Graph Sparsification III: Ramanujan Graphs, Lifts, and Interlacing Families

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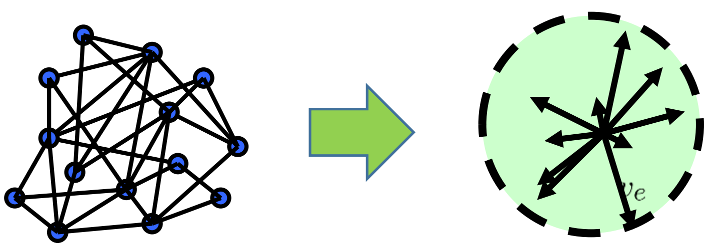
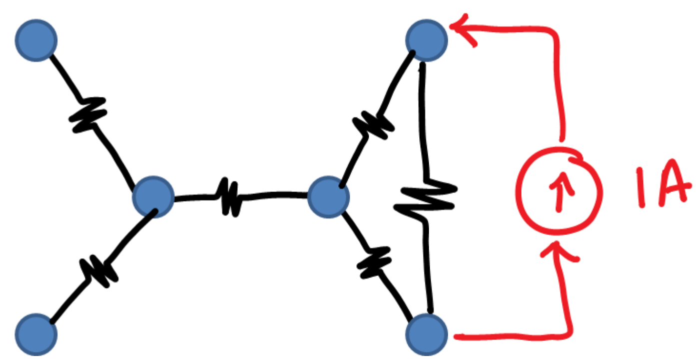
Simons Institute, August 27, 2014

The Last Two Lectures

Lecture 1. Every weighted undirected G has a weighted subgraph H with $O\left(\frac{n \log n}{\epsilon^2}\right)$ edges which satisfies

$$L_G \preccurlyeq L_H \preccurlyeq (1+\epsilon) L_G$$





random sampling

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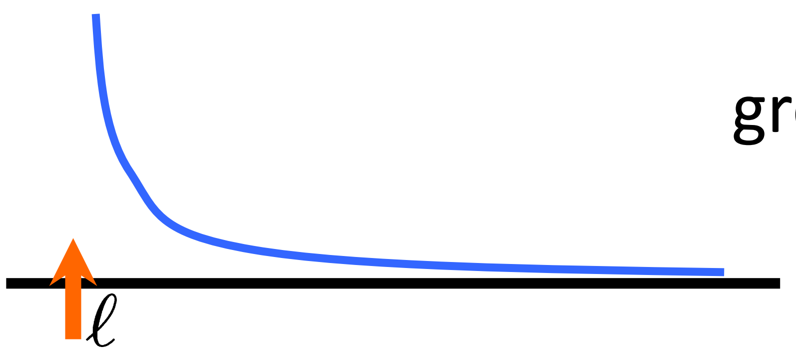
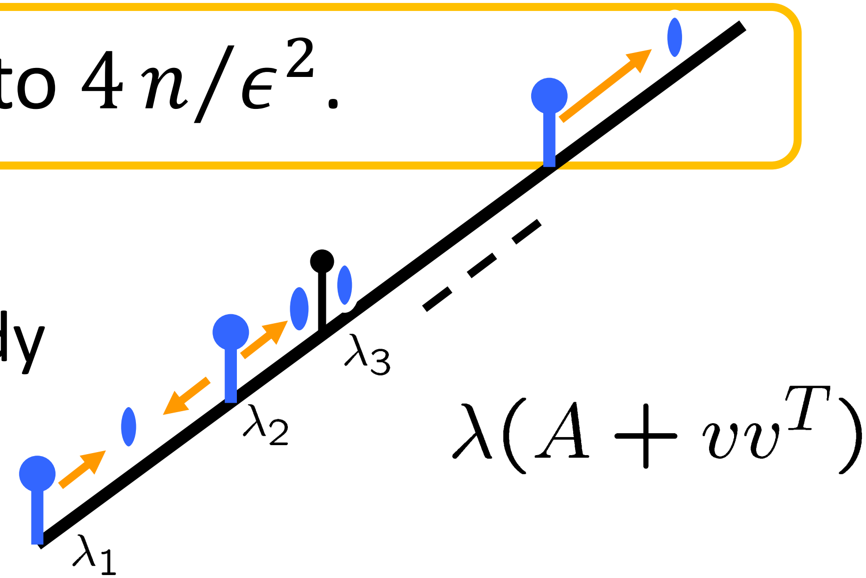
Lecture 2. Improved this to $4\,n/\epsilon^2$.



greedy



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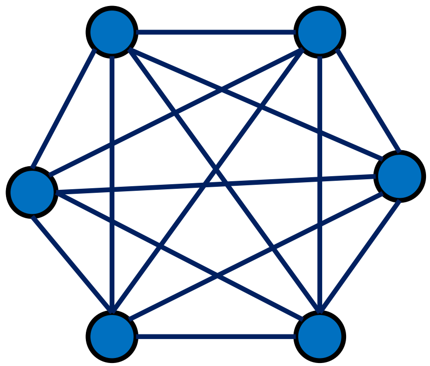
Lecture 2. Improved this to $4\,n/\epsilon^2$.

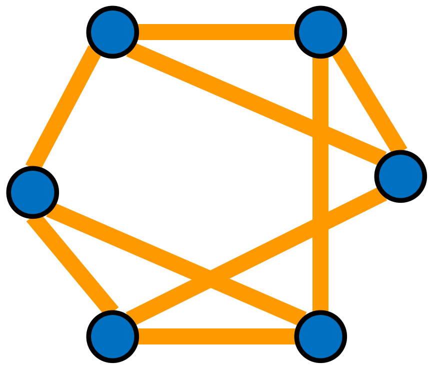
Suboptimal for K_n in two ways: weights, and $2n/\epsilon^2$.

Good Sparsifiers of K_n

G=K$_n$

H = random d-regular x **(n/d)**





$|E_G| = O(n^2)$

$|E_H| = O(dn)$

$$d = 2/\epsilon^2$$

$$L_{K_n}(1-\epsilon) \preccurlyeq L_H \preccurlyeq L_{K_n}(1+\epsilon)$$

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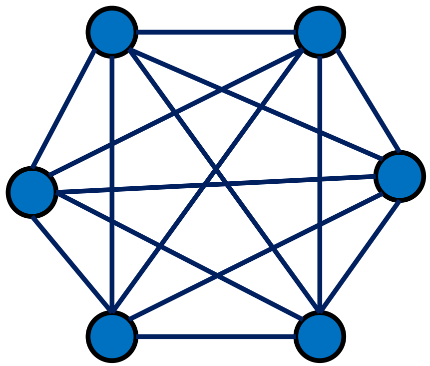

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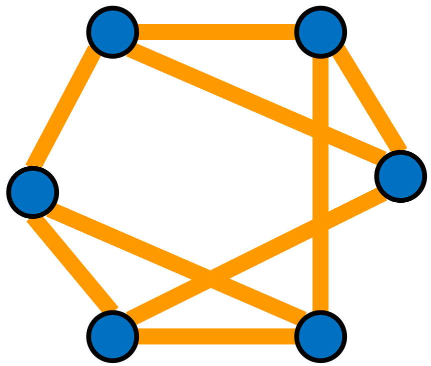


$|E_H| = O(dn)$

$$d = 2/\epsilon^2$$

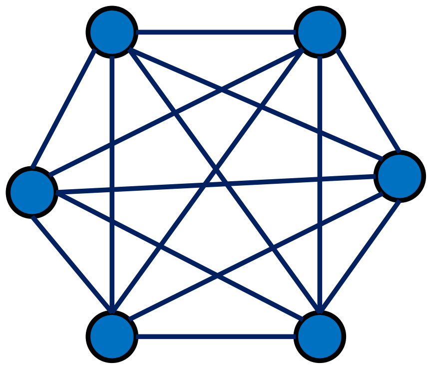
$$n(1-\epsilon) \preccurlyeq L_H \preccurlyeq n(1+\epsilon)$$

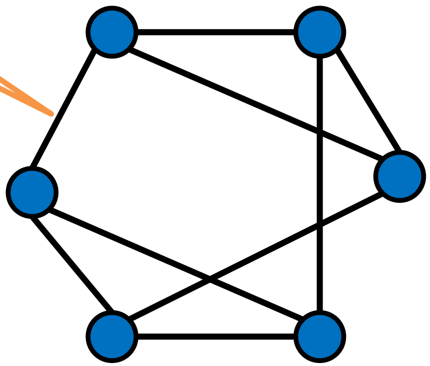
Regular Unweighted Sparsifiers of K_n

G=K$_n$

Rescale weights back to 1

H = random d-regular





$|E_G| = O(n^2)$

$|E_H| = dn/2$

$$d = 2/\epsilon^2$$

$$d(1-\epsilon) \preccurlyeq L_H \preccurlyeq d(1+\epsilon)$$

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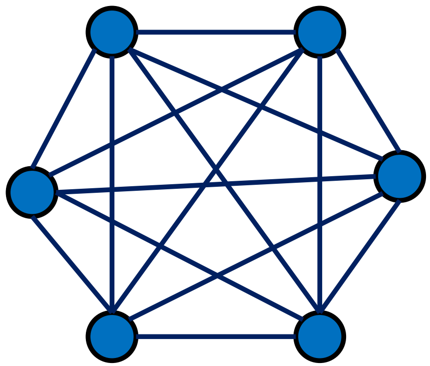

$|E_G| = O(n^2)$

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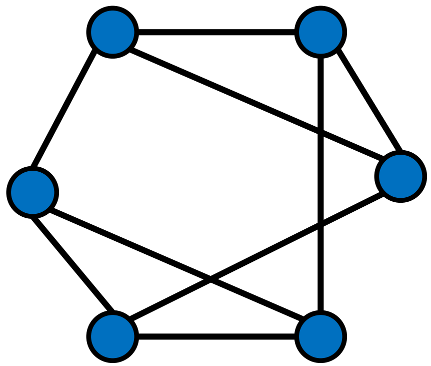

$|E_H| = dn/2$

[Friedman'08]

$$d - 2\sqrt{d-1} \lesssim L_H \lesssim d + 2\sqrt{d-1} + o(1)$$

Regular Unweighted Sparsifiers of K_n

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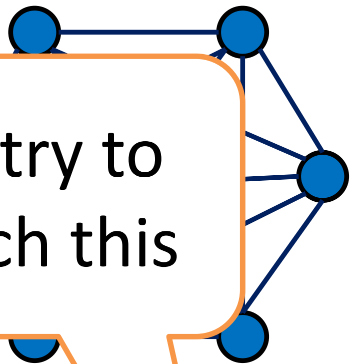

will try to match this

$|E_G| = O(n^2)$

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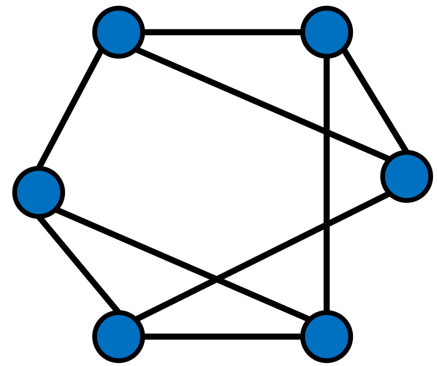


$|E_H| = dn/2$

[Friedman'08]

$$d - 2\sqrt{d-1} \leqslant L_H \leqslant d + 2\sqrt{d-1} + o(1)$$

Why do we care so much about K_n?

Unweighted d-regular approximations of K_n are called **expanders.**

They behave like random graphs:

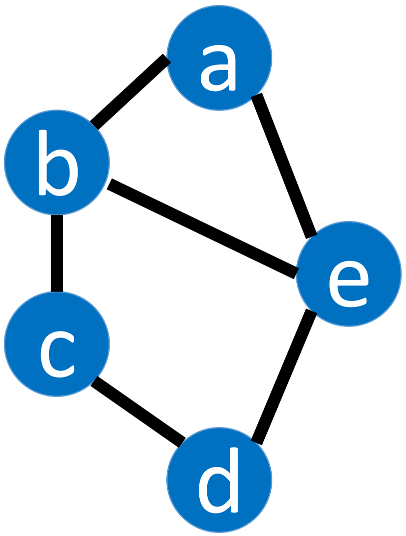
the right # edges across cuts

fast mixing of random walks

Prototypical 'pseudorandom object'. Many uses in CS and math (Routing, Coding, Complexity…)

Switch to Adjacency Matrix

Let G be a graph and A be its adjacency matrix



0	1	0	0	1
1	0	1	0	1
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eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$

$$L = dI - A$$

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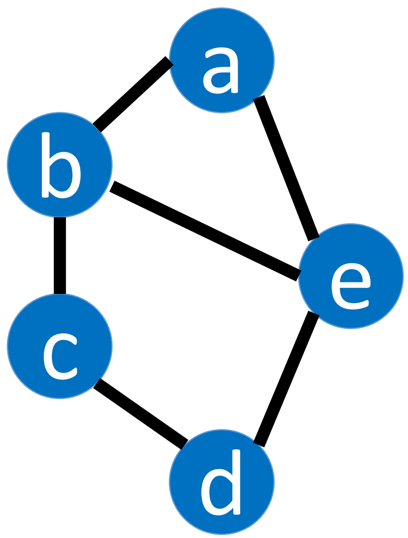

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eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$

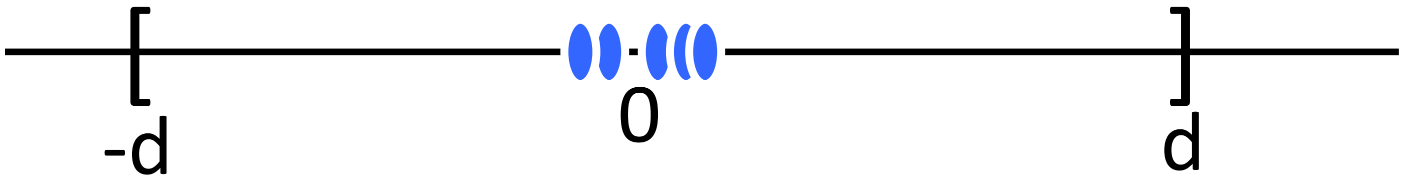
"trivial"

If d-regular, then $A\mathbf{1} = d\mathbf{1}$ so $\lambda_1 = d$

If bipartite then eigs are symmetric about zero so $\lambda_n = -d$

Spectral Expanders

Definition: *G* is a good expander
if all non-trivial eigenvalues are small



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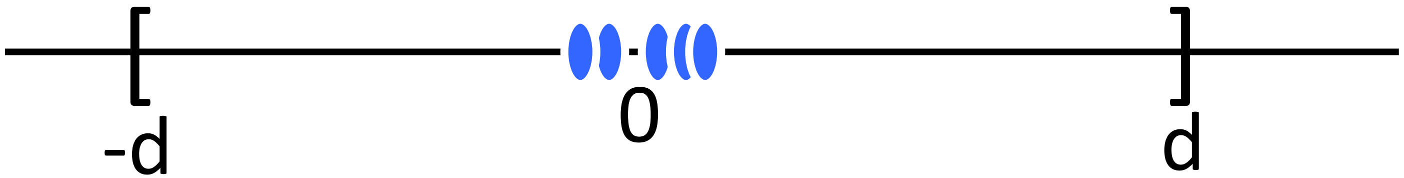

e.g. K_d and $K_{d,d}$ have all nontrivial eigs 0.

