An approximate superselection principle for entanglement

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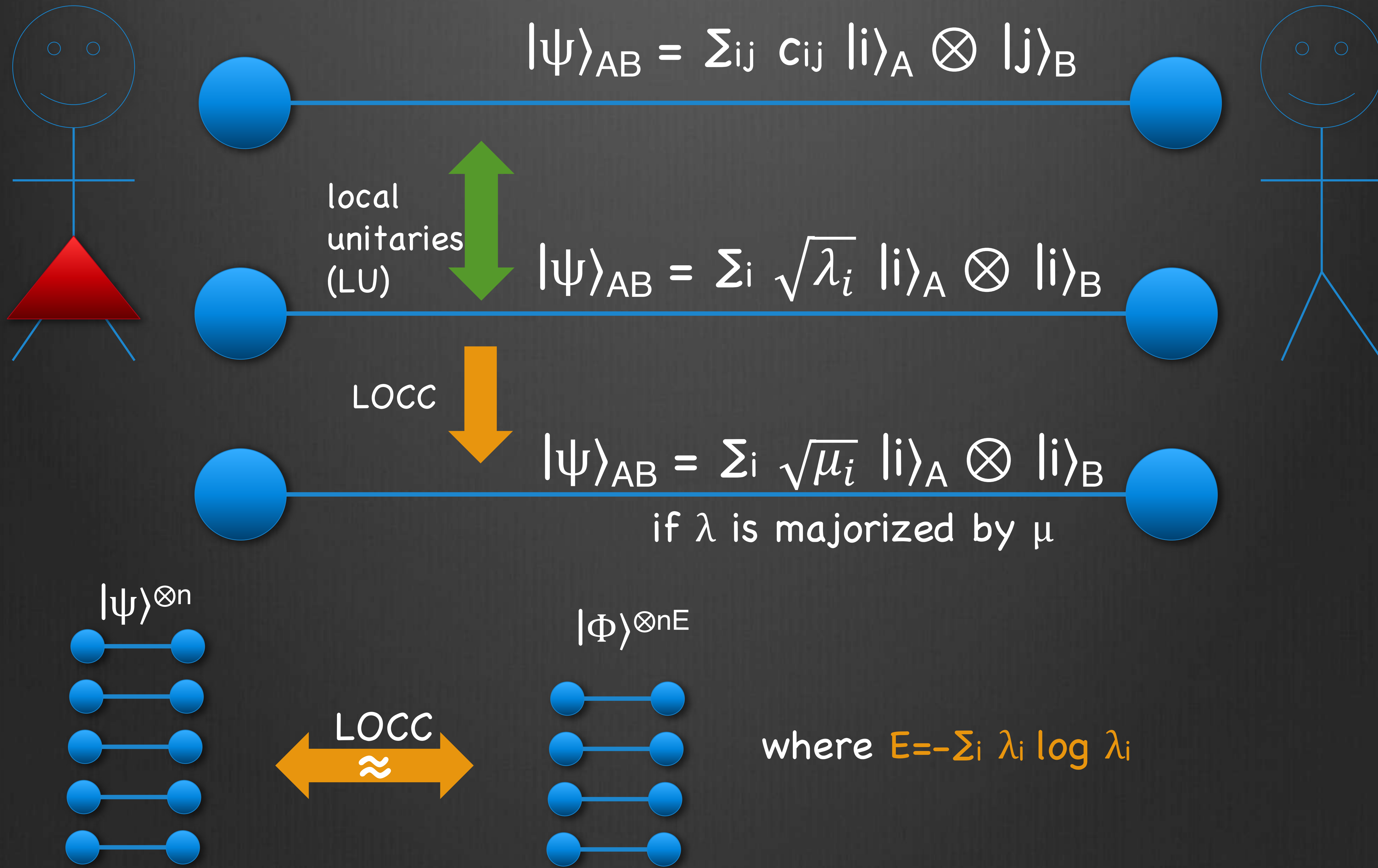
Simons reunion

