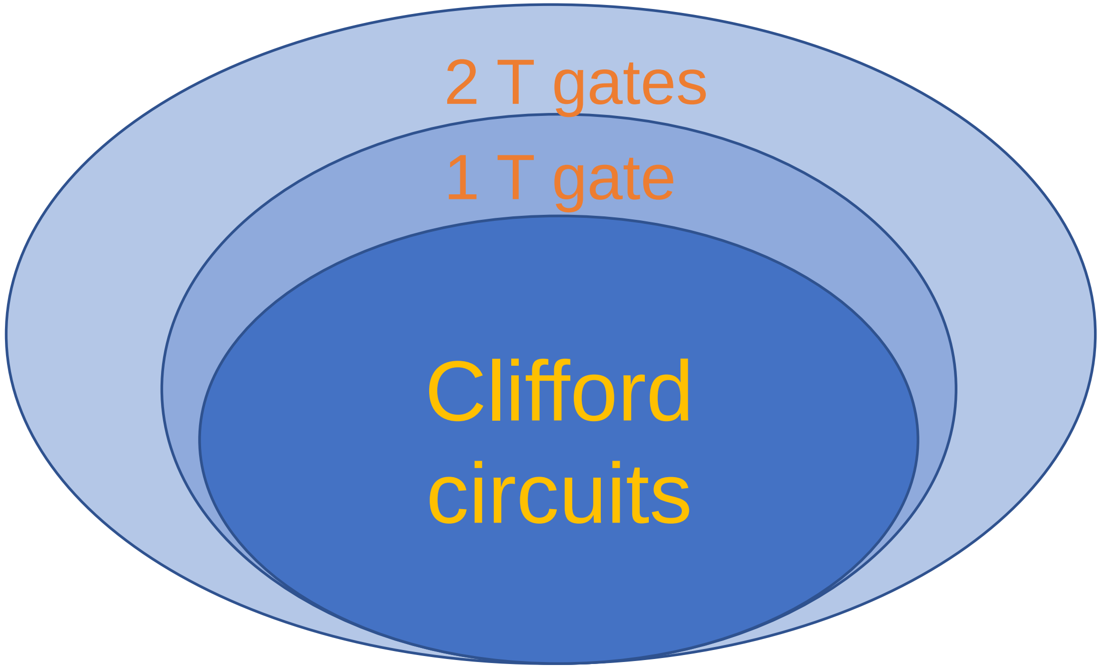
Can we extend Gottesman-Knill to circuits with a few non-Clifford gates?...

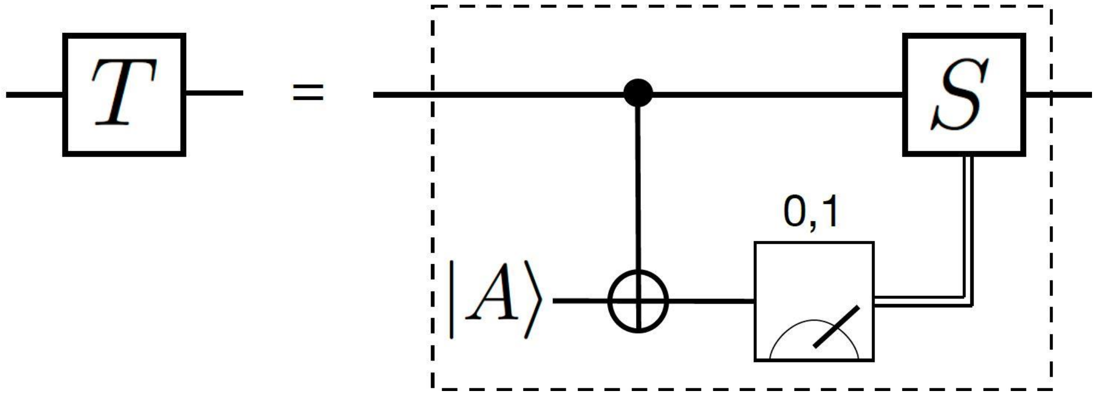
Outward from the Cliffords

Clifford+T gate set $\{H, S, CNOT, T\}$ $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$



What is the classical simulation cost of a circuit with m **T gates?**

Gadgetized Clifford+T circuit



T gate gadget
[Zhou Leung Chuang 2000]

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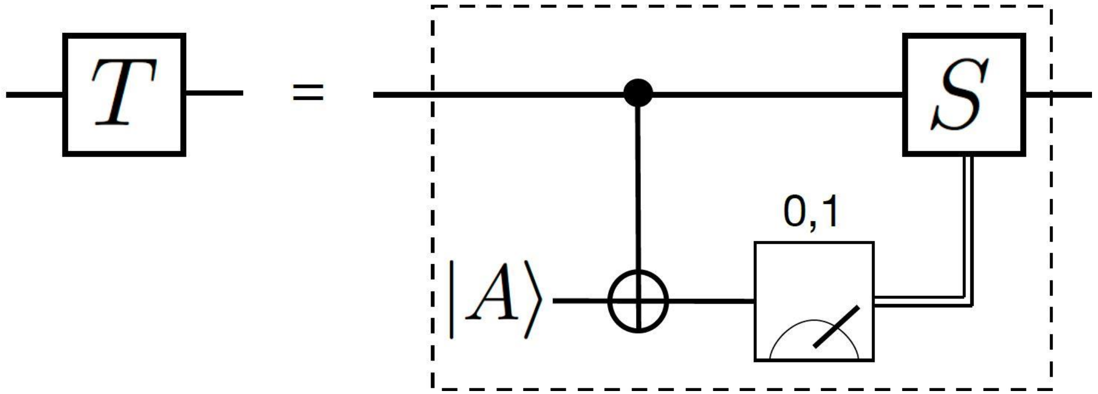

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$$|A\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\pi/4}|1\rangle\right)$$