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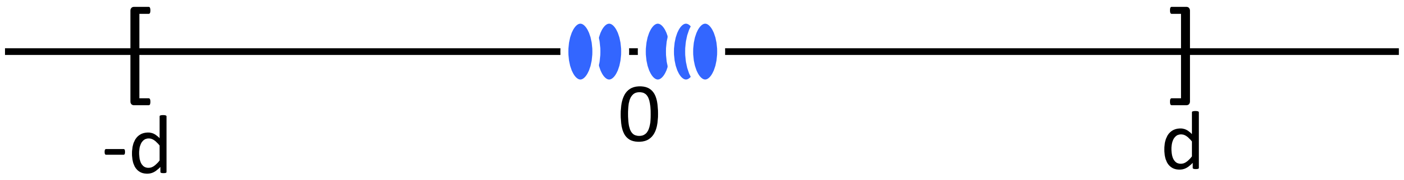

Challenge: construct infinite families.

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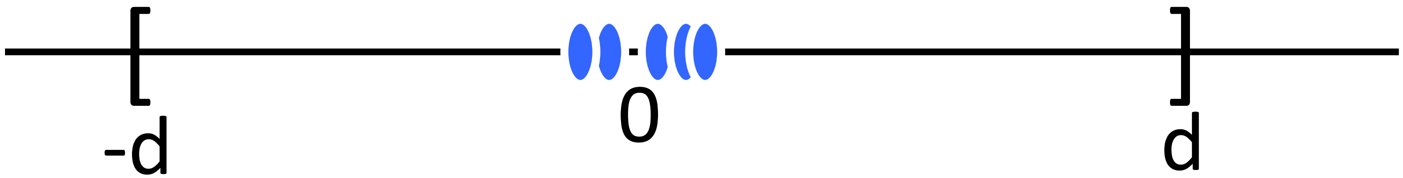
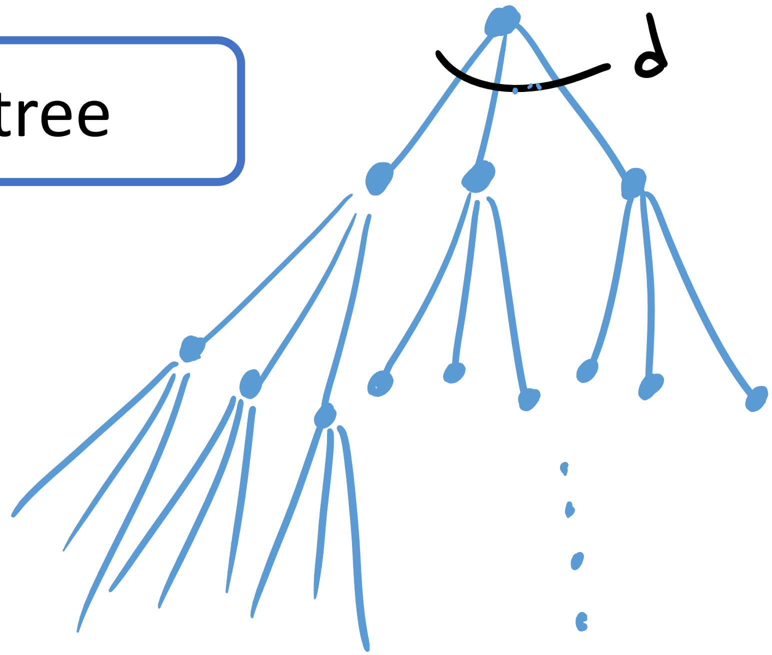


Challenge: construct infinite families.

Alon-Boppana'86: Can't beat
$$[-2\sqrt{d-1}, 2\sqrt{d-1}]$$

The meaning of $2\sqrt{d-1}$

The infinite d-ary tree



$$\lambda(A_T) = [-2\sqrt{d-1}, 2\sqrt{d-1}]$$

The meaning of $2\sqrt{d-1}$

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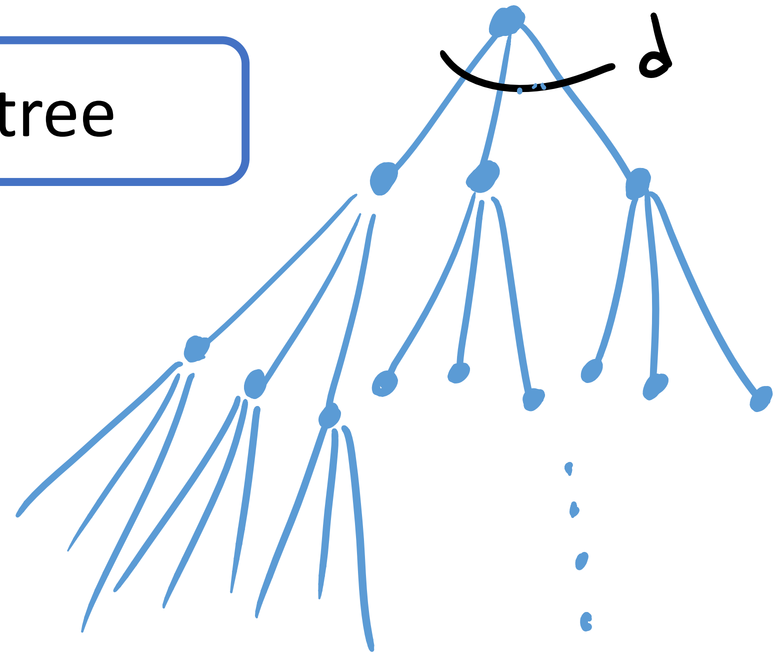
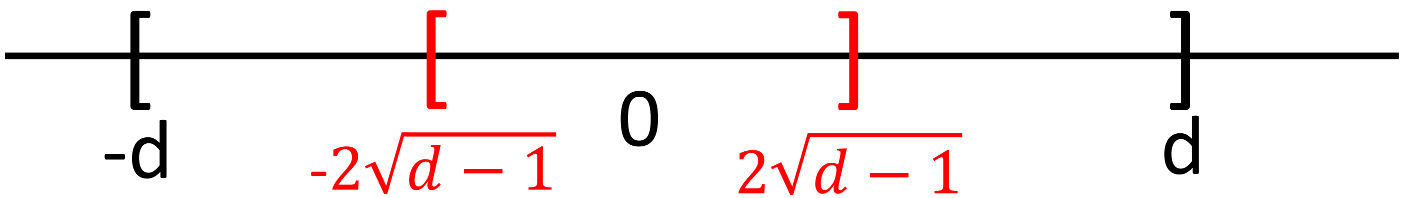


$$\lambda(A_T) = [-2\sqrt{d-1}, 2\sqrt{d-1}]$$

Alon-Boppana'86: This is the best possible spectral expander.

Ramanujan Graphs: $2\sqrt{d-1}$

Definition: G is **Ramanujan** if all non-trivial eigs have absolute value at most $2\sqrt{d-1}$



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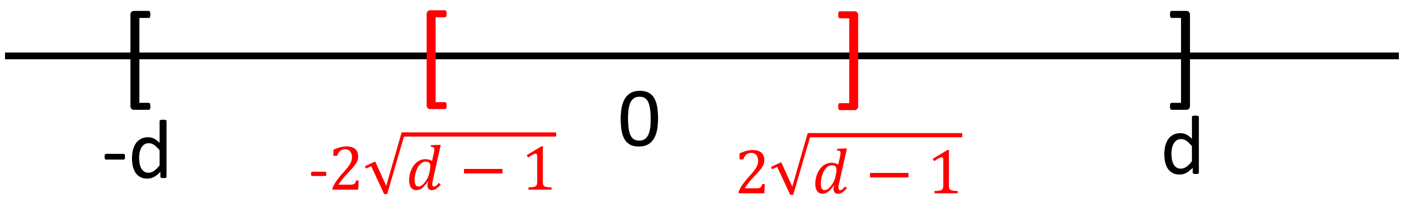

Friedman'08: A random d-regular graph is almost Ramanujan : $2\sqrt{d-1} + o(1)$

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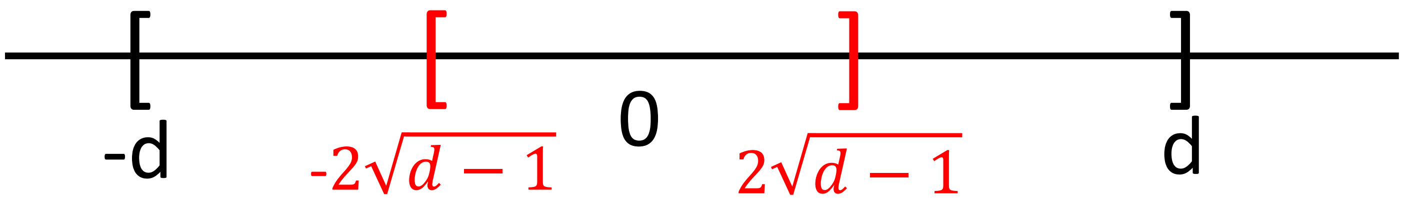

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Margulis, Lubotzky-Phillips-Sarnak'88: Infinite sequences of Ramanujan graphs exist for $d = p + 1$

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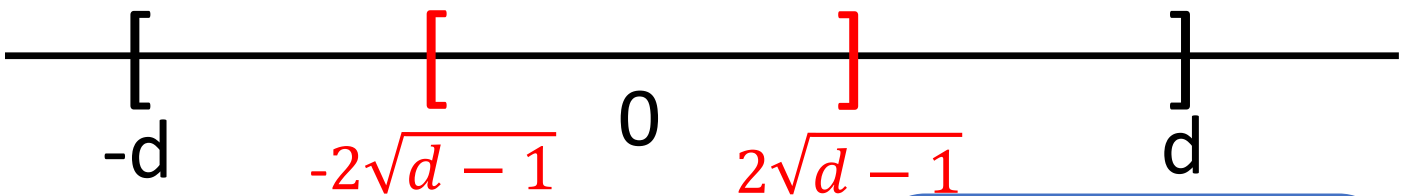

Friedman'08: A random d-regu st Ramanujan : $2\sqrt{d-1} + o(1)$

What about $d \neq p+1$?

Margulis, Lubotzky-Phillips-Sarnak'88: Infinit sequences of Ramanujan graphs exist for $d = p+1$

[Marcus-Spielman-S'13]

Theorem. Infinite families of bipartite Ramanujan graphs exist for every $d \geq 3$.

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Proof is elementary, doesn't use number theory.

Not explicit.

Based on a new existence argument: method of **interlacing families of polynomials**.

[Marcus-Spielman-S’13]

Theorem. Infin amilies of manujan graphs ex

$$\mathbb{E}p^{(k)} = \left(1 - \frac{1}{m}\frac{\partial}{\partial x}\right)^k x^n$$

Proof is elementary, doesn’t use number theory.

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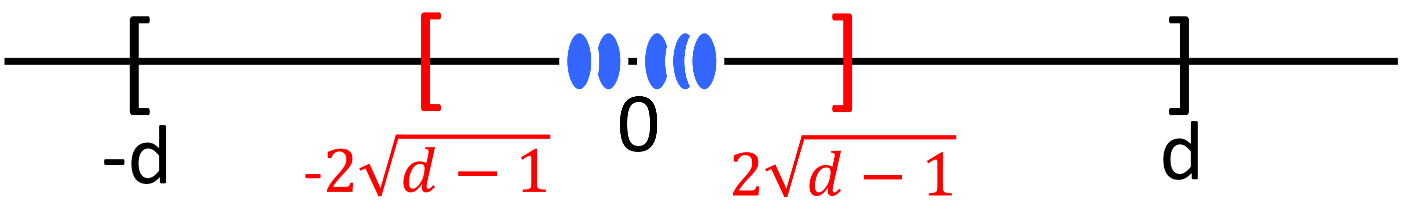
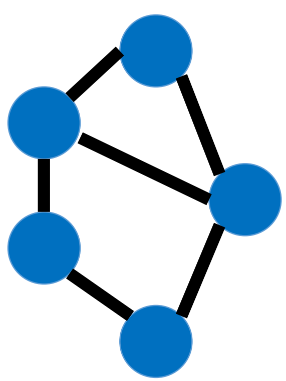
Based on a new existence argument: method of **interlacing families of polynomials**.

Bilu-Linial'06 Approach

Find an operation which doubles the size of a graph without blowing up its eigenvalues.