16 July 2021

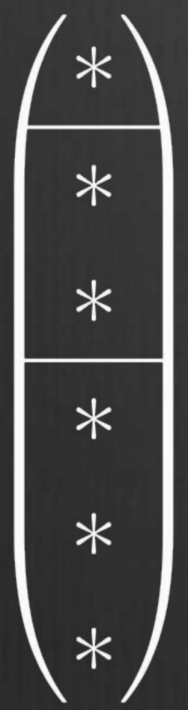
entanglement as resource

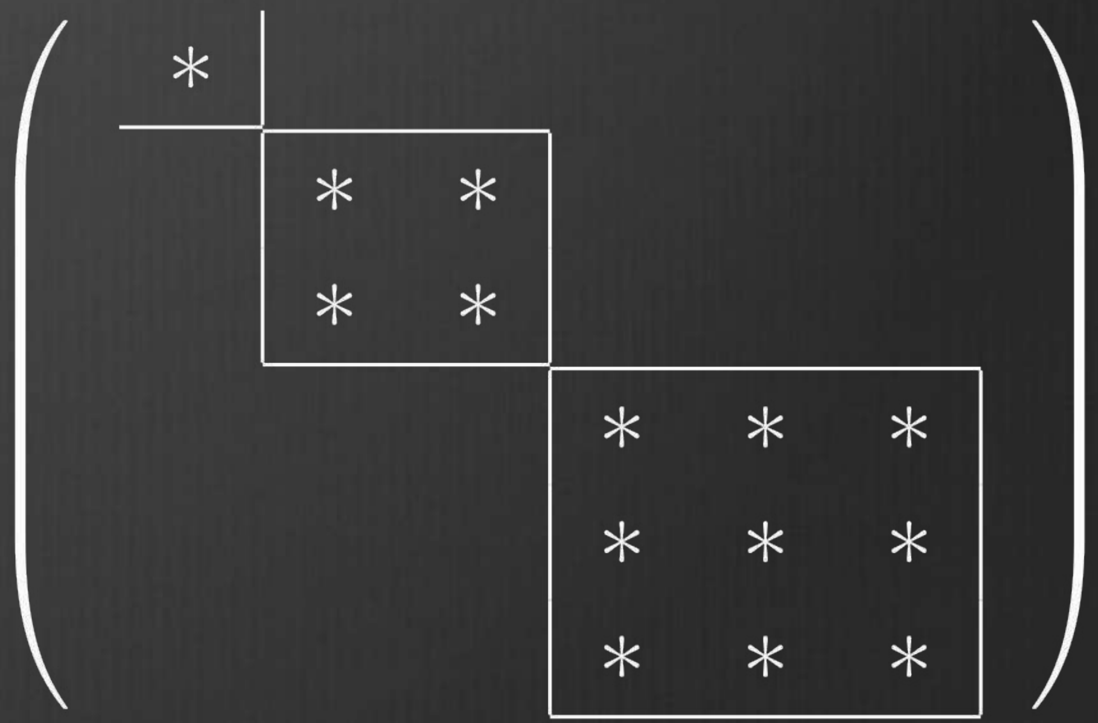


a different metaphor: conserved quantities

The state space is partitioned according to some observable, such as total particle number.

Measurements and unitary evolutions are constrained to respect this partition.





conserved entanglement

$$|\psi\rangle = \sum_k \sqrt{p_k}\, |k\rangle_A\, |k\rangle_B\, |\Phi_2\rangle_{AB}^{\otimes k}\, |00\rangle_{AB}^{\otimes n-k}$$

LU approximately preserves p

because different $|\Phi_2\rangle^{\otimes k}$ are only ≈ orthogonal

<u>Caveat</u>

To approximate any state we need

$$\sum_{k\geq 0} \sqrt{p_k} |k\rangle_A |k\rangle_B |\Phi_{\lfloor 2^{\epsilon k} \rfloor}\rangle_{AB}$$

implications

1. Any transformation using local unitaries and Q qubits of communication has off-diagonal blocks $\leq 2^{Q-\frac{|k-\ell|}{2}}$ decaying as

$$\begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{pmatrix}$$

2. States like $|01\rangle^{\otimes n} \pm |\Phi_2\rangle^{\otimes n} / \sqrt{2}$ are unusual and perhaps valuable.

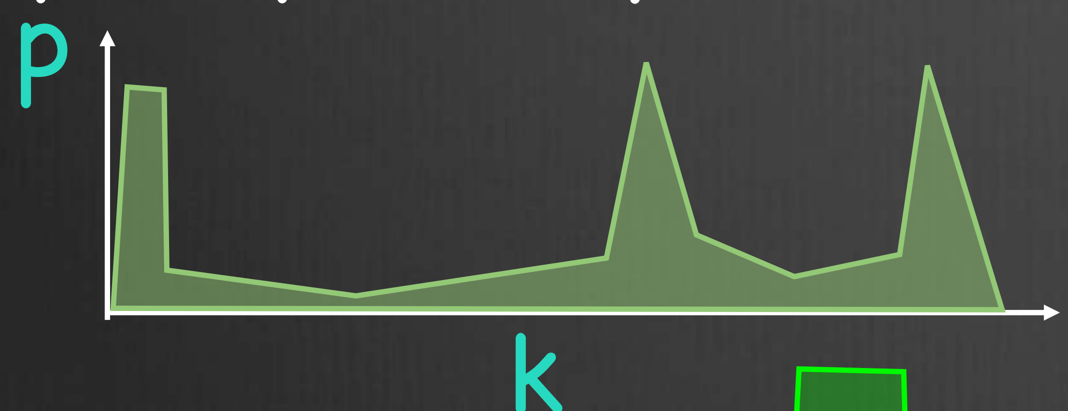
state transformation cost

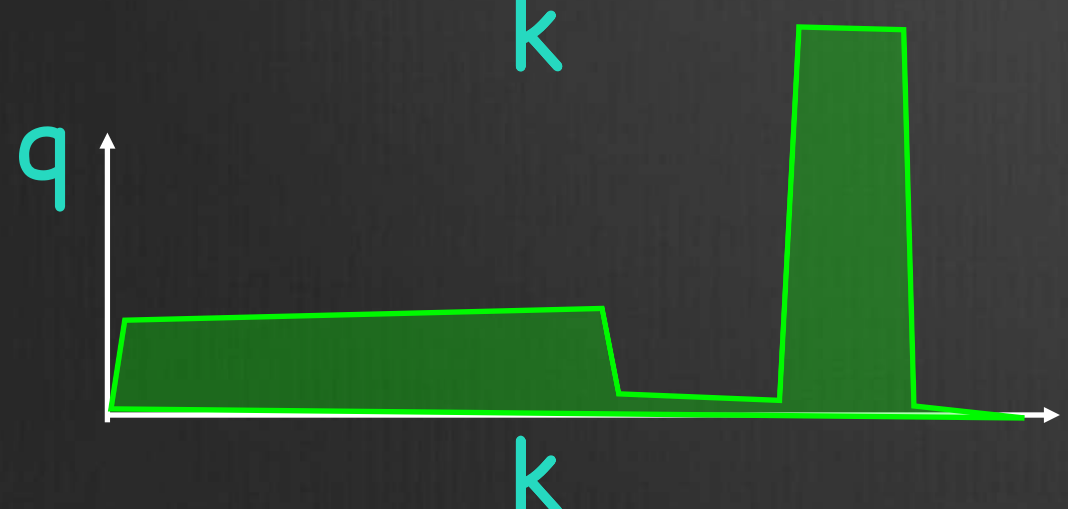
What is the communication cost to transform $|\psi\rangle \rightarrow |\varphi\rangle$ up to error ε?