Magic state

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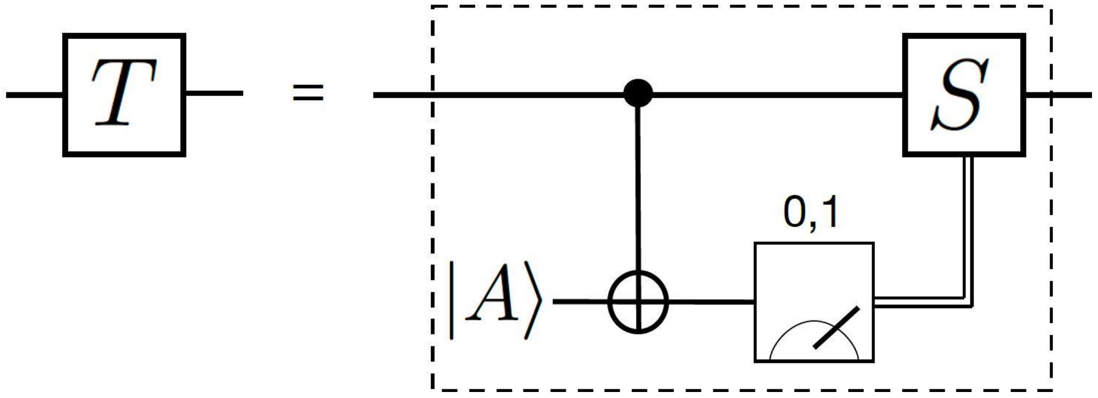

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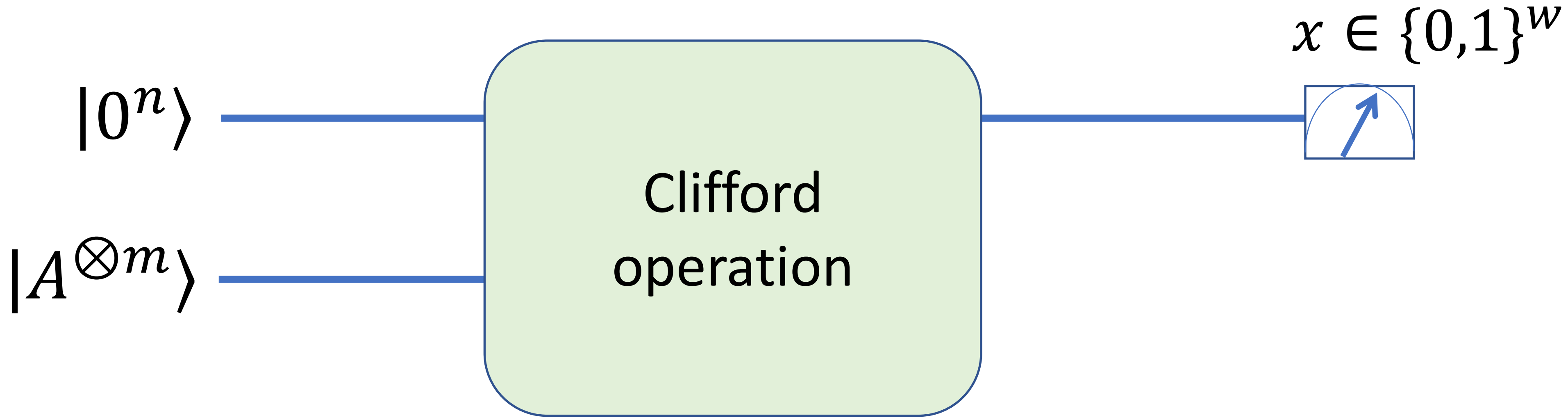
Magic state

We can **gadgetize** any Clifford+T circuit by replacing all T gates with the above gadget.

This gives an adaptive Clifford circuit with input state

$$|0^n\rangle|A^{\otimes m}\rangle$$

Simulating the gadgetized circuit [Bravyi Smith Smolin 2015]



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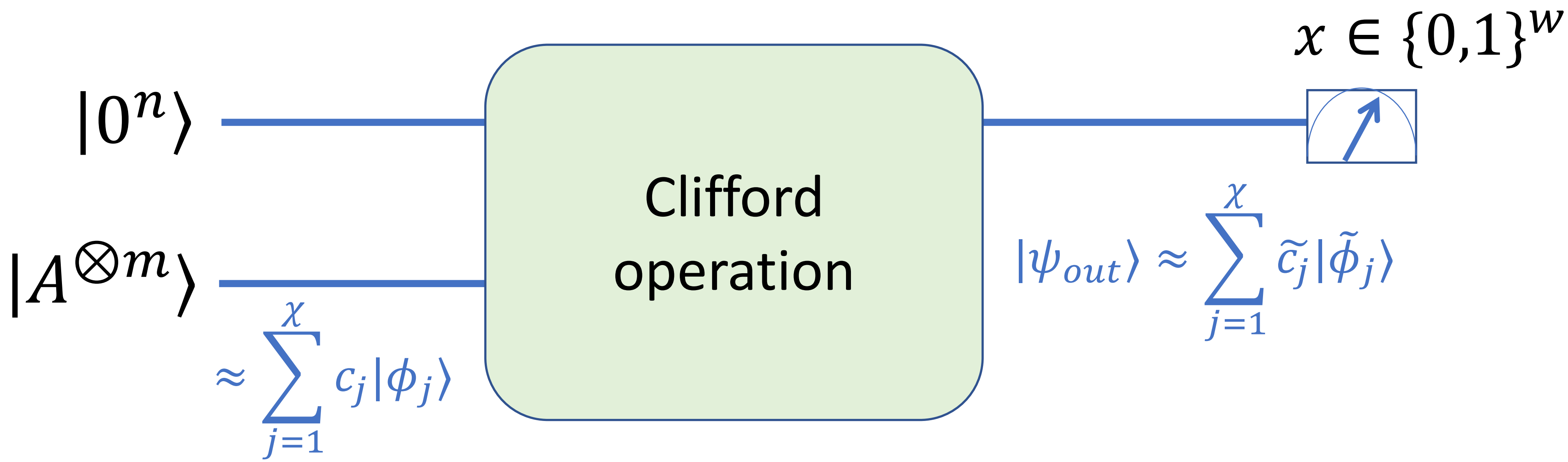

Simulation strategy (roughly):