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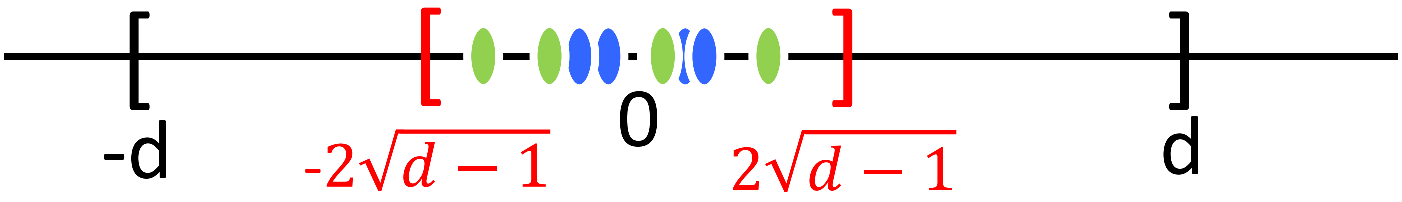

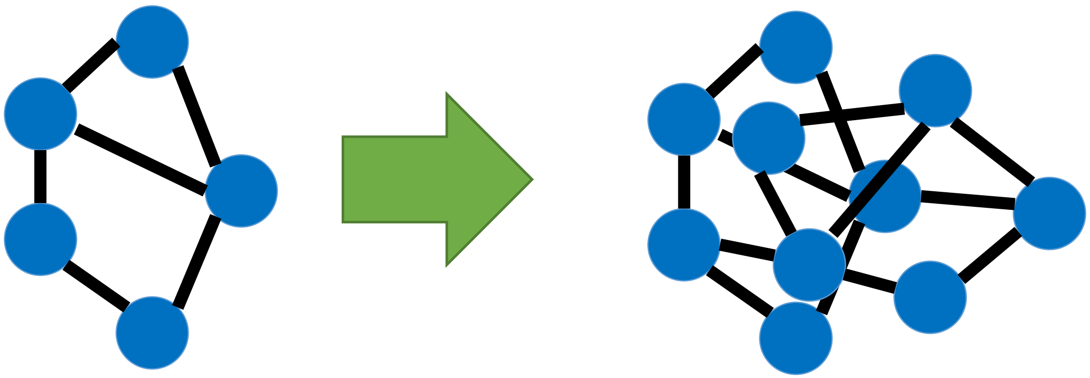

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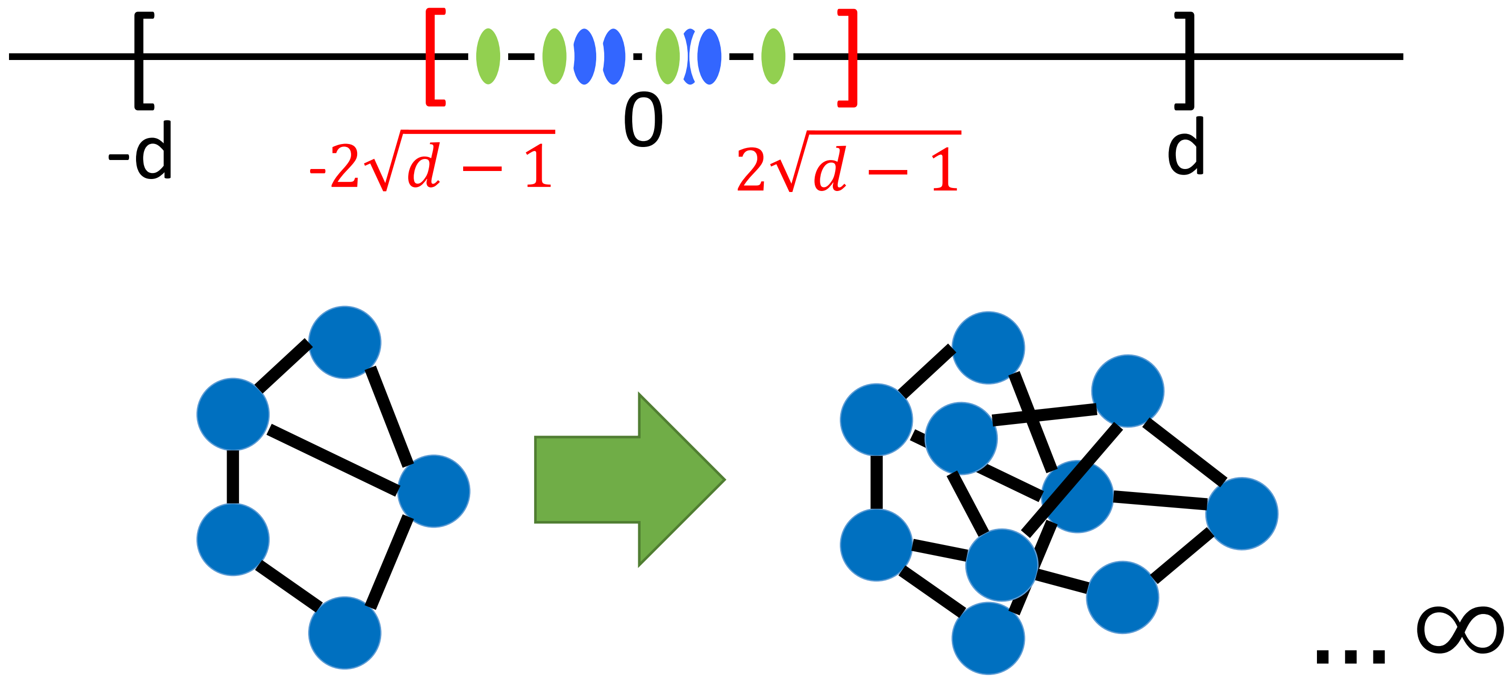
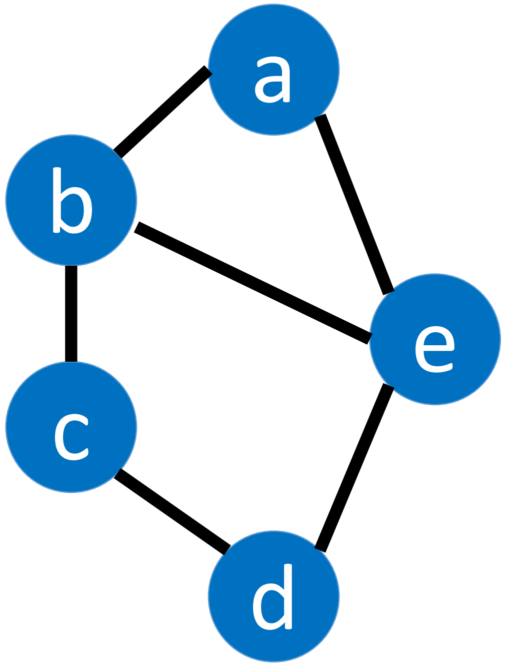


2-lifts of graphs



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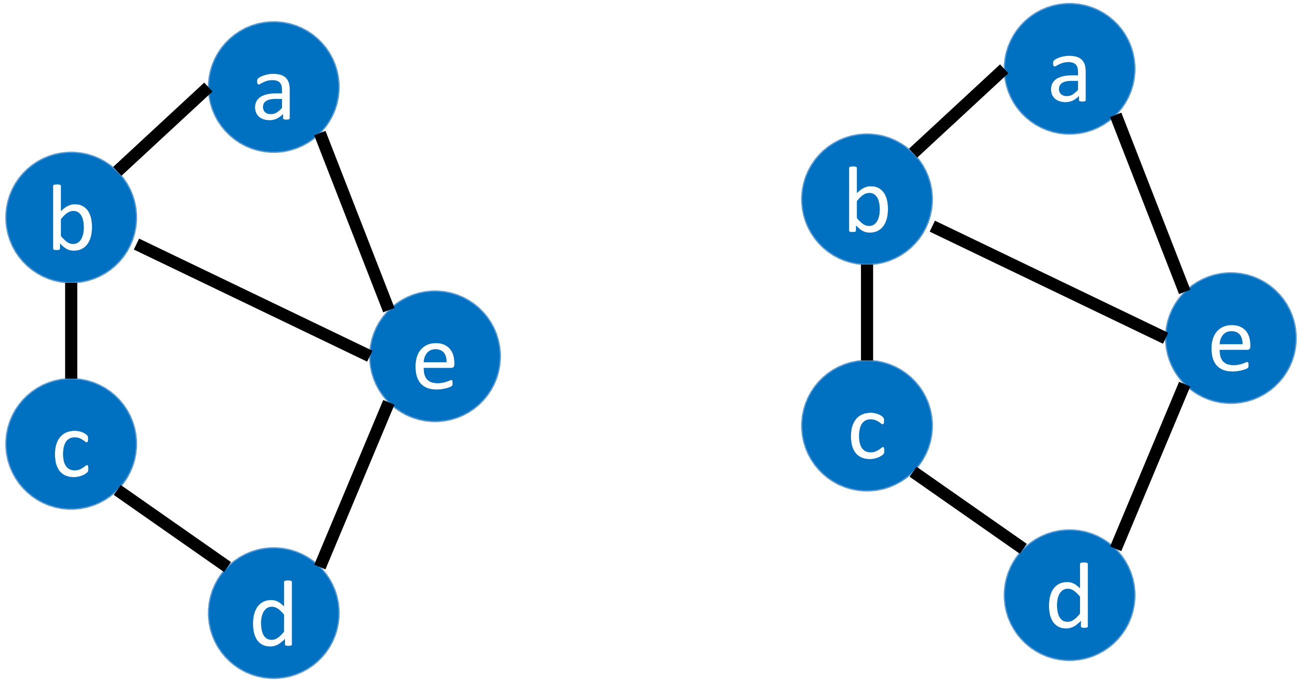
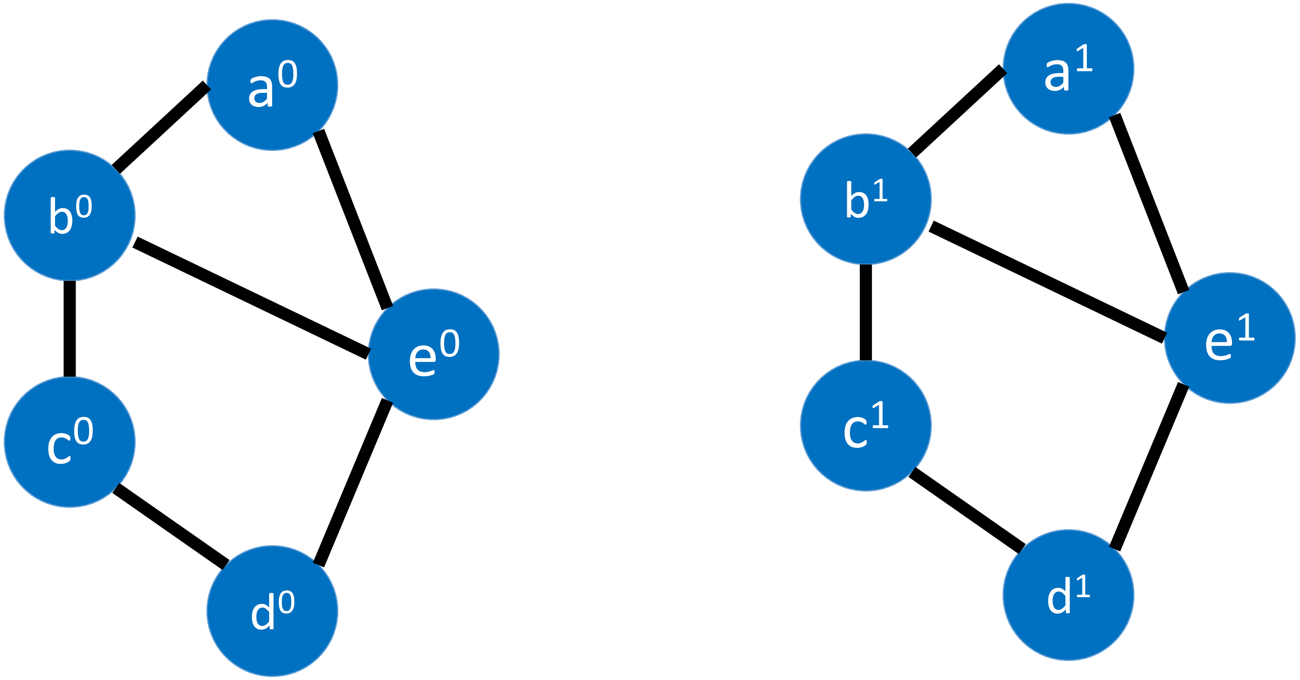


duplicate every vertex

2-lifts of graphs



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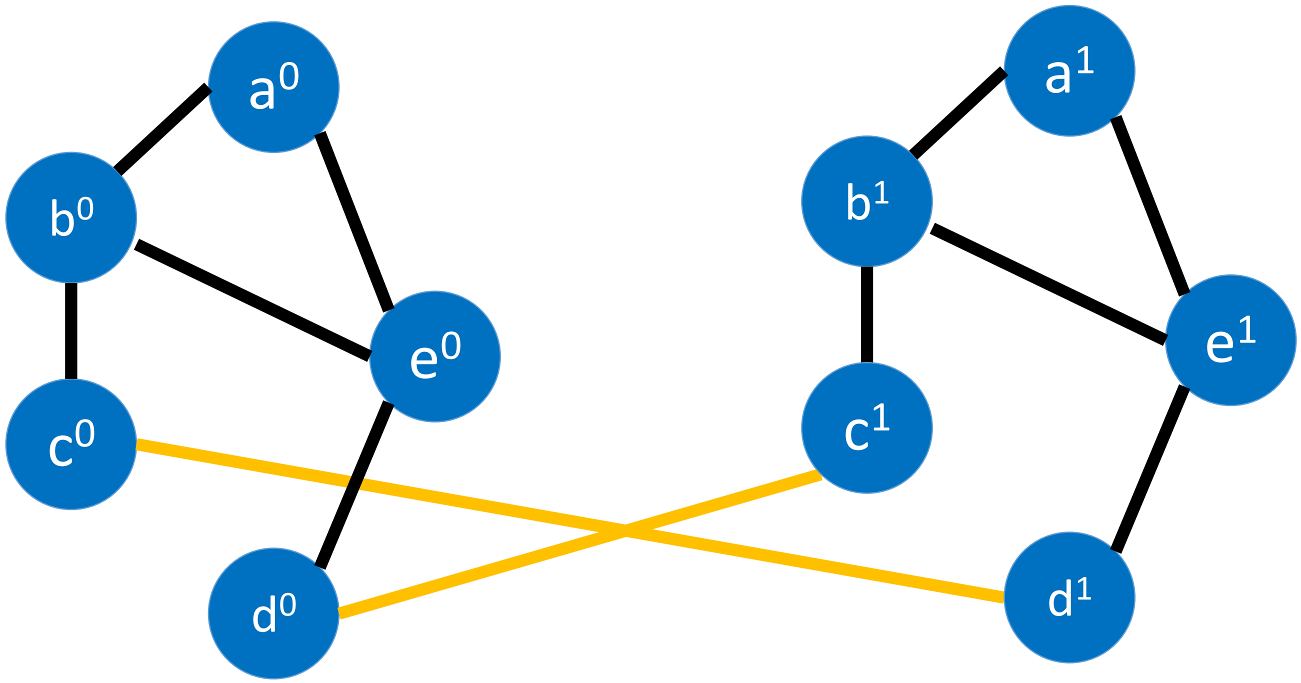
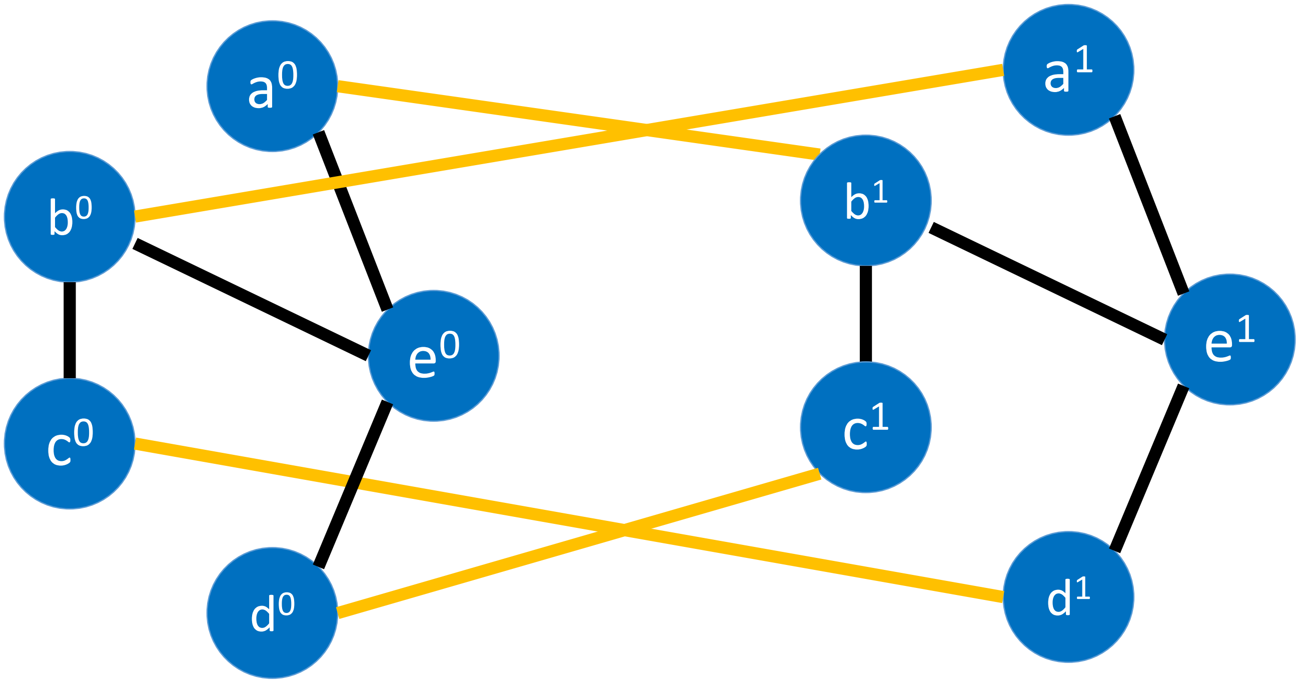