1. rephrase as

$$\sum_{k\geq 0}\sqrt{p_k}|k\rangle_A|k\rangle_B|\Phi_{\lfloor 2^{\epsilon k}\rfloor}\rangle_{AB} \rightarrow \sum_{k\geq 0}\sqrt{q_k}\,|k\rangle_A\,|k\rangle_B\,|\Phi_{\lfloor 2^{\epsilon k}\rfloor}\rangle_{AB}$$

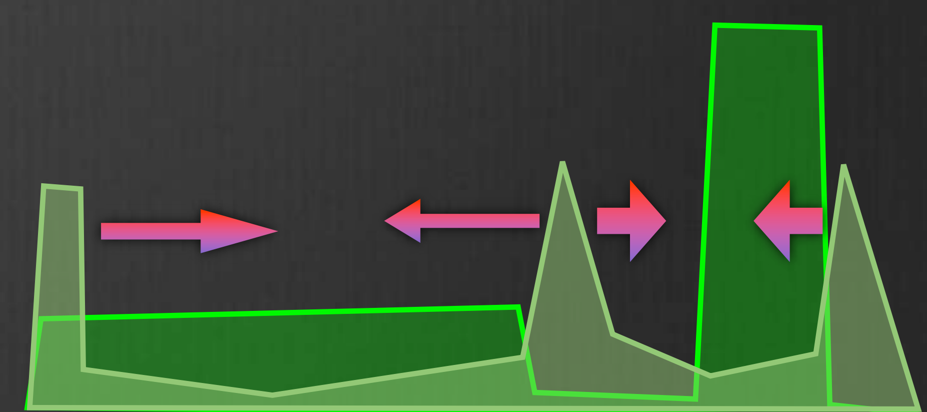
2. plot p and q





3. claim: cost ≈ 

l_∞-earth-mover distance



length of longest arrow

https://www.flickr.com/photos/ryanwick/3517668994

Lower bound from Rényi

If $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$, $|\Phi\rangle^{\otimes many} \rightarrow \sum_i \sqrt{\lambda_i}\, |i\rangle\, |i\rangle$
requires $\Delta(\psi) := \log(r\lambda_1)/2$ qubits of communication

$\Delta(\psi)$ = the "entanglement spread" of $|\psi\rangle$.

Proof: r and λ_1 each change by at most 2 per qubit sent.
For EPR pairs $r\lambda_1=1$.
[Hayden-Winter. quant-ph/0204092]

for $\sum_{k\geq 0} \sqrt{p_k} |k\rangle_A |k\rangle_B |\Phi_{\lfloor 2^{\epsilon k} \rfloor}\rangle_{AB}$ spread = width of support of p

Example: For $|01\rangle^{\otimes n} + |\Phi_2\rangle^{\otimes n} / \sqrt{2}$, $r\lambda_1 \approx 2^n$.
→ creating it requires ≈n/2 qubits of communication.

state testing

Thm: communication cost of

- measuring $\{|\psi\rangle\langle\psi|, I - |\psi\rangle\langle\psi|\}$
- performing $I - 2\,|\psi\rangle\langle\psi|$

is $O(\Delta(\psi)) + O(\log 1/\varepsilon)$.

Free EPR pairs don't help but other entangled states can.

Lower bound idea:

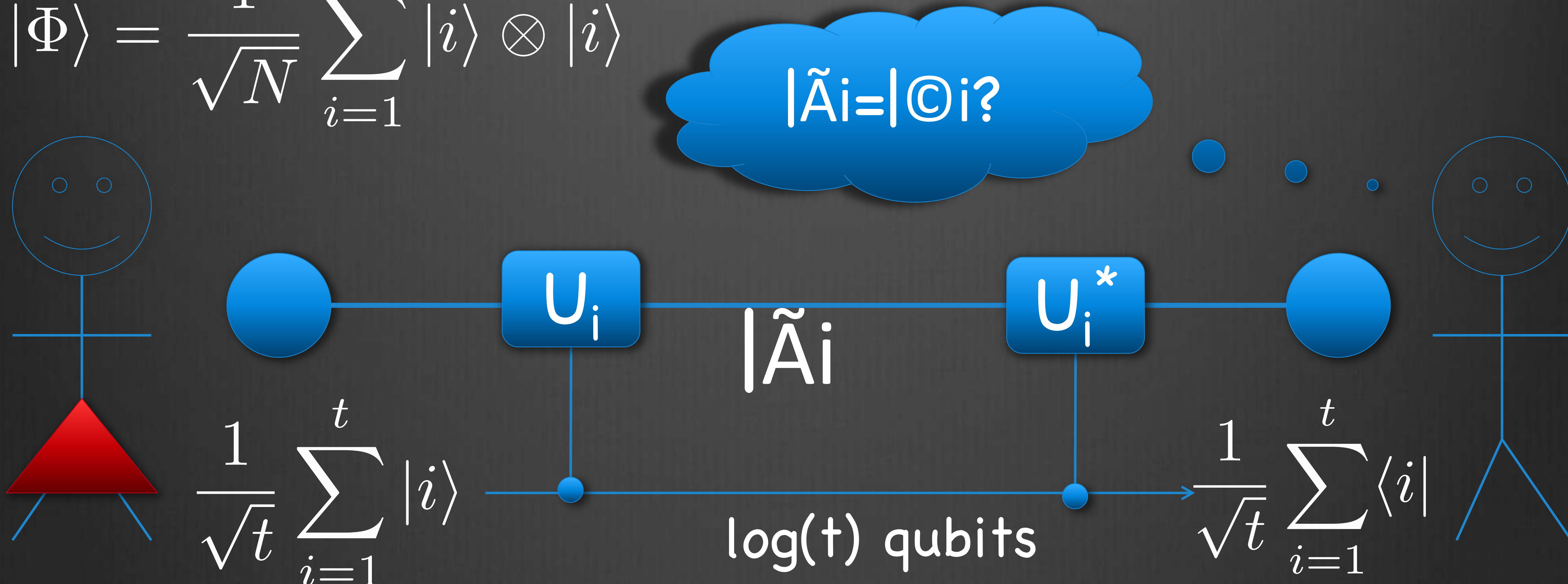
- Distinguishing $|01\rangle^{\otimes n} \pm |\Phi_2\rangle^{\otimes n} / \sqrt{2}$ is equivalent to $|01\rangle^{\otimes n} \leftrightarrow |\Phi_2\rangle^{\otimes n}$
- This is hard to do unitarily even with EPR assistance.