1. Approximate input magic state as superposition of $\chi \ll 2^n$ stabilizer states

2. Apply Clifford operation

3. Simulate final measurement

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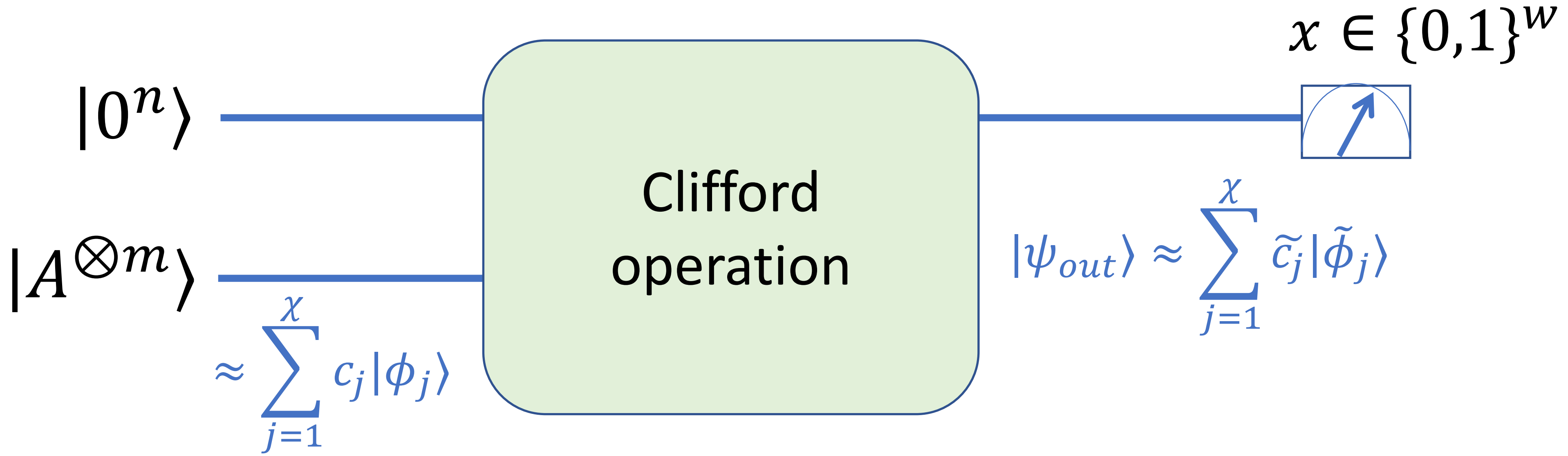

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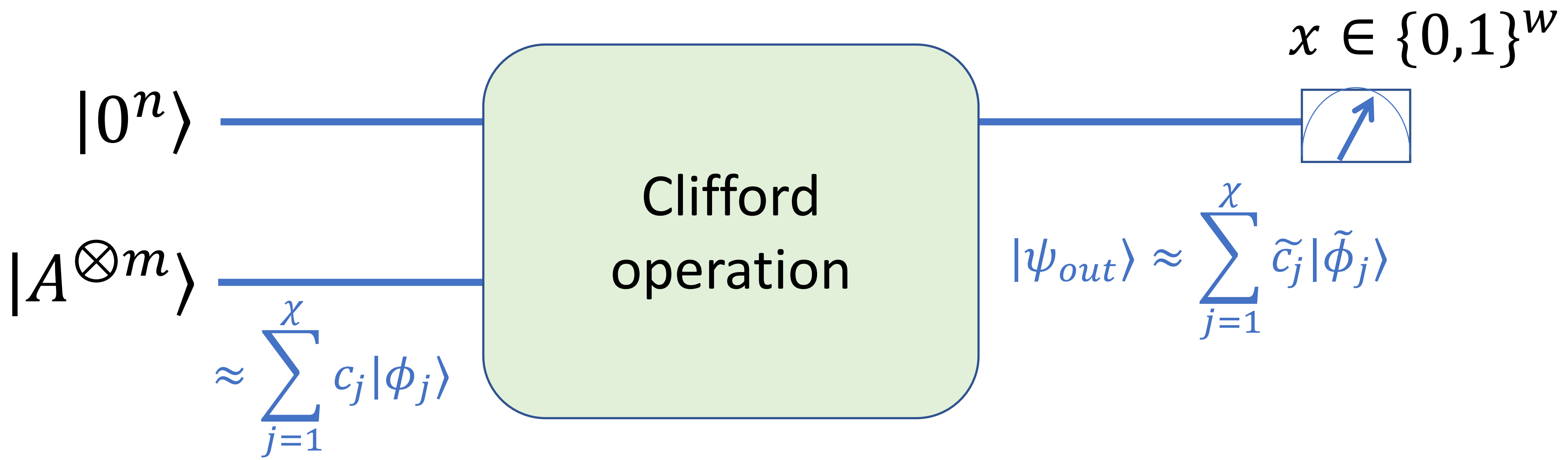

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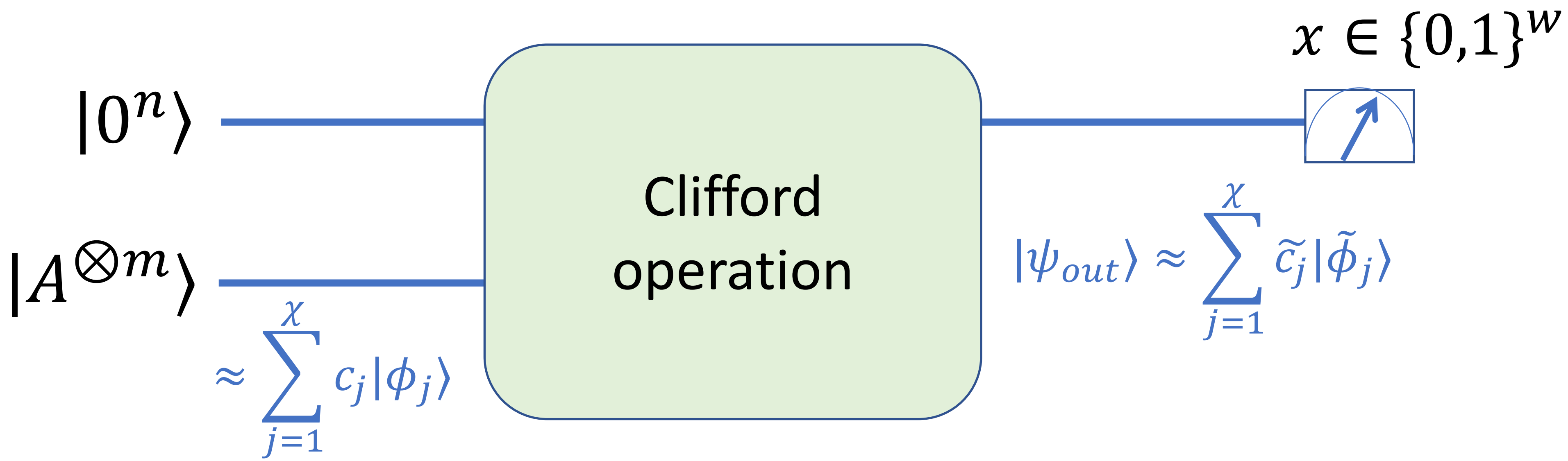