for every pair of edges:
leave on either side (parallel),
or make both cross

2-lifts of graphs



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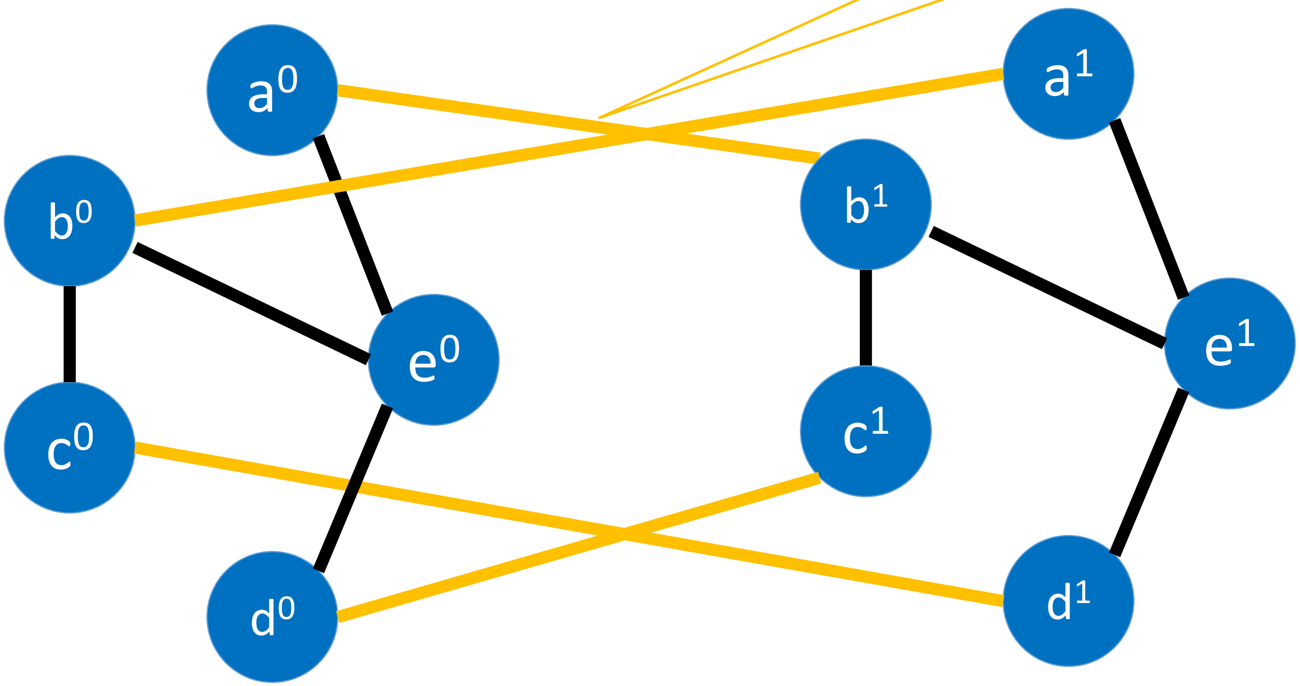
2-lifts of graphs

2^m possibilities



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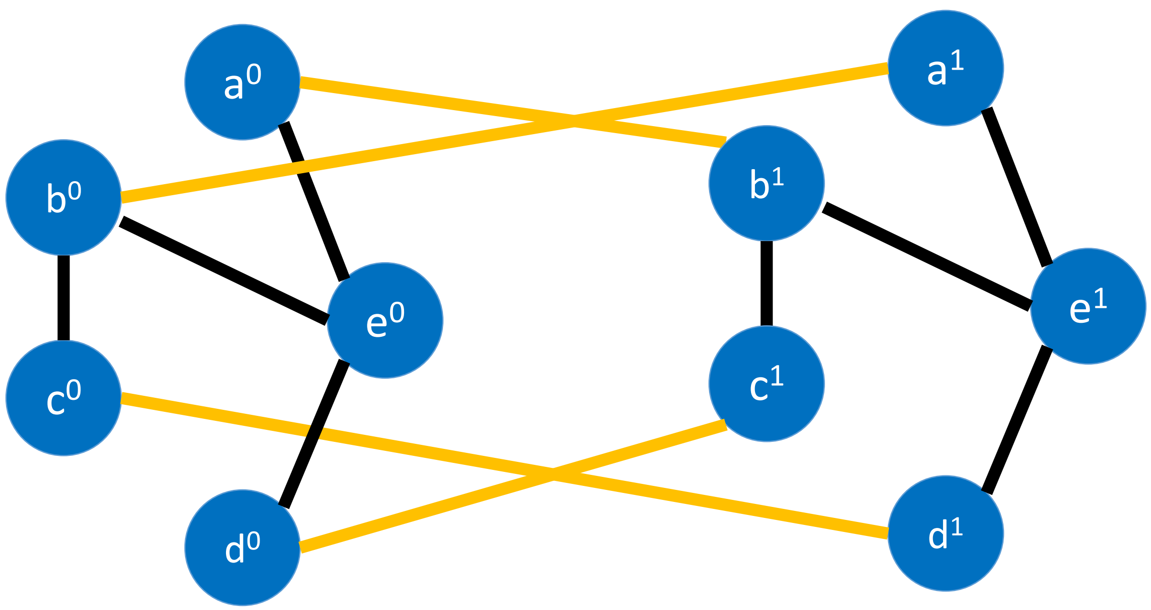
Eigenvalues of 2-lifts (Bilu-Linial)

Given a 2-lift of G,
create a signed adjacency matrix A_s
with a -1 for crossing edges
and a 1 for parallel edges



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Eigenvalues of 2-lifts (Bilu-Linial)

Theorem:

The eigenvalues of the 2-lift are:

$$\{\lambda_1, \ldots, \lambda_n\} = eigs(A)$$

$$\cup$$

$$\{\lambda'_1 \ldots \lambda'_n\} = eigs(A_s)$$

$$A_s = \begin{matrix} 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{matrix}$$

Eigenvalues of 2-lifts (Bilu-Linial)

Theorem:

The eigenvalues of the 2-lift are the union of the eigenvalues of A (old) and the eigenvalues of A_s (new)

Conjecture:

Every d-regular graph has a 2-lift in which all the new eigenvalues have absolute value at most $2\sqrt{d-1}$

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Every d-regular adjacency matrix A has a signing A_s with $||A_S|| \leq 2\sqrt{d-1}$

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Every d-regular adjacency matrix A has a signing A_s with $||A_S|| \leq 2\sqrt{d-1}$

Bilu-Linial'06: This is true with $O(\sqrt{d \log^3 d})$

Eigenvalues of 2-lifts (Bilu-Linial)

Conjecture:
Every d-regular adjacency matrix A has a signing A_s with $||A_S|| \leq 2\sqrt{d-1}$

We prove this in the bipartite case.

Eigenvalues of 2-lifts (Bilu-Linial)

Theorem:
Every d-regular adjacency matrix A
has a signing A_s with $\lambda_1(A_S) \leq 2\sqrt{d-1}$

Eigenvalues of 2-lifts (Bilu-Linial)

Theorem:
Every d-regular **bipartite** adjacency matrix *A* has a signing A_s with $||A_S|| \leq 2\sqrt{d-1}$

Trick: eigenvalues of bipartite graphs are symmetric about 0, so only need to bound largest

Random Signings

Idea 1: Choose $s \in \{-1,1\}^m$ randomly.

Random Signings

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Unfortunately,

$$\mathbb{E}\|A_s\| \gg 2\sqrt{d-1}$$

(Bilu-Linial showed $O(\sqrt{d \log^3 d})$ when

A is nearly Ramanujan)

Random Signings

Idea 2: Observe that $\lambda_1(A_s) = \lambda_{max}(\chi_{A_s})$

where $\chi_{A_s}(x) := \det(xI - A_s)$

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coefficient-wise average

Random Signings

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where $\chi_{A_s}(x) := \det(xI - A_s)$

Consider $\mathbb{E}_{s\in\{\pm 1\}^m}\chi_{A_s}(x)$

Usually useless, but **not here**!

$\{\chi_{A_s}\}_{s\in\{\pm 1\}^m}$ is an *interlacing family.*

$\exists s$ *such that* $\lambda_{max}(\chi_{A_s}) \leq \lambda_{max}(\mathbb{E}\chi_{A_s})$

3-Step Proof Strategy

1. Show that some poly does as well as the $\mathbb{E}$.

$$\exists s \text{ such that } \lambda_{max}(\chi_{A_s}) \leq \lambda_{max}(\mathbb{E}\chi_{A_s})$$

3-Step P

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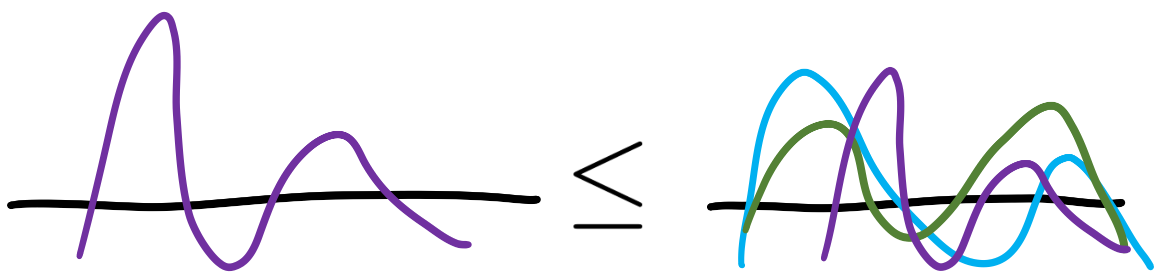

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$$\mathbb{E}\chi_{A_s}(x) = \mu_G(x)$$

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2. Calculate the expected polynomial.

$$\mathbb{E}\chi_{A_s}(x) = \mu_G(x)$$

3. Bound the largest root of the expected poly.

$$\lambda_{max}(\mu_G(x)) \leq 2\sqrt{d-1}$$

3-Step Proof Strategy

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Step 2: The expected polynomial

Theorem [Godsil-Gutman'81]

For any graph *G*,

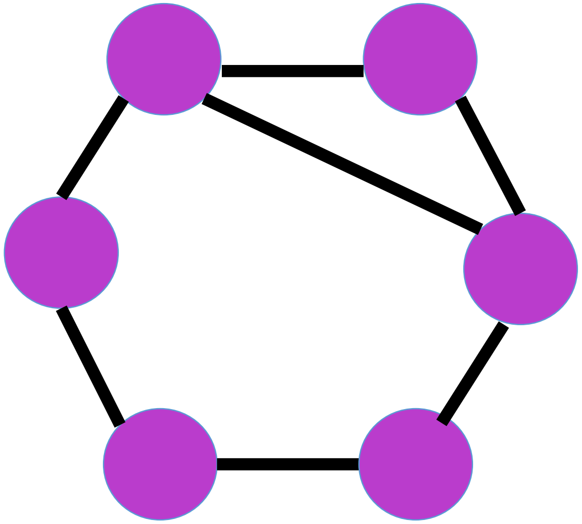
$$\mathop{\mathbb{E}}_{s}[\ \chi_{A_s}(x)\ ] = \mu_G(x)$$

the matching polynomial of G

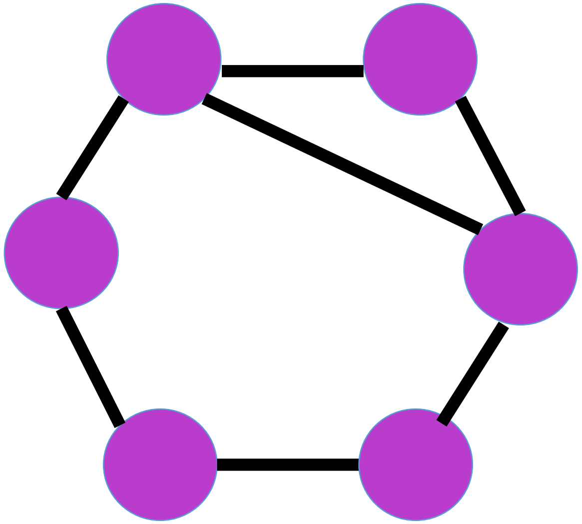
The matching polynomial (Heilmann-Lieb '72)

$$\mu_G(x) = \sum_{i \geq 0} x^{n-2i} (-1)^i m_i$$

m_i = the number of matchings with i edges



$$\mu_G(x) = x^6 - 7x^4 + 11x^2 - 2$$



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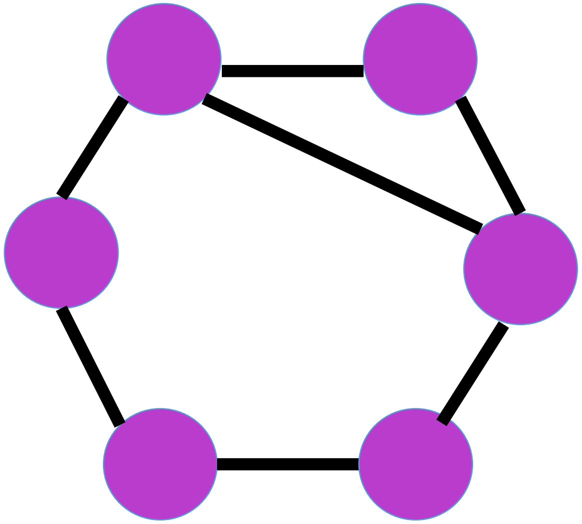
one matching with 0 edges