Helper states:

- Embezzling states + LU can approximately create *any* state
- $|\psi\rangle^{\otimes k}$ can be distinguished from $|\psi\rangle^{\otimes k-1}\,|\psi^{\perp}\rangle$ using $O(\log k)$ communication

EPR testing

$$|\Phi\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle \otimes |i\rangle$$



Idea: |©i is unique state invariant under UU*.

Result: Error ¸ with O(log 1/¸) qubits sent.

Previous work used O(loglog(N) + log(1/λ)) qubits [BDSW '96, BCGST '02]

EPR testing protocol

steps	state
1. Initial state:	$\|\psi\rangle^{AB}$
2. Alice adds ancilla in $\frac{1}{\sqrt{t}}\sum_{i=1}^{t}\|i\rangle^{A'}$ state	$\frac{1}{\sqrt{t}}\sum_{i=1}^{t}\|i\rangle^{A'}\|\psi\rangle^{AB}$
3. Alice applies controlled U_i i.e. $\sum_i \|i\rangle\langle i\| \otimes U_i$	$\frac{1}{\sqrt{t}}\sum_{i=1}^{t}\|i\rangle^{A'}(U_i\otimes I)\|\psi\rangle^{AB}$
4. Alice sends A′ to Bob and Bob applies controlled U_i^*	$\frac{1}{\sqrt{t}}\sum_{i=1}^{t}\|i\rangle^{B'}(U_i\otimes U_i^*)\|\psi\rangle^{AB}$
5. Bob projects B′ onto $t^{-1/2}\sum_i \|i\rangle$.	$\frac{1}{t}\sum_{i=1}^{t}(U_i\otimes U_i^*)\|\psi\rangle^{AB}$

Analyzing protocol

Subnormalized output state (given acceptance) is

$$M|\psi\rangle = \frac{1}{t}\sum_{i=1}^{t}(U_i \otimes U_i^*)|\psi\rangle$$

Pr[accept] = $\langle\psi|M^\dagger M|\psi\rangle$

Key claim: $\| M - |\Phi\rangle\langle\Phi| \| \leq \lambda$

Interpretation as super-operators:

$X = \Sigma_{a,b} X_{a,b} |a\rangle\langle b|$ → $|X\rangle = \Sigma_{a,b} X_{a,b} |a\rangle\otimes|b\rangle$

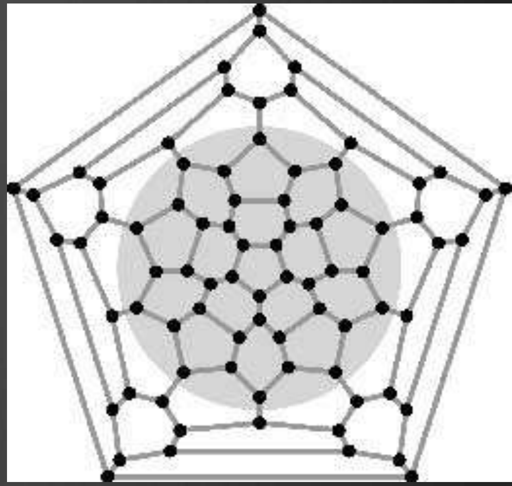
$T(X) = AXB$ → $T|X\rangle = (A \otimes B^T)|X\rangle$

$T(X) = UXU^y$ → $T|X\rangle = (U \otimes U^*)|X\rangle$

identity matrix → $|\Phi\rangle$

$\|M(X)\|_2 \leq \lambda\|X\|_2$ if $tr[X]=0$ ←→ $\| M - |\Phi\rangle\langle\Phi| \| \leq \lambda$

Quantum expanders



A collection of unitaries $U_1, \ldots, U_t$ is a quantum (N,t,λ) expander if

$$\left\| \frac{1}{t} \sum_{i=1}^{t} U_i X U_i^\dagger \right\|_2 \leq \lambda \|X\|_2 \quad \text{whenever } \mathrm{tr}[X]=0$$

(cf. classical t-regular expander graphs can be viewed as permutations $\pi_1,\ldots,\pi_t$ such that $t^{-1} \|\Sigma_i \pi_i x\|_2 \leq \lambda \|x\|_2$ whenever $\Sigma_i x_i = 0$.)

Random unitaries satisfy $\lambda \approx 1 / t^{1/2}$ [Hastings '07]
Efficient constructions (i.e. polylog(N) gates) achieve $\lambda \leq 1 / t^c$ for c>0. [various]
Recall communication is $\log(t) = O(\log 1/\lambda)$

state testing

Thm: communication cost of
- measuring $\{|\psi\rangle\langle\psi|, I - |\psi\rangle\langle\psi|\}$
- performing $I - 2\,|\psi\rangle\langle\psi|$

is $O(\Delta(\psi)) + O(\log 1/\varepsilon)$.