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3. Simulate final measurement — Involves computation of marginals $\langle\psi_{out}||x\rangle\langle x|\otimes I|\psi_{out}\rangle$ — $\chi^2 \cdot poly(m,n)$

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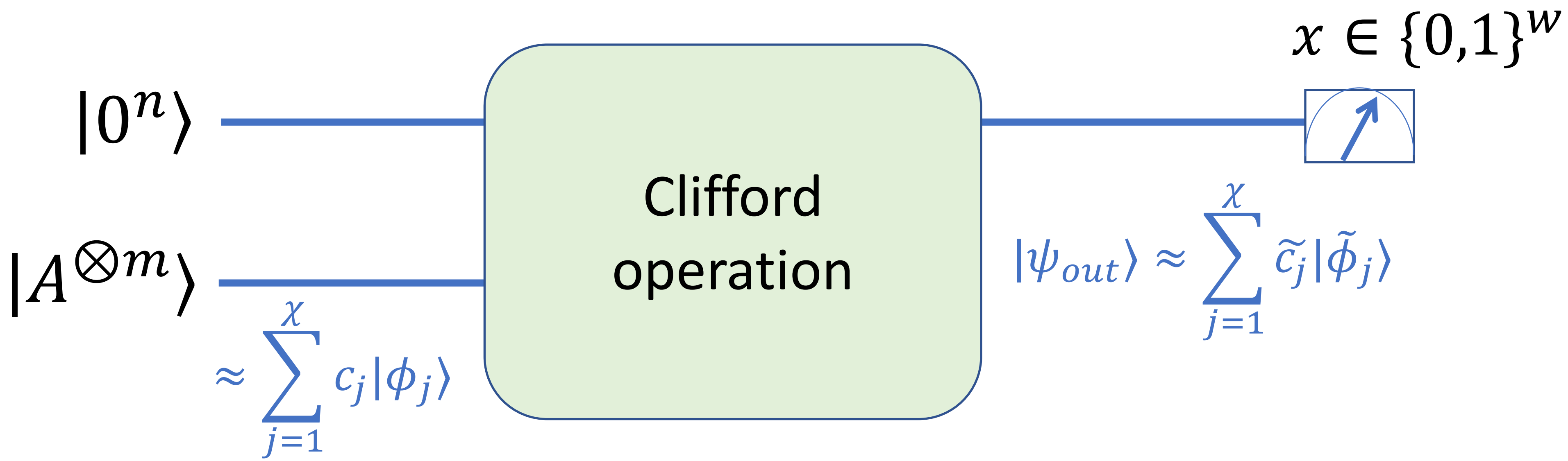

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"norm estimation algorithm"
(weak or ϵ-strong sim.)
[Bravyi DG 2016]

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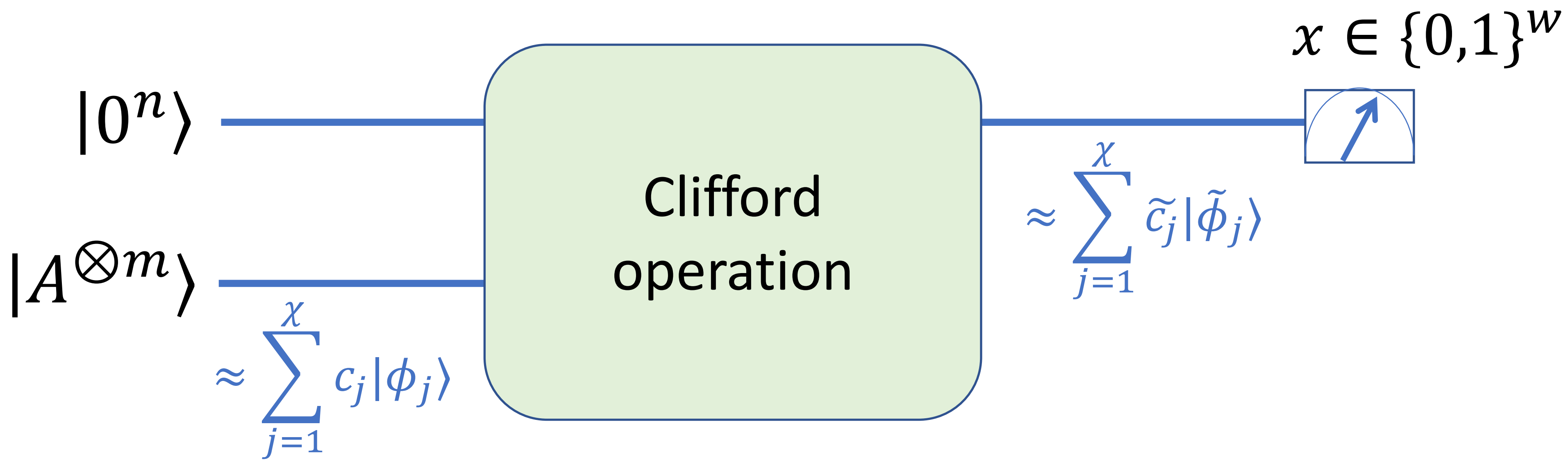

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What kind of approximation is needed? How big is χ?...