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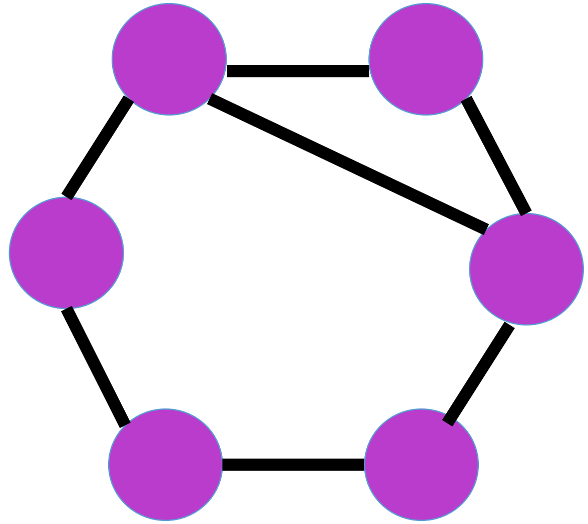

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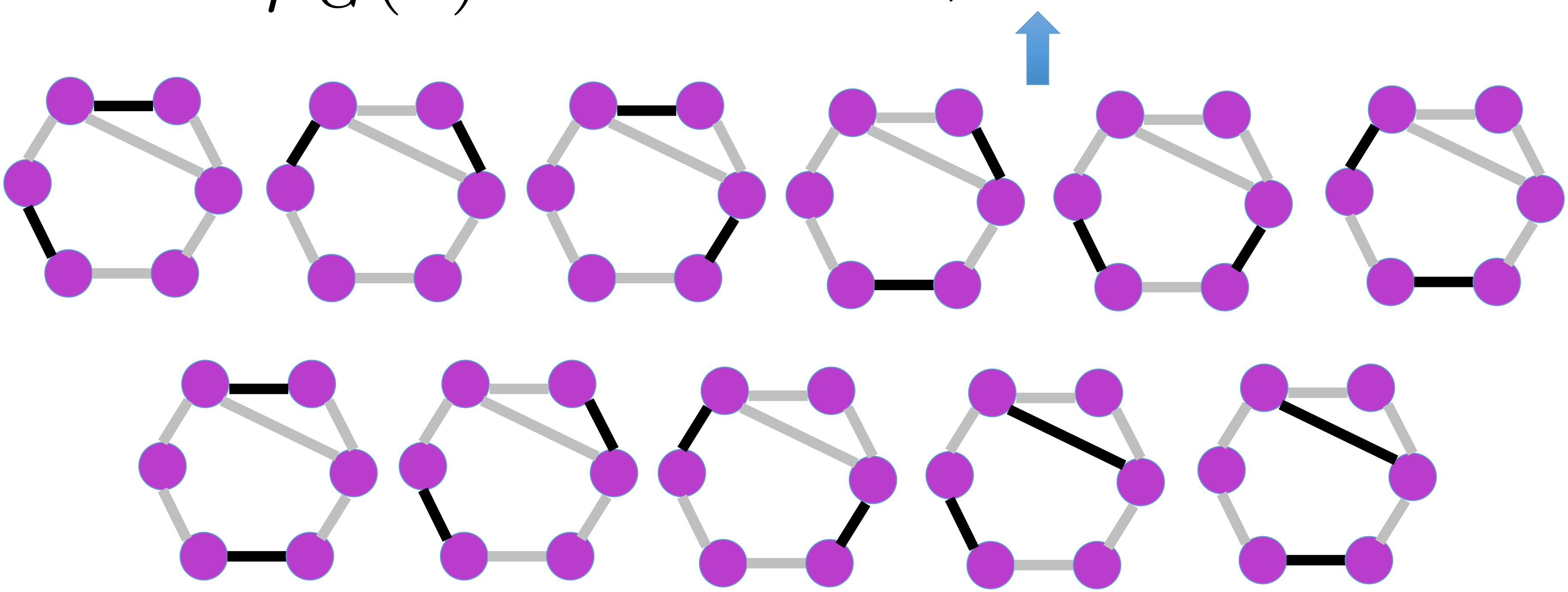

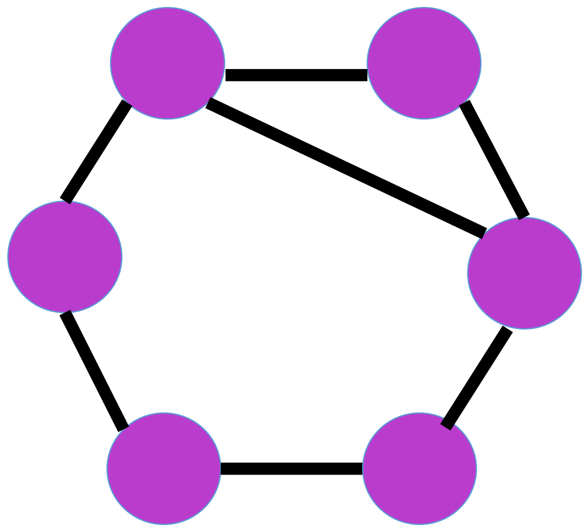

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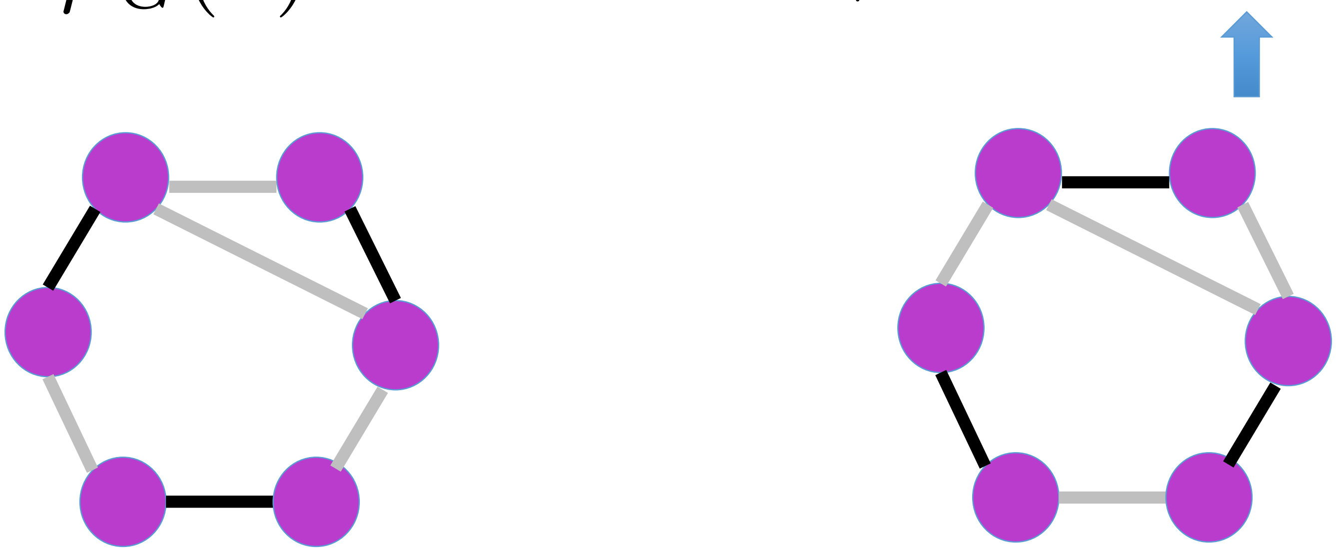

Proof that $\mathbb{E}_s[\ \chi_{A_s}(x)\ ] = \mu_G(x)$

Expand $\mathbb{E}_s[\ \det(xI - A_s)\ ]$ using permutations

x	±1	0	0	±1	±1
±1	x	±1	0	0	0
0	±1	x	±1	0	0
0	0	±1	x	±1	0
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1. Show that some poly does as well as the $\mathbb{E}$.

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$\mathbb{E}\chi_{A_s}(x) = \mu_G(x)$ **[Godsil-Gutman'81]**

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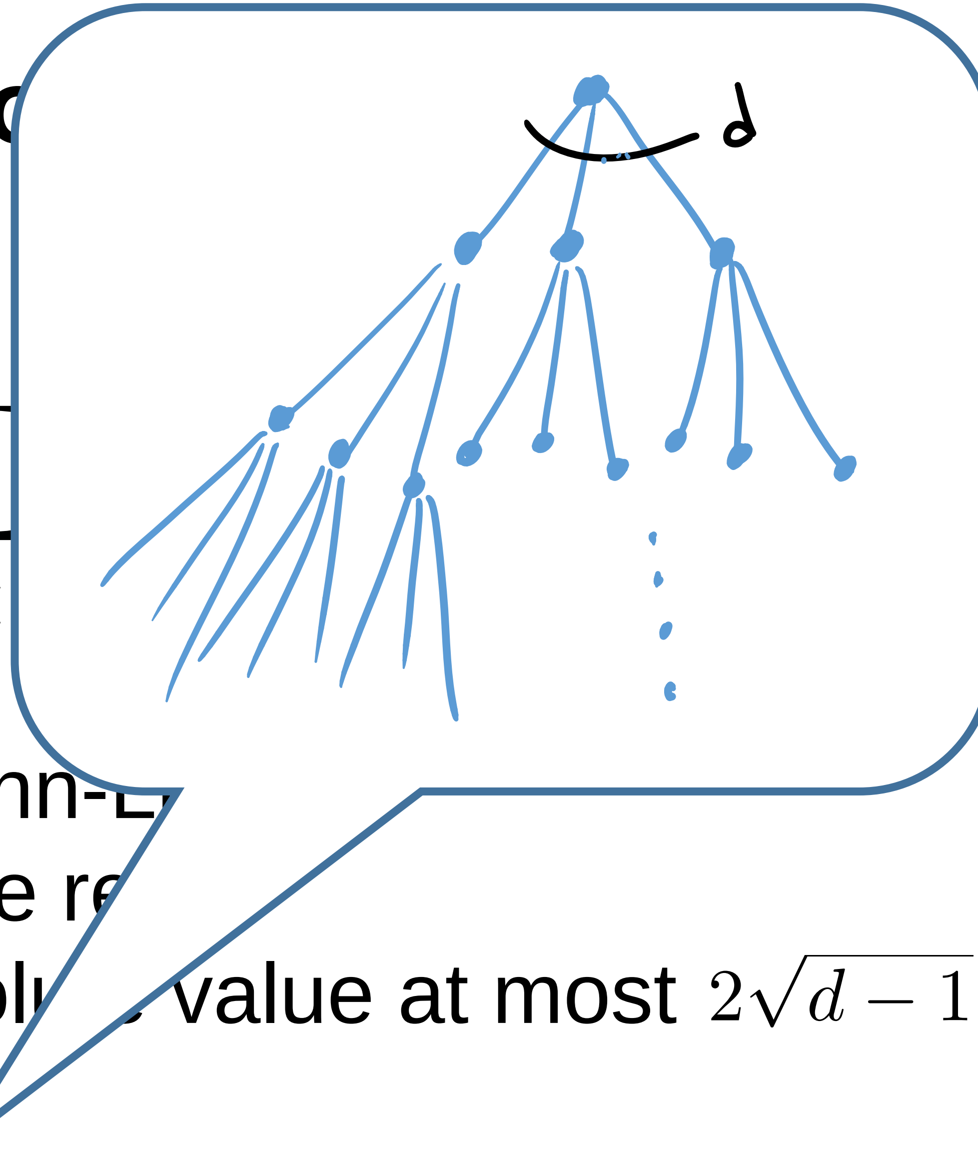

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The number $2\sqrt{d-1}$ comes by comparing
to an infinite d –ary tree [Godsil]. □

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Implied by:

" $\{\chi_{A_s}\}_{s\in\{\pm 1\}^m}$ is an interlacing family."

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Basic Question: Given p_0, p_1 when are the roots of the $p_i(x)$ related to roots of $\mathbb{E}_i p_i(x)$?

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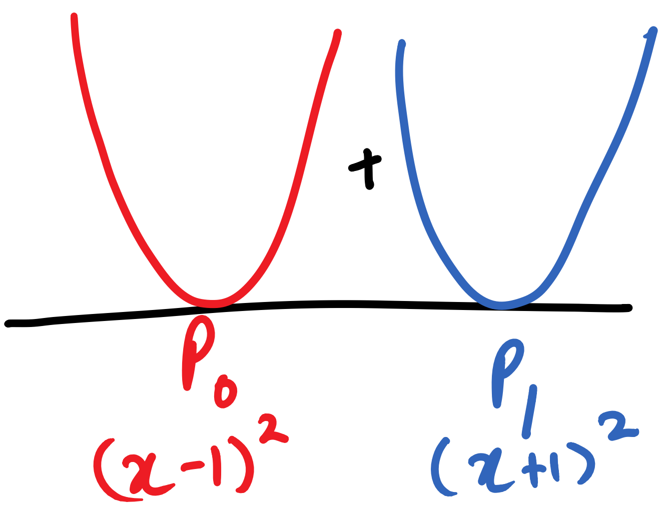

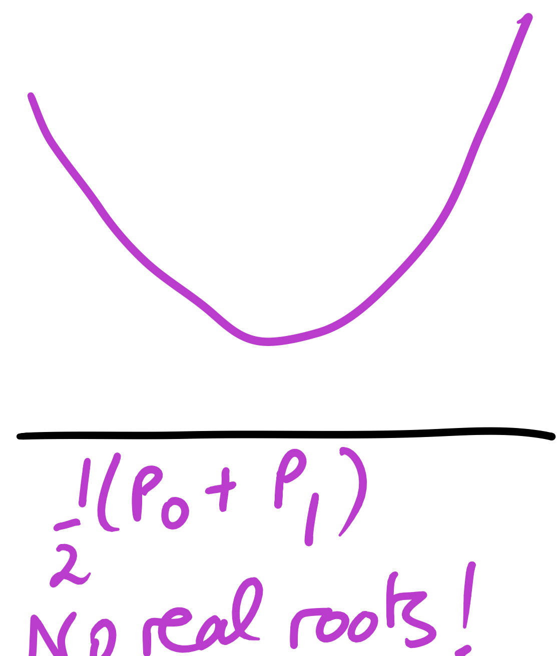

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Answer: Certainly not always…

$$\frac{1}{2}\times \quad p(x) = (x-1)(x-2) = x^2 - 3x + 2$$

$$\frac{1}{2}\times \quad q(x) = (x-3)(x-4) = x^2 - 7x + 12$$

$$\left(x - 2.5 + \sqrt{3}i\right)\left(x - 2.5 - \sqrt{3}i\right) = x^2 - 5x + 7$$

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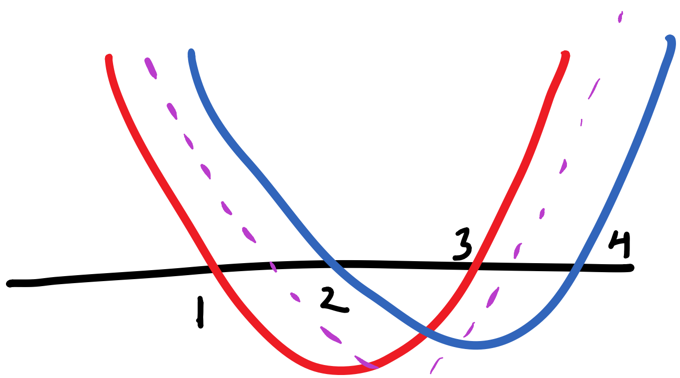
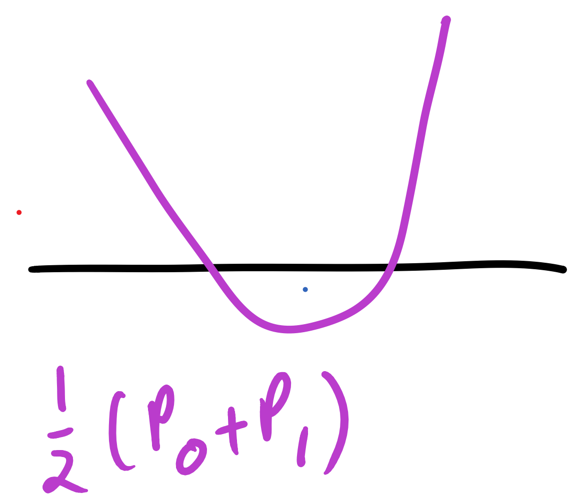
But sometimes it works:





A Sufficient Condition

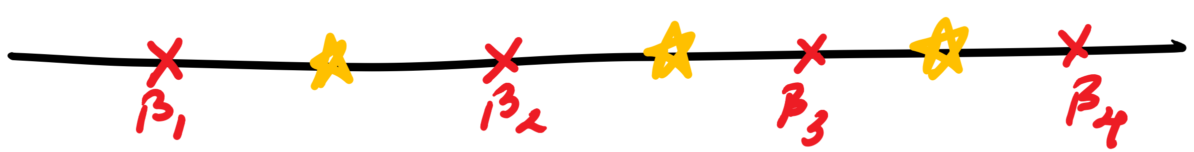
Basic Question: Given p_0, p_1 when are the roots of the $p_i(x)$ related to roots of $\mathbb{E}_i p_i(x)$?