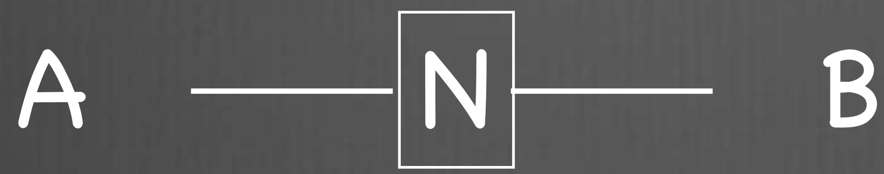
Free EPR pairs don't help but other entangled states can.

Upper bound:
- We have $O(\log 1/\varepsilon)$ for any maximally entangled state.
- $|\psi\rangle \leftrightarrow |\Phi_N\rangle$ using $\Delta(\psi)/2$ communication

Application to information theory

- Traditionally spread has been thought of as a "sublinear" phenomenon, and as a result, has been neglected.
- <u>Example</u>: If $|\psi\rangle$ is an entangled state, then $|\psi\rangle^{\otimes n}$ is very close to a state with spread $O(\sqrt{n})$. Therefore, $O(\sqrt{n})$ bits of communication are necessary and sufficent to prepare $|\psi\rangle^{\otimes n}$ from EPR pairs. (a.k.a. entanglement dilution.) [Hayden-Winter '02, Harrow-Lo '02]
- However, even in i.i.d. settings, entanglement spread can be size O(n).

Example: Channel simulation



Shannon's (noisy coding) theorem:

Any noisy channel N using input distribution p^A can code at rate $C_{N,p} = H(A)_p + H(B)_p - H(AB)_p$.

(Classical) Reverse Shannon Theorem: N can be simulated on $p^{\otimes n}$ using communication $C_{N,p}$ and shared randomness $R_{N,p} = H(AB)_p - H(A)_p$.

[BSST01,Cuff08]

On general inputs:

The capacity and simulation cost are replaced by $C(N) = \max_p C_{N,p}$. Randomness cost for simulation is $\max_p H(B)_p - C(N)$.

Simulating quantum channels

- Coding with quantum channels: Using shared EPR pairs, a quantum channel N can send noiseless qubits at rate $\max_\rho Q_{N,\rho} = \max_\rho (H(A)_\rho + H(B)_\rho - H(AB)_\rho) / 2$.

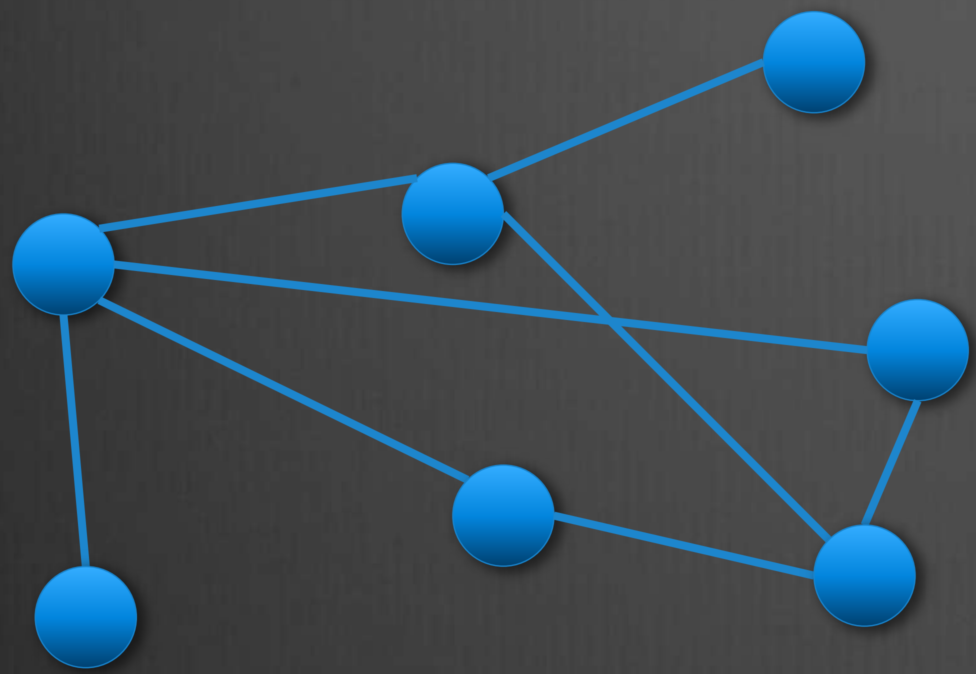
- Quantum Reverse Shannon Theorem: For a quantum channel N and an input distribution ρ, $N^{\otimes n}$ can be simulated on $\rho^{\otimes n}$ using $Q_{N,\rho}$ qubits of communication and $E_{N,\rho} = H(B)_\rho - Q_{N,\rho}$ shared EPR pairs.

- However, it does not follow that $N^{\otimes n}$ can be simulated on arbitrary inputs using $\max_\rho(Q_{N,\rho})$ qubits of communication and $\max_\rho(E_{N,\rho})$ shared EPR pairs!

- Problem: suppose that the input to $N^{\otimes n}$ is $(|\rho\rangle^{\otimes n} + |\sigma\rangle^{\otimes n})/\sqrt{2}$ with $Q_{N,\rho} = Q_{N,\sigma}$ but $E_{N,\rho} > E_{N,\sigma}$. Then the naive method of combining the two simulations will require creating $n(E_{N,\rho} - E_{N,\sigma})$ entanglement spread.

- This requires either extra communication (forward or back) or embezzling states.

Local Hamiltonians



$$H = \sum_{(i,j)\in E} H_{i,j}$$

$\| H_{i,j} \| \leq 1$

Define: eigenvalues $\lambda_0 \leq \lambda_1 \leq \ldots$
eigenstates $|\psi_0\rangle, |\psi_1\rangle, \ldots$

Assume: degree ≤ const, gap := $\lambda_1 - \lambda_0 \geq$ const.