Stabilizer decompositions

Stabilizer rank $\boldsymbol{\chi(\psi)}$ is the minimum r such that

$$|\psi\rangle = \sum_{j=1}^{r} c_j |\phi_j\rangle$$

Stabilizer states

[Bravyi Smith Smolin 2015]

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Stabilizer states

Approximate stabilizer rank $\chi_\delta(\psi)$ is the minimum r such that

$$\left\| |\psi\rangle - \sum_{j=1}^{r} c_j |\phi_j\rangle \right\| \le \delta$$

[Bravyi Smith Smolin 2015] [Bravyi, DG 2016]

Remark on stabilizer decompositions

Recall that a stabilizer state can be parameterized as

$$|\phi\rangle \propto \sum_{x \in V} (-1)^{q(x)} i^{\ell(x)} |x\rangle$$

V: affine subspace of $\mathbb{F}_2^n$

q: quadratic function $q(x) = x^T B x \mod 2$

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The decomposition of a given state into the overcomplete basis of stabilizer states generalizes the decomposition of a Boolean function into quadratic Boolean functions (**quadratic Fourier analysis)**

Stabilizer rank simulators

Classical simulation cost scales with stabilizer rank of magic state input

For circuits with $\boldsymbol{n}$ **qubits, g total gates,** $\boldsymbol{m}$ **T gates:**

[Bravyi Smith Smolin 2015]	$\left(\chi\left(A^{\otimes m}\right)\right)^2 \mathrm{poly}(n,g)$	**Strong simulation**
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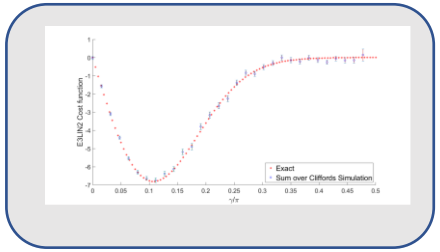
Remarks:

Techniques can be extended to circuits with other non-Clifford gates [Bravyi Browne Calpin Campbell DG Howard 18]



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Remarks:

Up to polynomial factors, classical simulation has linear scaling with stabilizer rank.

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The best unconditional lower bound is $\chi\left(A^{\otimes m}\right) = \Omega(m)$ [Peleg, Shpilka, Volk 2021]

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Remarks:

Rest of this talk: upper bounds…

Upper bounds on stabilizer rank of magic states

Upper bound on approximate SR [Bravyi DG 2016]

$$\chi_\delta(A^{\otimes m}) \leq O\left(\frac{1}{\delta^2} 2^{\alpha m}\right)$$

$\alpha = -2\log_2(\cos \pi/8) \approx 0.228 \ldots$

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Proof sketch $|H\rangle = \frac{1}{2\cos(\frac{\pi}{8})}(|0\rangle + |+\rangle)$ **Clifford equivalent to $|A\rangle$**

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A low stabilizer rank approximation is given by restricting the sum to $x \in L \subseteq \{0,1\}^m$ where L is a randomly chosen linear subspace of size $|L| \approx \delta^{-2} 2^{\alpha m}$

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Can also show the corresponding stabilizer decomposition has minimal 1-norm of the coefficients in the decomposition

("stabilizer extent") [Bravyi Browne Calpin Campbell DG Howard 2018]

In contrast, the known upper bounds on the **exact** stabilizer rank of magic states seem unlikely to be optimal…

Upper bounds on exact stabilizer rank

All previously known upper bounds follow a similar strategy

1. **Upper bound SR of a constant number of magic states**

Upper bound $\chi\left(A^{\otimes c}\right)$ for some small constant number of magic states c

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n	2	3	4	5	6
$\chi_n \leq$	2	3	4	6	7

We believe that these upper bounds are tight. A lower bound $\chi_n \geq \Omega(n^{1/2})$ is proved in Appendix C.

[From Bravyi Smith Smolin 2015]

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# of 6-qubit stabilizer states: 315057600

[From Bravyi Smith Smolin 2015]

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# of 6-qubit stabilizer states: 315057600

# of size-7 subsets of 6-qubit stabilizer states: > 10^48

[From Bravyi Smith Smolin 2015]

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The best bound from [Bravyi Smith Smolin 15] is $\chi\left(A^{\otimes 6}\right) \leq 7$

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[Kocia 20] shows (by inspection?) $\chi\left(A^{\otimes 12}\right) \leq 47$

Upper bounds on exact stabilizer rank

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Using the bound $\chi(A^{\otimes 6}) \leq 7$ from [Bravyi Smith Smolin 15] gives

$$\chi(A^{\otimes m}) \leq 2^{\alpha m} \qquad \alpha = \frac{\log_2 7}{6} \approx 0.4679 \ldots$$

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Using the bound $\chi(A^{\otimes 12}) \leq 47$ from [Kocia 20] gives

$$\chi(A^{\otimes m}) \leq 2^{\alpha m} \qquad \alpha = \frac{\log_2 47}{12} \approx 0.4629\ldots$$

In [Qassim Pashayan DG 21] we improve on this bound using a different strategy…

Magic cat states

Consider the magic cat state

Image source: Etsy

$$|\mathrm{Cat}_m\rangle = \frac{1}{\sqrt{2}}\left(|A\rangle^{\otimes m} + \left|A^{\perp}\right\rangle^{\otimes m}\right)$$

$$|A\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\pi/4}|1\rangle\right)$$

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Stabilizer projector—does not increase SR

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$$|A\rangle^{\otimes m} = (|A\rangle\langle A| \otimes I)|\mathrm{Cat}_m\rangle$$

This single-qubit operator can be written as a sum of two Cliffords, each of which does not increase SR

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Stabilizer rank of magic cat has same asymptotic scaling with m

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Stabilizer rank of magic cat has same asymptotic scaling with m

Stabilizer decompositions of magic cats with small m are easier to find…

Stabilizer rank of small magic cats

$$|\text{Cat}_1\rangle = \frac{1}{\sqrt{2}}\left(|A\rangle + \left|A^{\perp}\right\rangle\right) = |0\rangle$$

Cat1 is a stabilizer state

Stabilizer rank of small magic cats

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Cat2 is also a stabilizer state

$$\chi(\mathrm{Cat}_2) = 1$$

Stabilizer rank of small magic cats

The largest magic cat state for which we can exactly compute the stabilizer rank is

$$\chi(\mathrm{Cat}_6) = 3$$

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$$|\mathrm{Cat}_6\rangle = \frac{1}{2}|\phi_1\rangle + \frac{e^{\frac{3i\pi}{4}}}{\sqrt{2}}(|\phi_2\rangle + i|\phi_3\rangle)$$

$$|\phi_1\rangle = \frac{1}{\sqrt{2}}(|0^6\rangle - i|1^6\rangle)$$

$$|\phi_2\rangle = 2^{-5/2}\sum_{|x|\ even}|x\rangle$$

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(We establish a matching lower bound by showing that no state with stabilizer rank 2 can have the same set of Pauli expectation values)

Low hanging fruit

$\chi(A^m) \leq 2\chi(\mathrm{Cat}_m)$

$\chi(\mathrm{Cat}_6) = 3$

$\chi\left(\mathrm{A}^{\otimes 6}\right) \leq 6$

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Falsifies conjecture that

$$\chi(A^{\otimes 6}) = 7$$

[Bravyi Smith Smolin 2015]

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[Bravyi Smith Smolin 2015]

Submultiplicativity →

$$\chi\left(\mathrm{A}^{\otimes \mathrm{m}}\right) \leq O(2^{\alpha m})$$

$$\alpha = \frac{\log_2(6)}{6} \approx 0.4308$$

Improves the previously known upper bounds from

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We can construct better stabilizer decompositions of $|A\rangle^{\otimes m}$ by using magic cats in a different way…

Stabilizer rank of large magic cats

Can we build big magic cat states out of small ones?