Answer: When they have a *common interlacing*.

Definition. $q = \prod_{i=1}^{n-1}(x - \alpha_i)$ *interlaces*

$p = \prod_{i=1}^{n}(x - \beta_i)$ if

$$\beta_n \leq \alpha_{n-1} \leq \beta_{n-1} \ldots \leq \alpha_1 \leq \beta_1.$$



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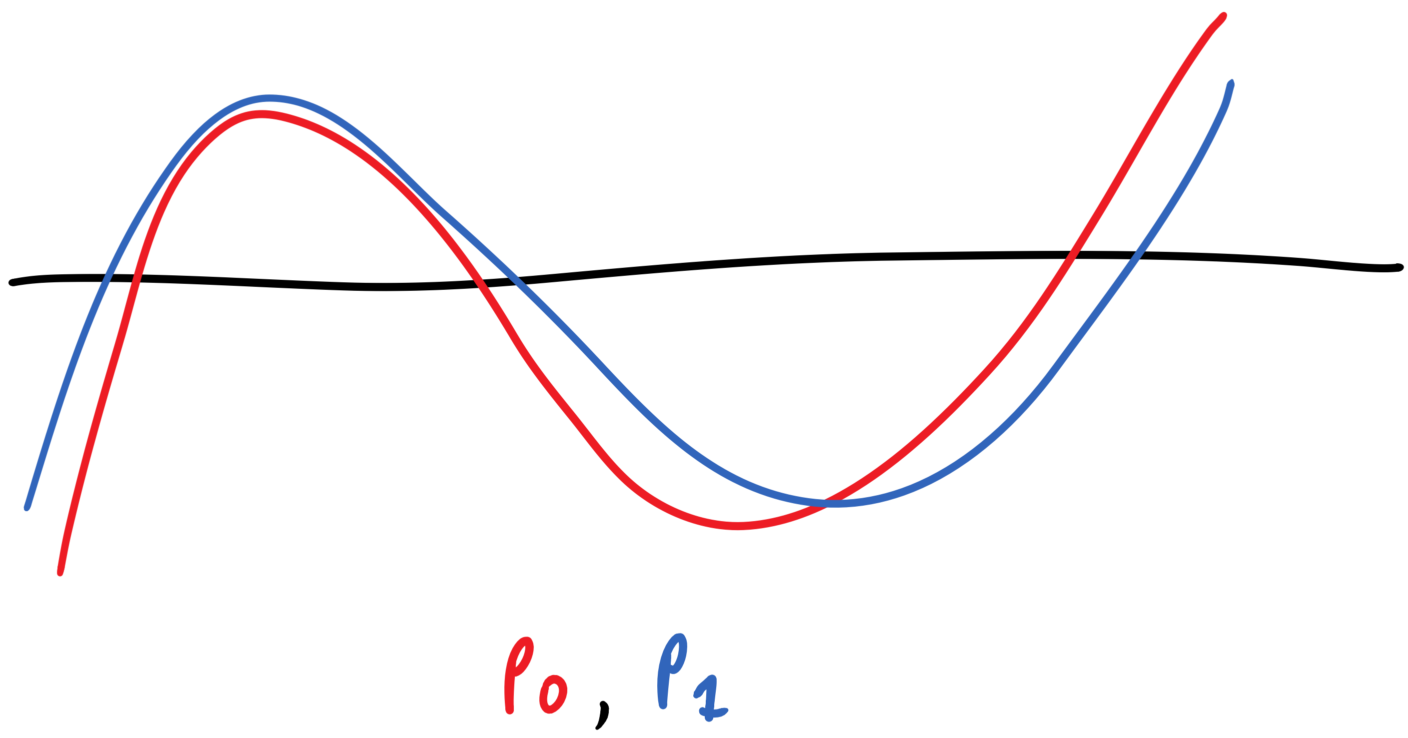

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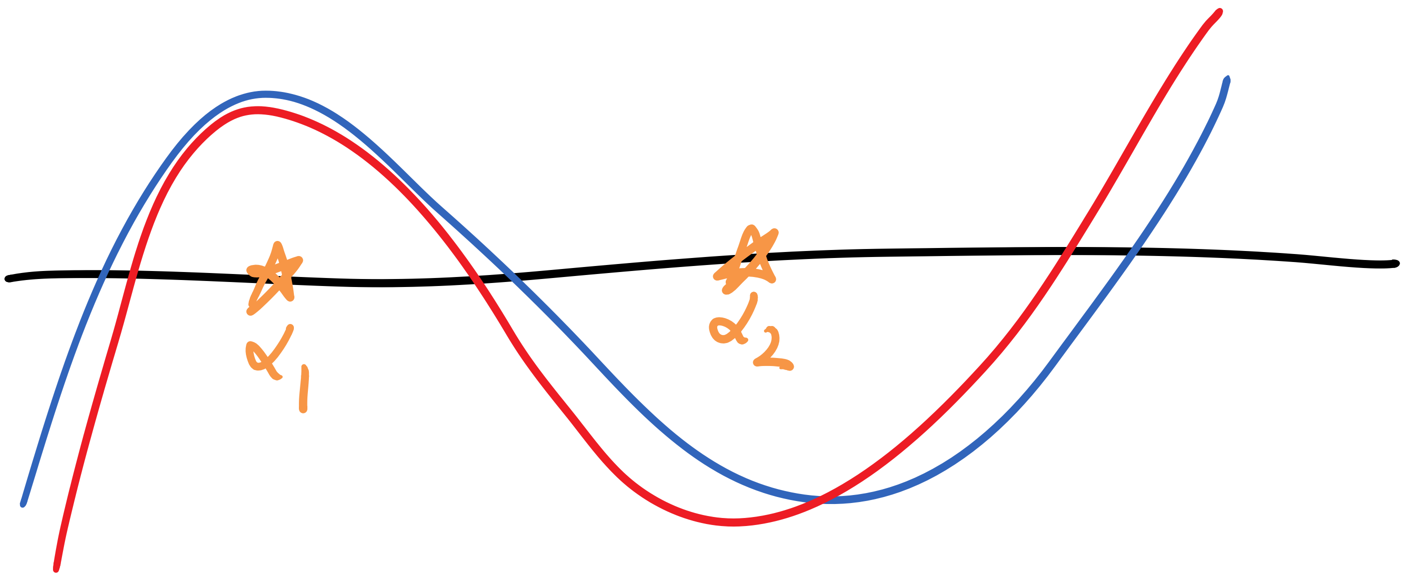

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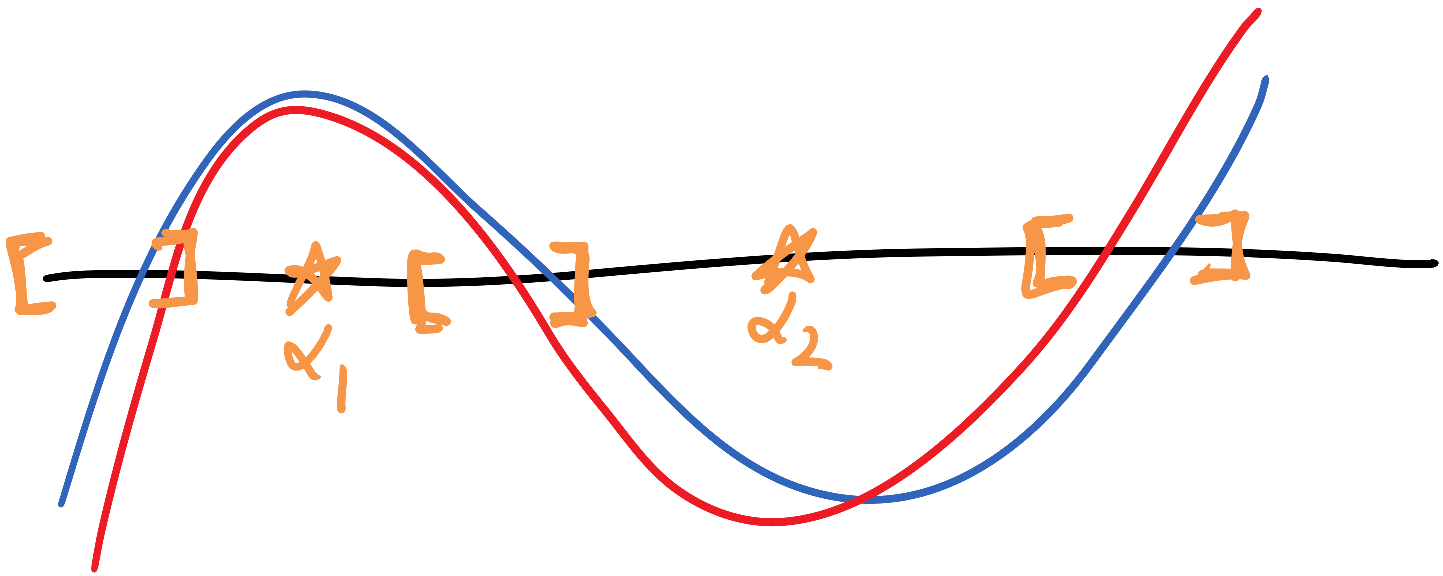

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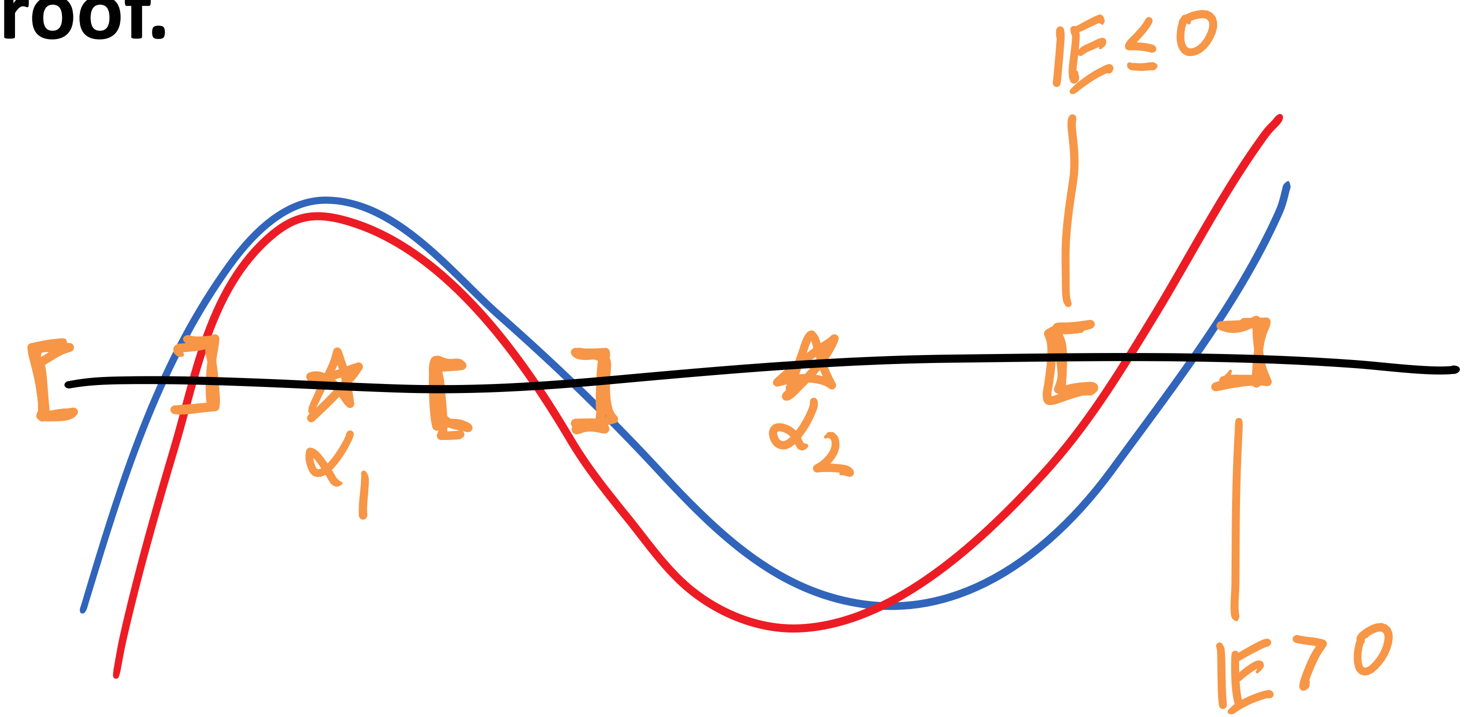