Known: $|\langle AB\rangle - \langle A\rangle\langle B\rangle| \lessapprox \|A\| \|B\| \exp(-\text{dist}(A,B) / \xi)$
"correlation decay" [Hastings '04, Hastings-Kumo '05, ...]

Intuition: $((1+\lambda_0)I - H)^{\log(1/\varepsilon)/\text{gap}} \approx |\psi_0\rangle\langle\psi_0|$ $\langle X\rangle := \text{tr}[X\psi_0]$
[Arad-Kitaev-Landau-Vazirani, 1301.1162]

Area "law"?

Conjecture: For any set of systems A⊆V

$$S(\psi_0^A) \le O(|\partial A|)$$

Or even, with variable dimensions $d_1, \ldots, d_n$.

$$S(\psi_0^A) \le O(1) \cdot \sum_{\substack{(i,j)\in E \\ i\in A, j\in A^c}} \log(d_i) + \log(d_j) \quad := |\partial A|$$

Known: in 1-D

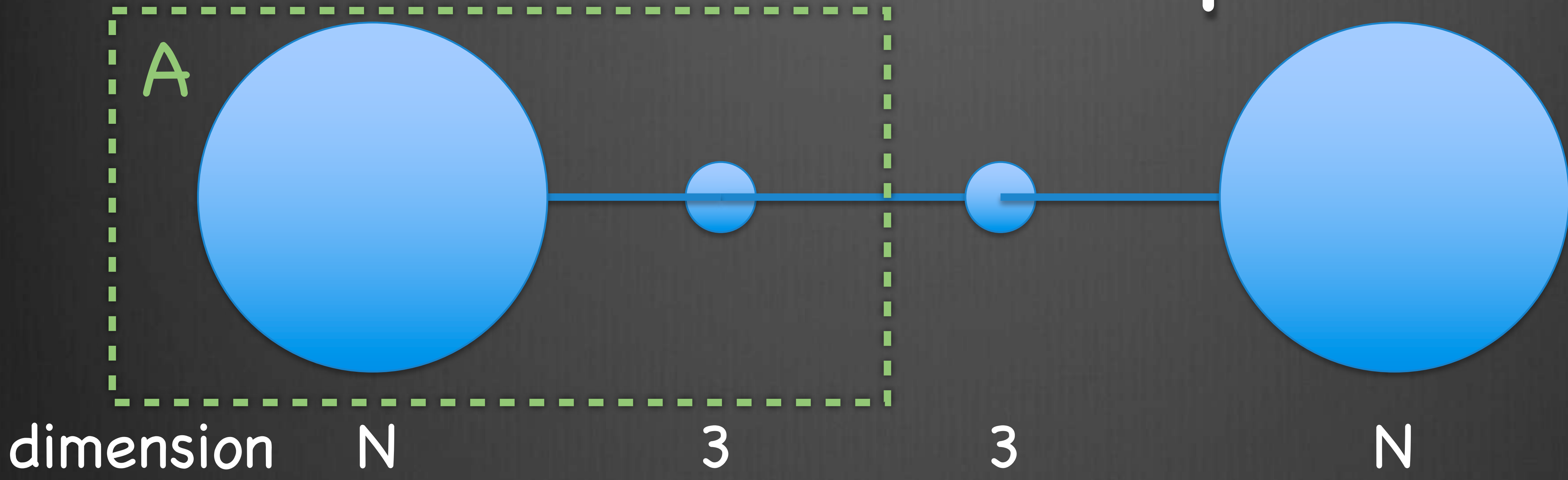
gap → (Hastings '04) → correlation decay → (Brandão-Horodecki '12) → area law

gap → area law (Hastings '07, Arad-Kitaev-Landau-Vazirani '13)