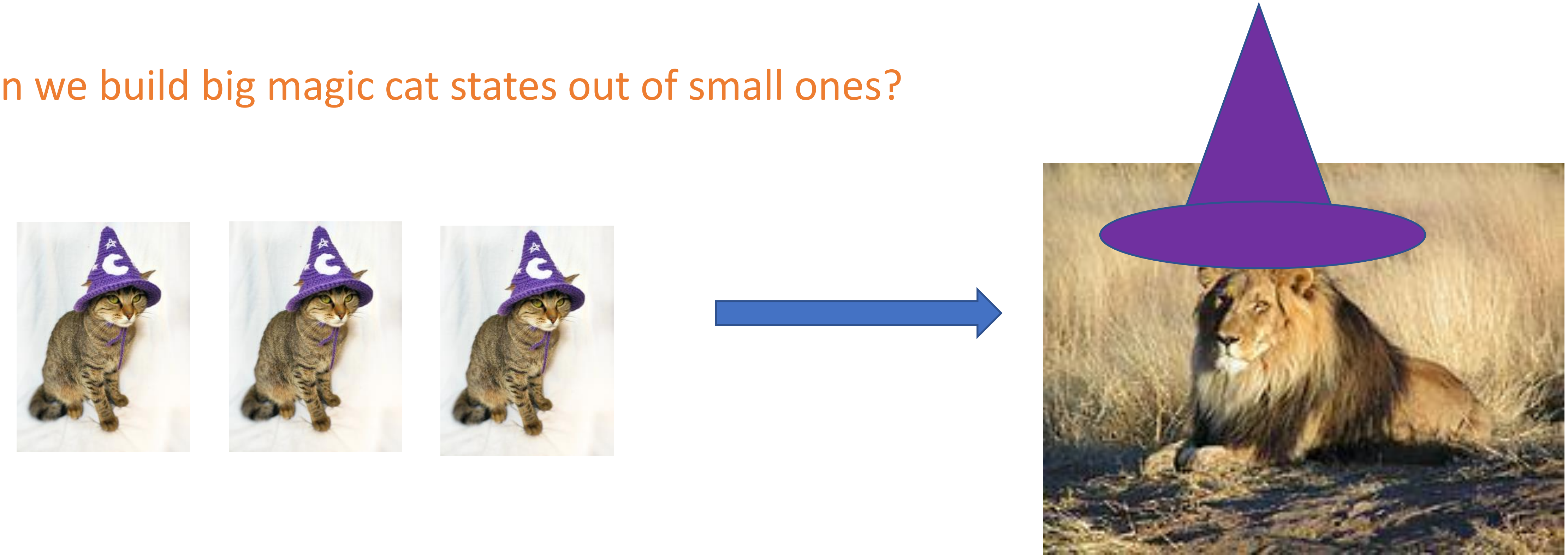

Image source: Wikipedia

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$$\langle AA|_{m,m+1}|\mathrm{Cat}_m\rangle\,|\mathrm{Cat}_m\rangle = \frac{1}{2}|A\rangle^{2m-2}$$

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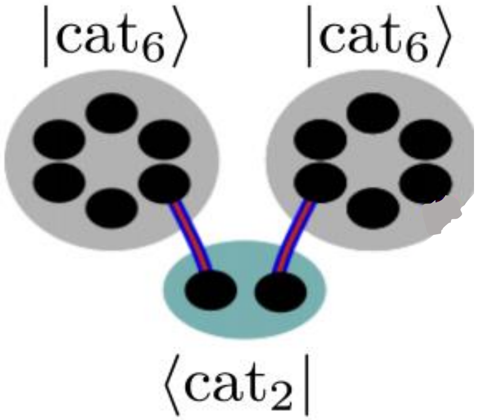
$$\sqrt{2}\underbrace{\langle \mathrm{Cat}_2|_{m,m+1}}_{\chi=1}\underbrace{|\mathrm{Cat}_m\rangle\,|\mathrm{Cat}_m\rangle}_{\leq(\chi(\mathrm{Cat}_m))^2} = \frac{1}{\sqrt{2}}|\mathrm{Cat}_{2m-2}\rangle$$

$\Rightarrow$

$$\chi(\mathbf{Cat}_{2m-2}) \leq \left(\chi(\mathbf{Cat}_m)\right)^2$$

Stabilizer rank of large magic cats

So for example, contracting two copies of Cat6 with one copy of Cat2 in this way gives



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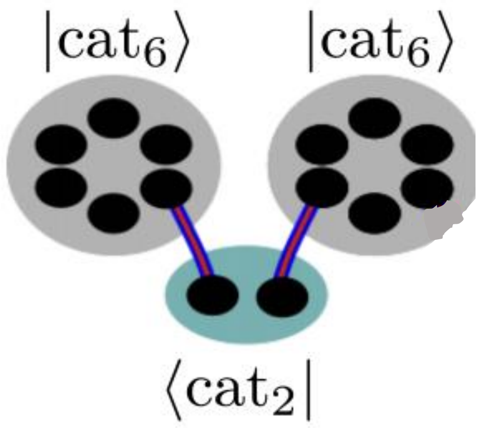

$$\chi(\mathbf{Cat_{10}}) \leq \left(\chi(\mathbf{Cat_6})\right)^{\mathbf{2}} = \mathbf{9}$$