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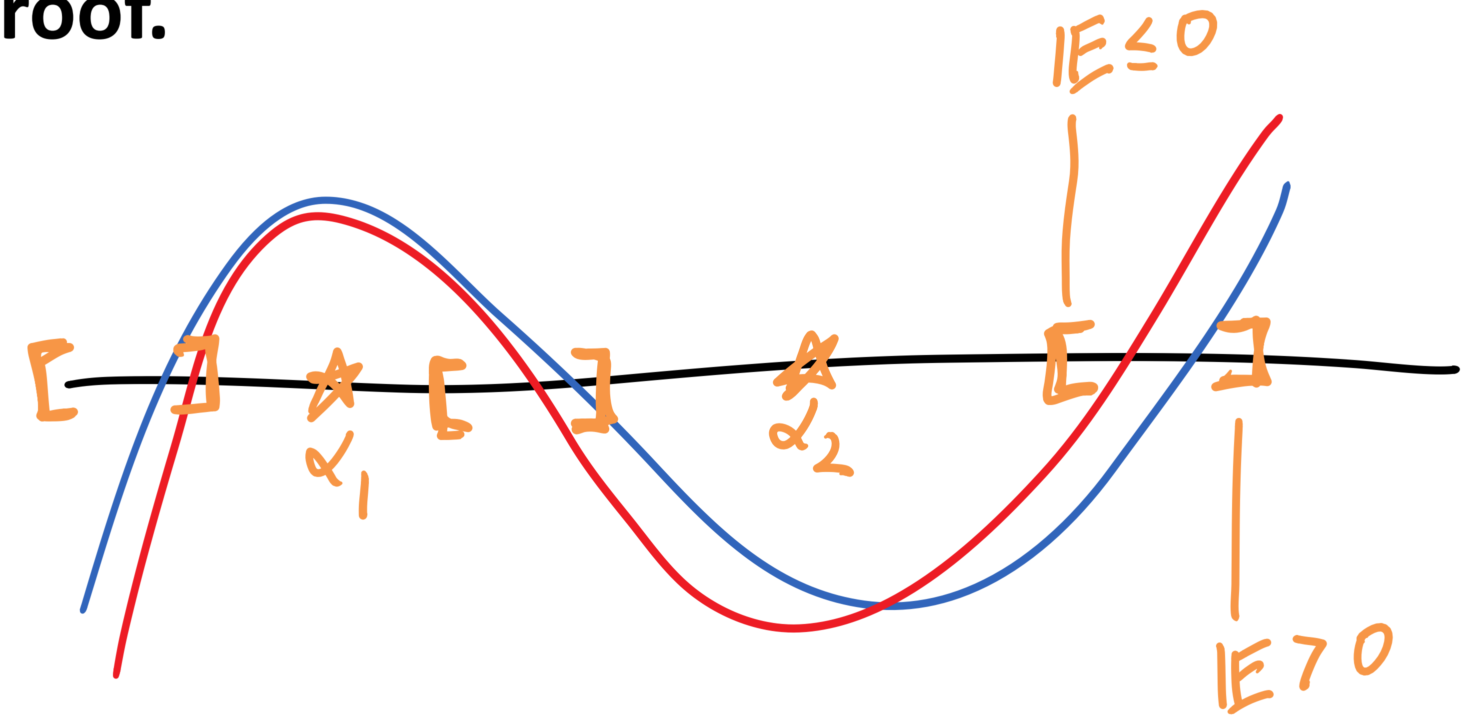

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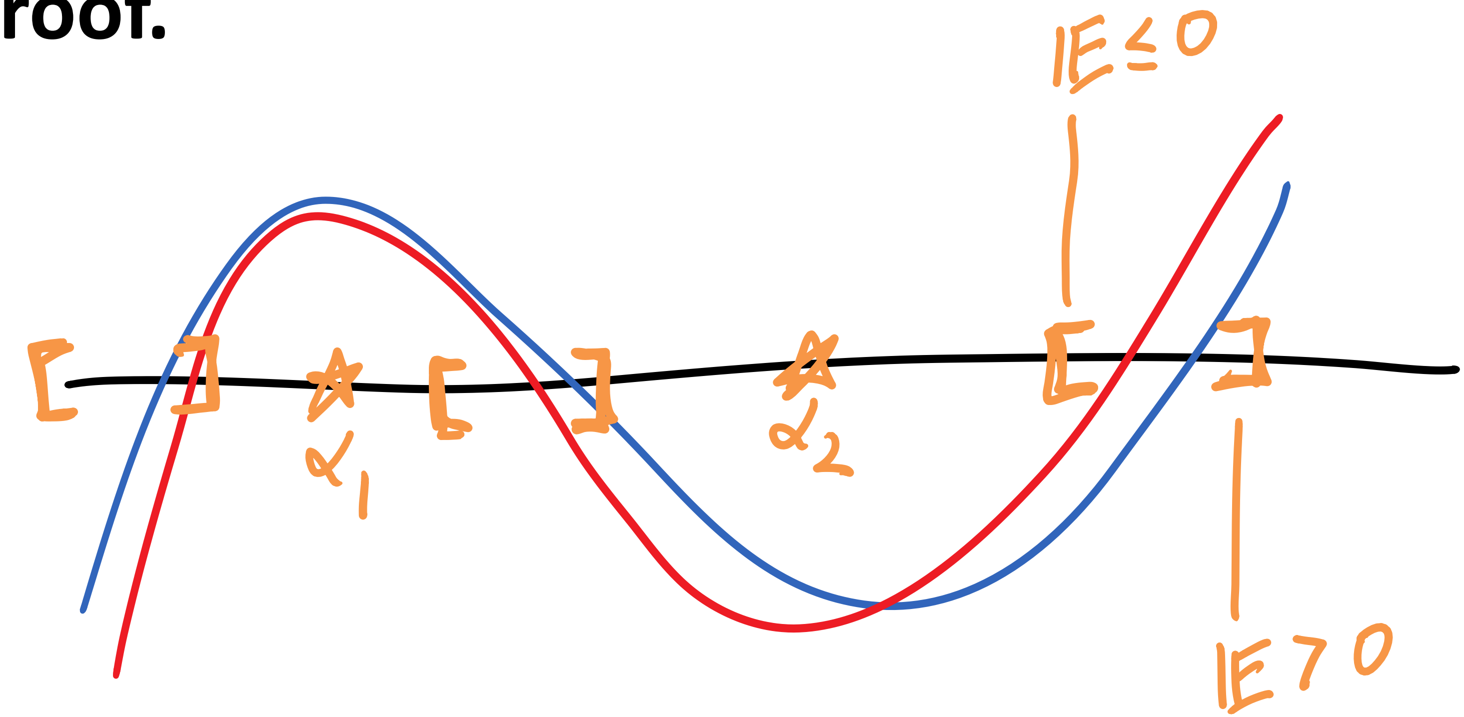

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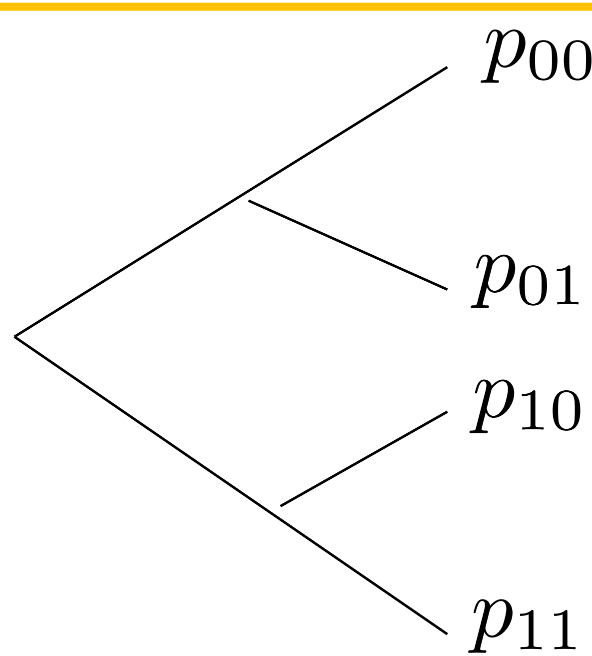
Interlacing Family of Polynomials

Definition: $\{p_s\}_{s \in \{0,1\}^m}$ is an interlacing family

if can be placed on the leaves of a tree so that when every node is the sum of leaves below & sets of siblings have common interlacings



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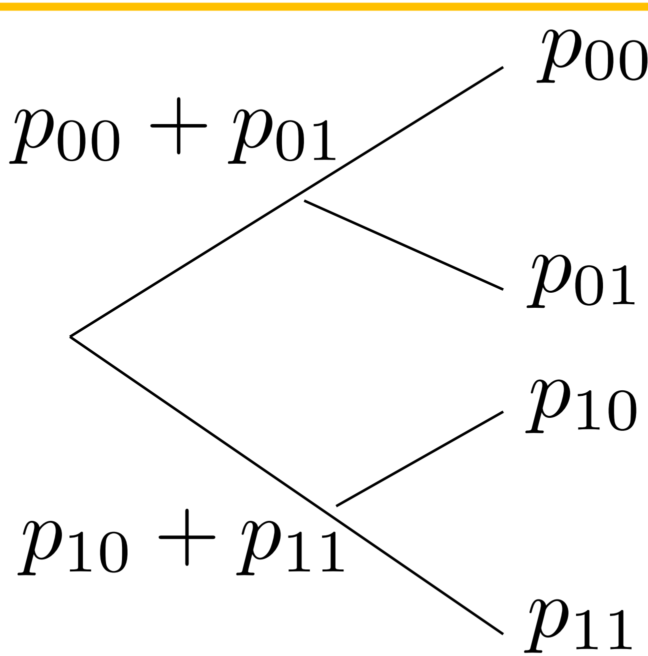

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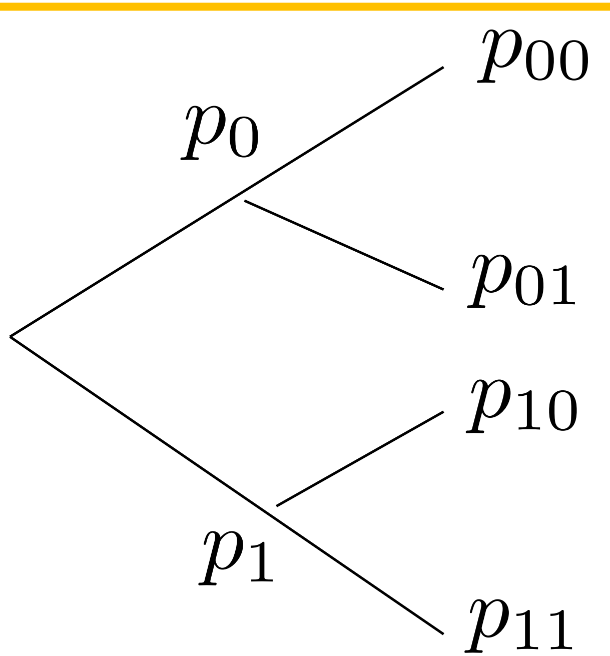

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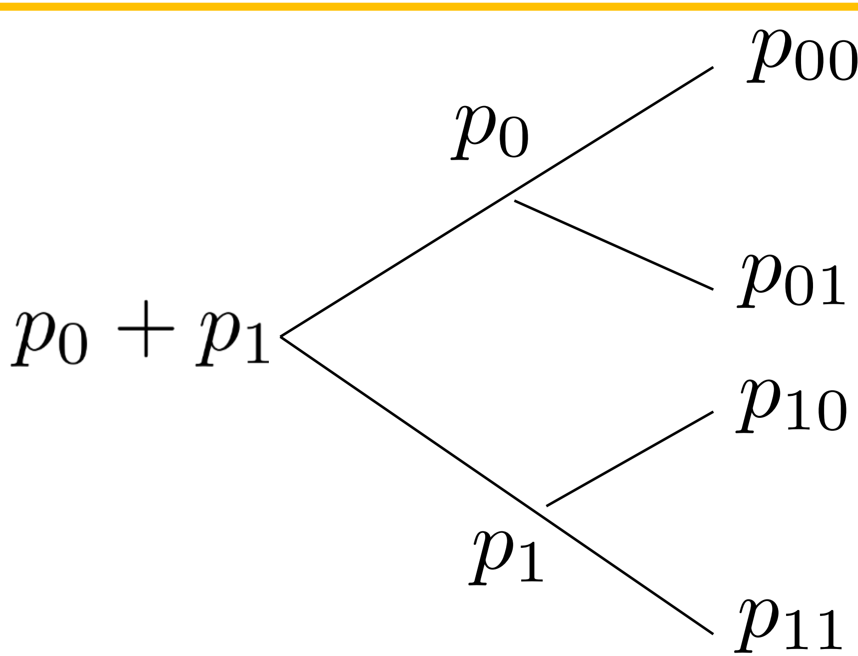

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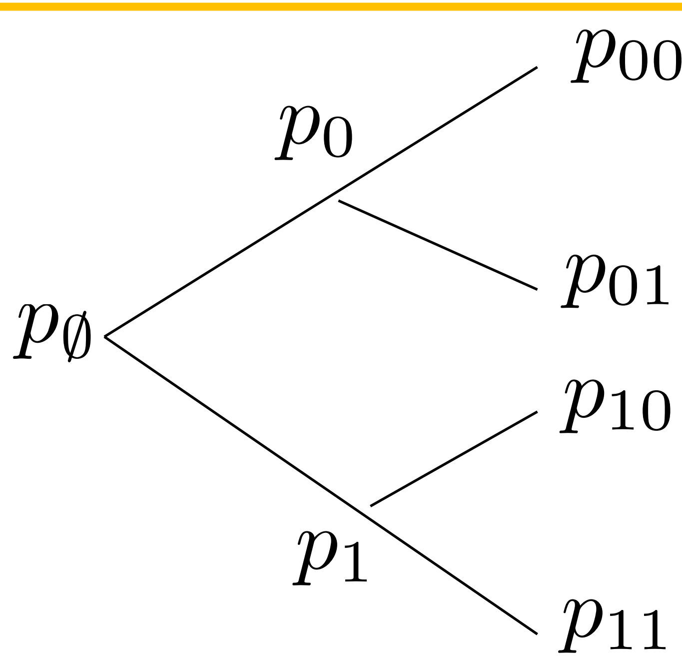

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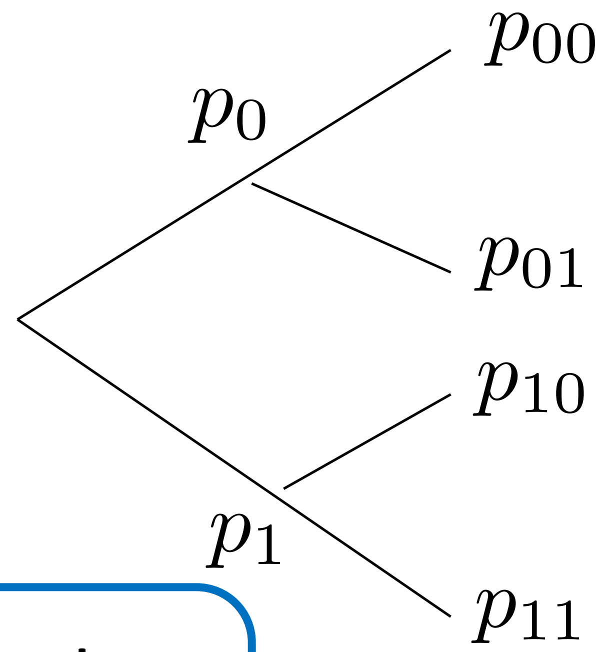