further implications: efficient description (MPS), algorithms

1410.0951
Aharonov
Harrow
Landau
Nagaj
Szegedy
Vazirani

counter-examples

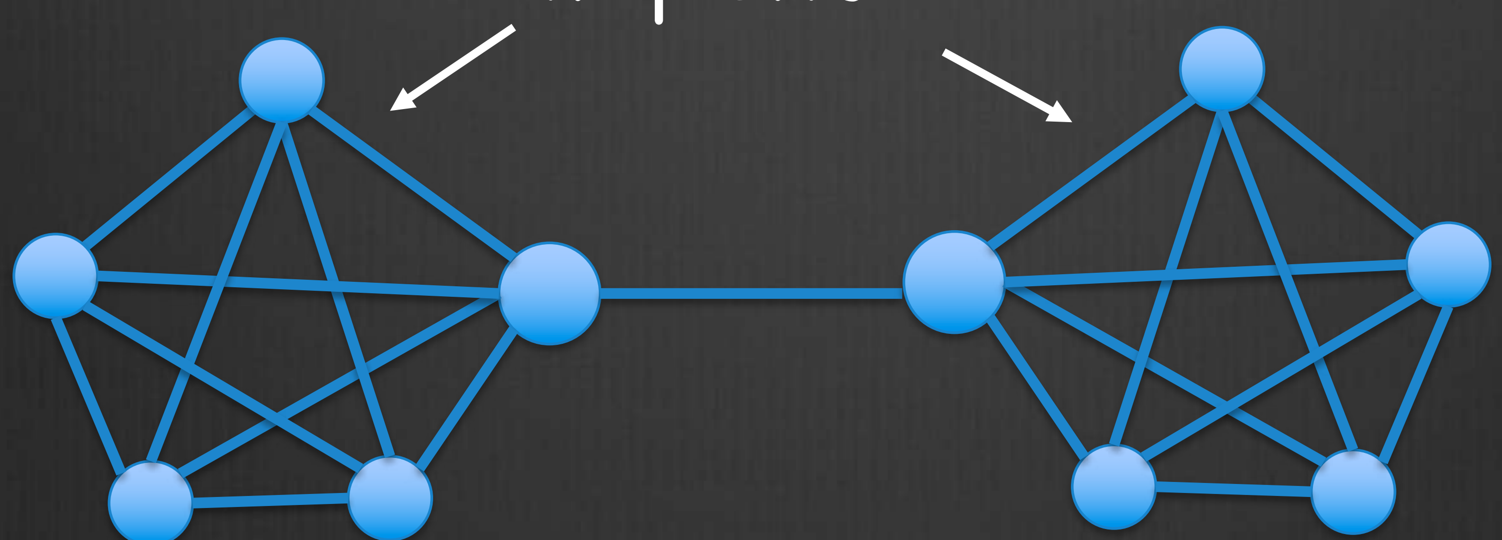


Entanglement $\propto$ log(N)

$|\partial A| \leq O(1)$

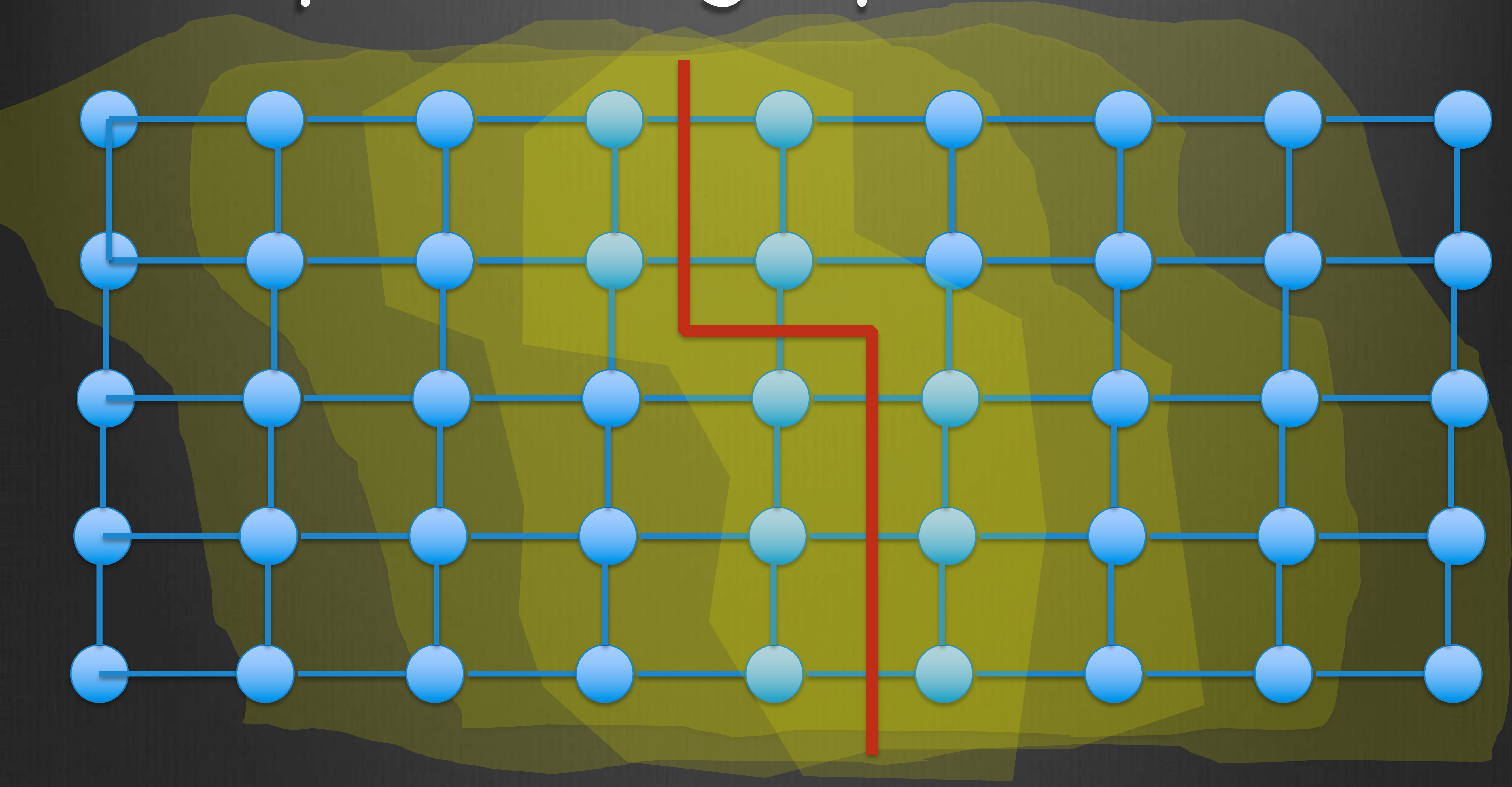
Qubit version:

n qubits



entanglement $\propto n^c$

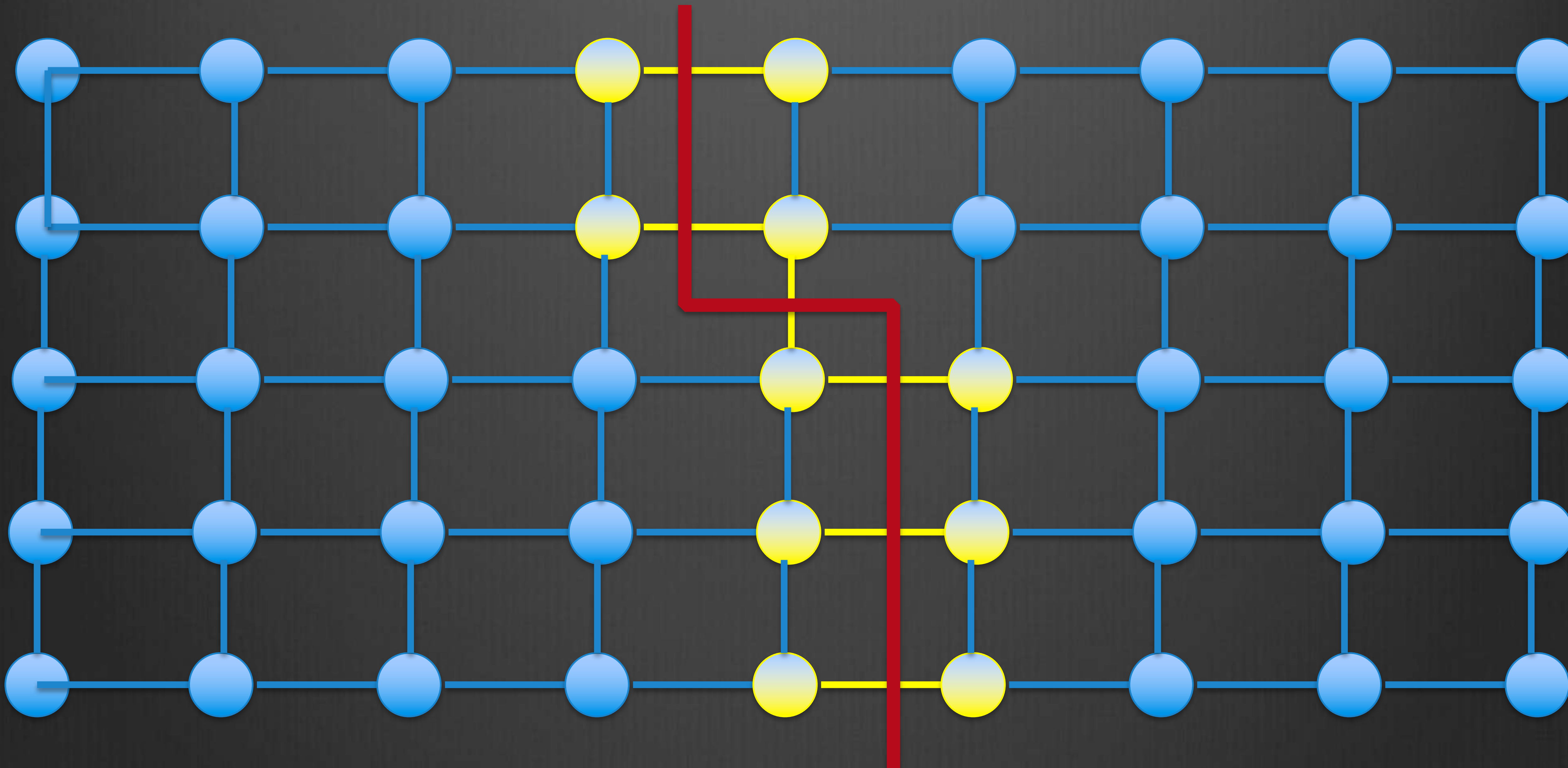
possible graph area law



conjecture: entanglement ≤ Σ_v log(dim(v)) exp(-dist(v, cut) / ξ)

Anshu, Harrow, Soleimanifar

exact area law for spread



claim: spread ≤ $O(|\partial A| / \text{gap})$

previously known: $I(A:B)_T \leq O(|\partial A| / T)$ for temperature T

Wolf, Verstraete,Hastings,Cirac 0704.3906

exact area law for spread

claim: spread $\leq O(|\partial A| / \text{gap})$

Lemma: Alice and Bob can estimate the energy of a state to precision δ using $O(|\partial A|/\delta)$ q. communication.

Proof of lemma: Use phase estimation.
Requires applying e^{-iHt} for $0 \leq t \leq 1/\delta$.
Use Low-Wiebe interaction-picture simulation

- $H = H_A + H_B + H_{\partial A}$
- H_A and H_B are free
- $H_{\partial A}$ needs $O(|\partial A|\ t)$ qubits of communication

Proof of main result:
Implement $I-2|gs\rangle\langle gs|$ with δ = gap.
Cost is $O(|\partial A|/\text{gap}) \geq \Omega(\text{spread}(gs))$.

Communication complexity

- Alice gets $x\in\{0,1\}^n$, Bob gets $y\in\{0,1\}^n$ and they would like to compute $f(x,y)$ using as little communication as possible, allowing a small chance of error.
- Communication can be one-way or two-way.
- Shared randomness is known to help, but by Newman's theorem, $O(\log n)$ bits of shared randomness always suffice.
- Free EPR pairs are known to help, although all known examples simply use them to turn classical communication into quantum communication.
- Can non-standard entanglement (e.g. embezzling states) save even more communication?

Communication complexity

Claim: General entanglement is not much better than EPR pairs in reducing communication complexity.

Proof: Let $\sum_k \sqrt{p_k}\,|k\rangle|k\rangle|\Phi_2\rangle^{\otimes k}$ be our starting state for a protocol that uses Q qubits of communication. Then Pr[accept] is of the form

$$\begin{pmatrix} \sqrt{p_1} & \sqrt{p_2} & \sqrt{p_3} & \cdots & \end{pmatrix} \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{pmatrix} \begin{pmatrix} \sqrt{p_1} \\ \sqrt{p_2} \\ \sqrt{p_3} \\ \vdots \end{pmatrix}$$

Thus we can replace $|\psi\rangle$ with a mixture of states with spread $O(Q/\varepsilon)$ and incur error $\leq\varepsilon$.

Open questions

Does entanglement help in communication complexity?

- Problem reduces to EPR pairs

Do ground states satisfy an area law?

- Can assume WLOG that they are “EPR-like”

Do these reductions help?

References

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