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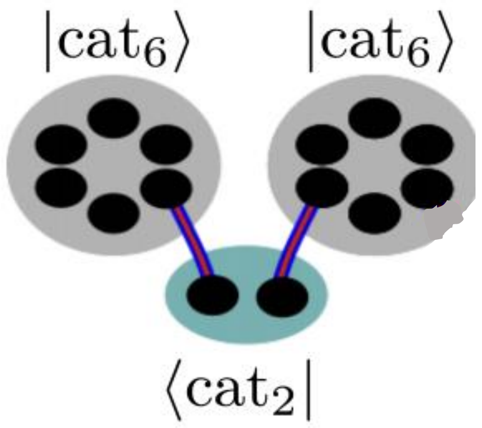

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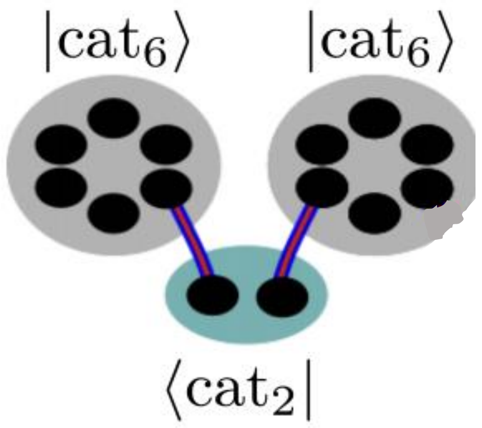

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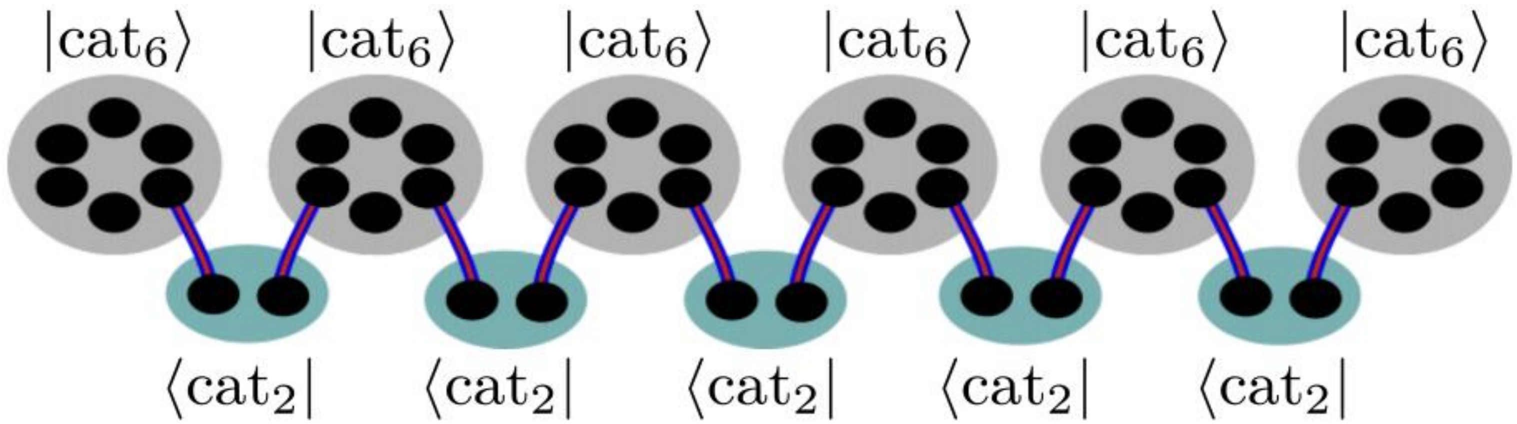
$$\chi\left(A^{10}\right) \leq 18$$

$$\chi(A^m) \leq 2^{\alpha m}$$

$$\alpha \leq \frac{\log_2(18)}{10} = 0.41699\ldots$$

Stabilizer rank of large magic cats

An even better strategy is to contract many copies of Cat6 arranged in a chain



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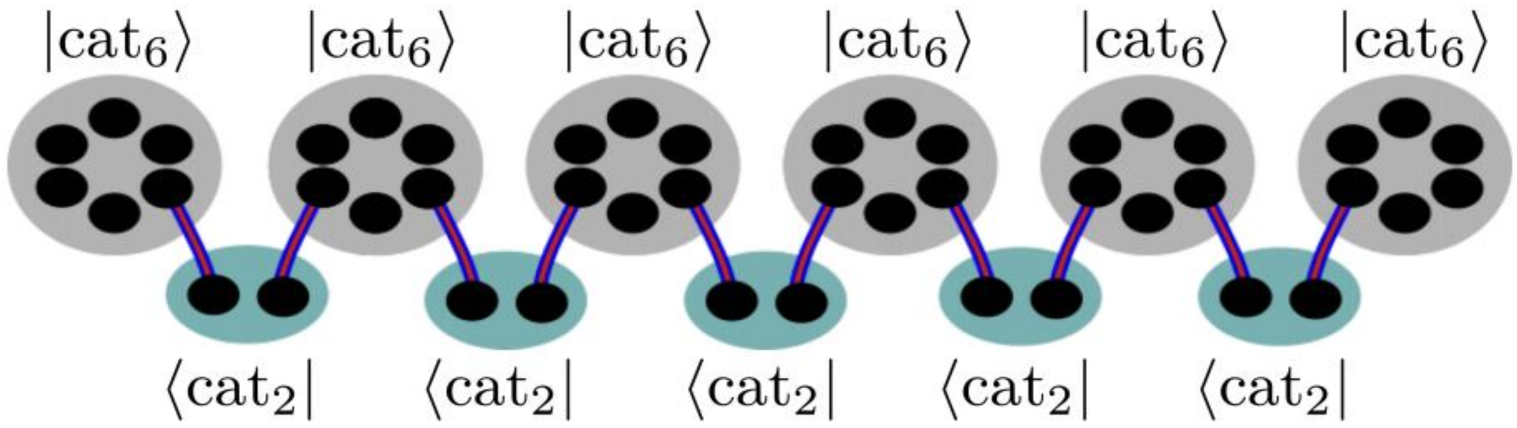

$$\chi(\mathbf{Cat}_{4L-2}) \leq \left(\chi(\mathbf{Cat}_6)\right)^L = 3^L$$

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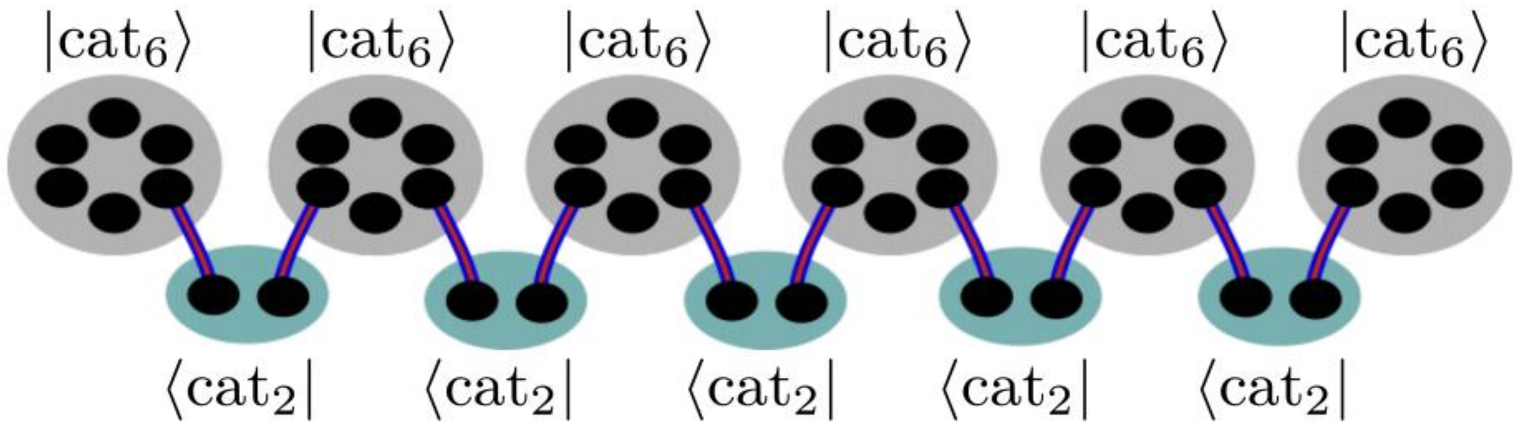

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Taking L to be large and setting $m = 4L - 2$ we get

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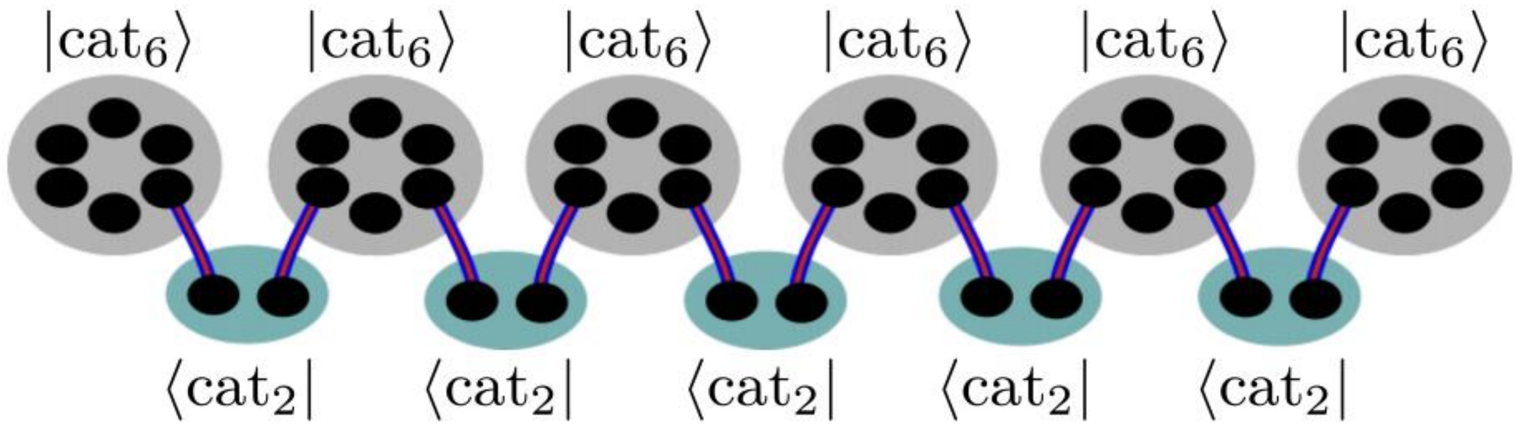

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Upper bounds (up to polynomial factors) the runtime of ϵ-strong simulation of Clifford+T circuits as a function of the number of T gates m

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Upper bounds (up to polynomial factors) the runtime of ϵ-strong simulation of Clifford+T circuits as a function of the number of T gates m

There is still a gap between this scaling and that of approx. SR (weak simulation cost) where the best upper bound has $\alpha \approx 0.228 \ldots$

Extensions

Stabilizer rank of symmetric states: Using a similar technique we upper bound the stabilizer rank of many copies of any given equatorial single-qubit state. Using the fact that such states span the symmetric subspace we get

$$\chi(\psi^{\otimes m}) \leq O\left(2^{m/2}\right)$$

Any single-qubit state

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Magic code states: We can generalize the magic cat state to a family of states which are superpositions over the codewords of a linear code $L \subseteq \{0,1\}^m$

$$|L\rangle = \sum_{x \in L} |\hat{x}\rangle \qquad \left|\hat{0}\right\rangle = |A\rangle, \quad \left|\hat{1}\right\rangle = |A^{\perp}\rangle$$

Cat state is the special case where L is the repetition code

We show how an upper bound on the stabilizer rank of magic states follows from an upper bound on $\chi(|L\rangle)$ for any code with dimension $k \leq m/2$

Open questions

Describe any nontrivial criterion for certifying that a given state has stabilizer rank at least k

Establish superpolynomial lower bound on $\chi(A^m)$

Improve upper bounds on $\chi(A^m)$

Applications to classical counting problems?

m	1	2	3	4	5	6	7	8
$\chi(A^{\otimes m})$	2	2	3	≤ 4	≤ 6	≤ 6	≤ 12	≤ 12
$\chi(\mathrm{Cat_m})$	1	1	2	2	3	3	≤ 6	≤ 6

[Bravyi Smith Smolin 15][Bravyi et al 18][Qassim Pashayan DG 21]

Thanks!