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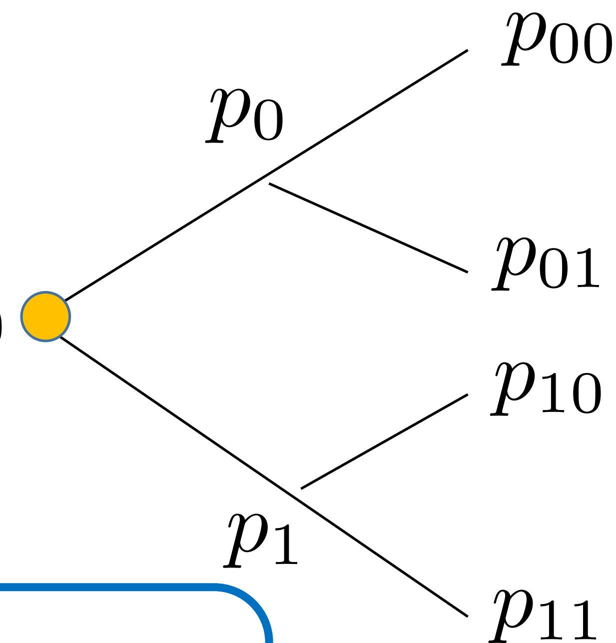

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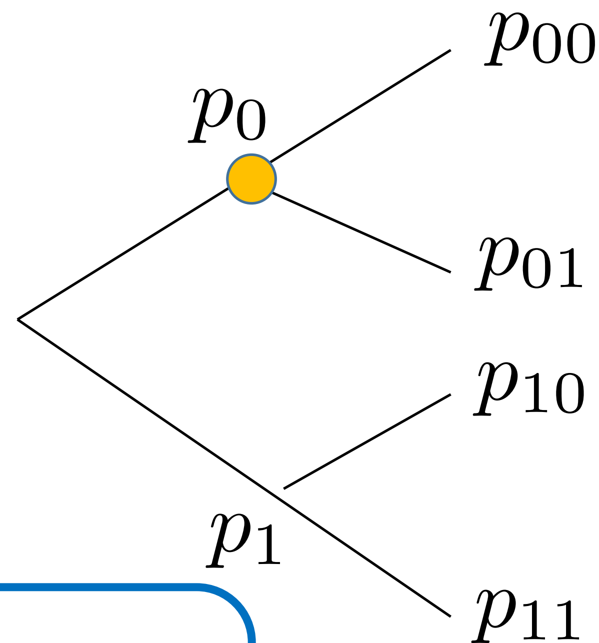

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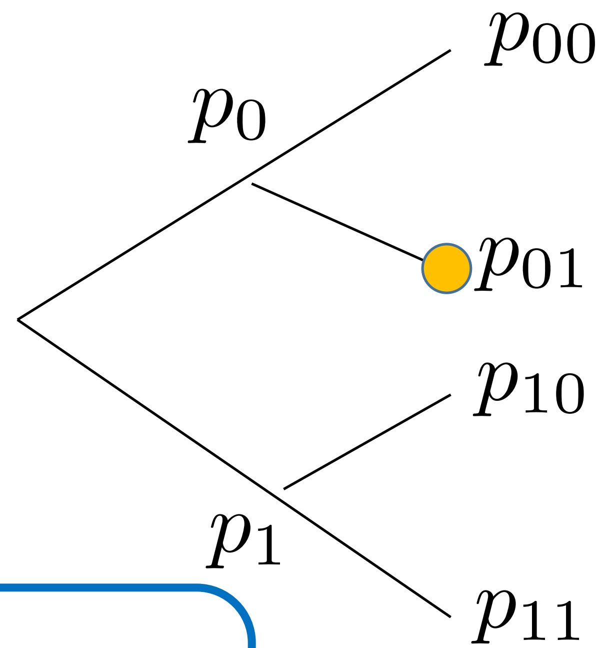

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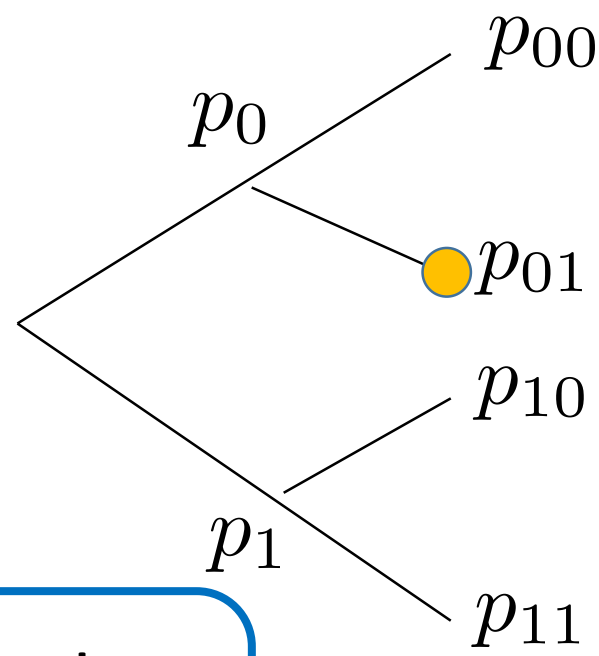

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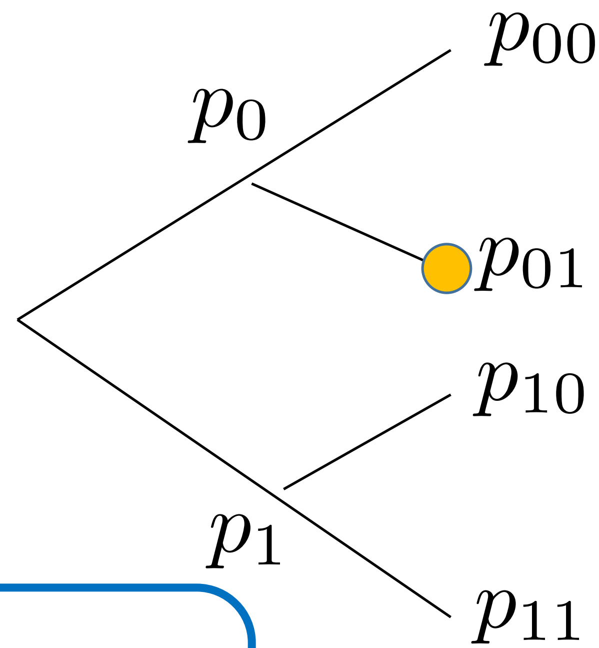

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An interlacing family

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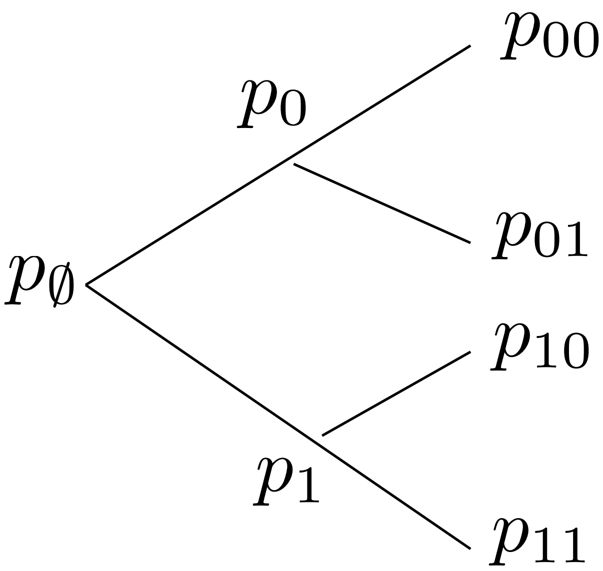
Let $p_s(x) = \chi_{A_s}(x)$

$\{p_s\}_{s \in \{\pm 1\}^m}$ is an interlacing family

To prove interlacing family

Let $p_{s_1,...,s_k}(x) = \mathbb{E}_{s_{k+1},...,s_m}\left[\, p_{s_1,...,s_m}(x) \,\right]$

Leaves of tree = signings $s_1, \ldots, s_m$
Internal nodes = partial signings $s_1, \ldots, s_k$



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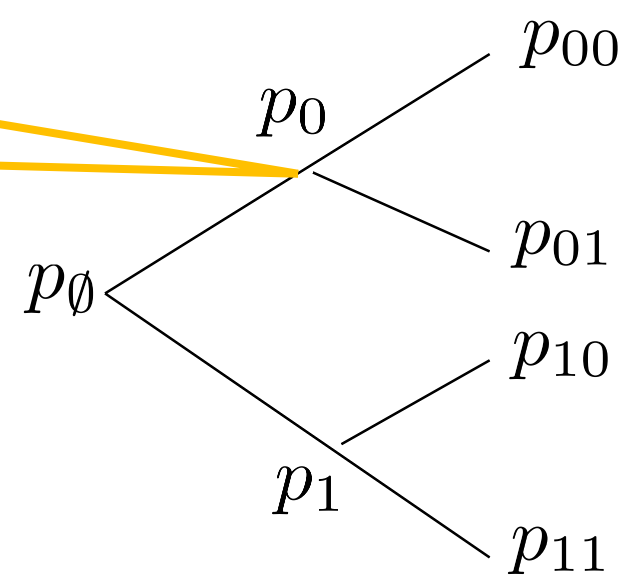
Leaves of tree = signings $s_1, \ldots, s_m$
Internal nodes = partial signings $s_1, \ldots, s_k$

Need to find common interlacing for every internal node



How to Prove Common Interlacing

Lemma (Fisk'08, folklore): Suppose $p(x)$ and $q(x)$ are monic and real-rooted. Then:

$\exists$ a common interlacing r of p and q

$\Updownarrow$

$\forall$ convex combinations,

$$\alpha p + (1 - \alpha) q$$

has real roots.

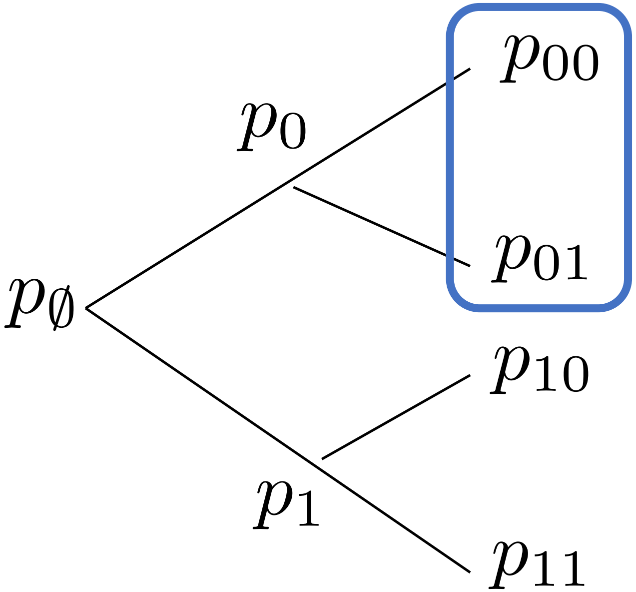
To prove interlacing family

Let $p_{s_1,...,s_k}(x) = \mathop{\mathbb{E}}_{s_{k+1},...,s_m}[\, p_{s_1,...,s_m}(x) \,]$

Need to prove that for all $s_1, \ldots, s_k$, $\lambda \in [0,1]$

$$\lambda p_{s_1,...,s_k,1}(x) + (1-\lambda) p_{s_1,...,s_k,-1}(x)$$

is real rooted



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$s_1, \ldots, s_k$ *are fixed*

s_{k+1} *is 1 with probability λ, -1 with $1-\lambda$*

$s_{k+2}, \ldots, s_m$ *are uniformly* ± 1

Generalization of Heilmann-Lieb

Suffices to prove that

$$\mathop{\mathbb{E}}_{s \in \{\pm 1\}^m} [\ p_s(x)\ ] \quad \text{is real rooted}$$

for **every** product distribution on the entries of *s*

Generalization of Heilmann-Lieb

Suffices to prove that

$$\mathop{\mathbb{E}}_{s\in\{\pm 1\}^m} [\ p_s(x)\ ] \quad \text{is real rooted}$$

for **every** product distribution on the entries of *s:*

$$\sum_{s\in\{\pm 1\}^m} p_s(x) \prod_{i:s_i=1} \lambda_i \prod_{i:s_i=-1} (1-\lambda_i)$$

$$\lambda_1, \ldots, \lambda_m \in [0,1]$$

Transformation to PSD Matrices

Suffices to show real rootedness of

$$\mathbb{E}_{s\in\{\pm 1\}^m} p_s(x-d) = \mathbb{E}_{s\in\{\pm 1\}^m} \det(xI - (dI - A_s))$$

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$$A_s = \sum_{ij\in E} s_{ij}(\delta_i \delta_j^T + \delta_j \delta_i^T)$$

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$$dI - A_s = \sum_{s_{ij}=1} (\delta_i - \delta_j)(\delta_i - \delta_j)^T + \sum_{s_{ij}=-1} (\delta_i + \delta_j)(\delta_i + \delta_j)^T$$

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$$\mathbb{E}_s \det\big(xI - (dI - A_s)\big) = \mathbb{E}\det\left(xI - \sum_{ij\in E} v_{ij} v_{ij}^T\right)$$

where $v_{ij} = \begin{cases} (\delta_i - \delta_j) \text{ with probability } \lambda_{ij} \\ (\delta_i + \delta_j) \text{ with probability } (1-\lambda_{ij}) \end{cases}$

Master Real-Rootedness Theorem

Given *any* independent random vectors $v_1, \ldots, v_m \in \mathbb{R}^d$, their expected characteristic polymomial

$$\mathbb{E}\det\left(xI - \sum_i v_i v_i^T\right)$$

has real roots.

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has real roots.

How to prove this?

The Multivariate Method

A. Sokal, 90's-2005:

> "…it is often useful to consider the multivariate polynomial … even if one is ultimately interested in a particular one-variable specialization"

Borcea-Branden 2007+: prove that univariate polynomials are real-rooted by showing that they are nice transformations of *real-rooted multivariate polynomials*.

Real Stable Polynomials

Definition. $p \in \mathbb{R}[x_1, \dots, x_n]$ is *real stable* if every univariate restriction in the strictly positive orthant:

$$p(t) := f(\vec{x} + t\vec{y}) \qquad \vec{y} > 0$$

is real-rooted.

If it has real coefficients, it is called *real stable.*

Real Stable Polynomials

Definition. $p \in \mathbb{C}[x_1, \ldots, x_n]$ is *real stable* if every univariate restriction in the strictly positive orthant:

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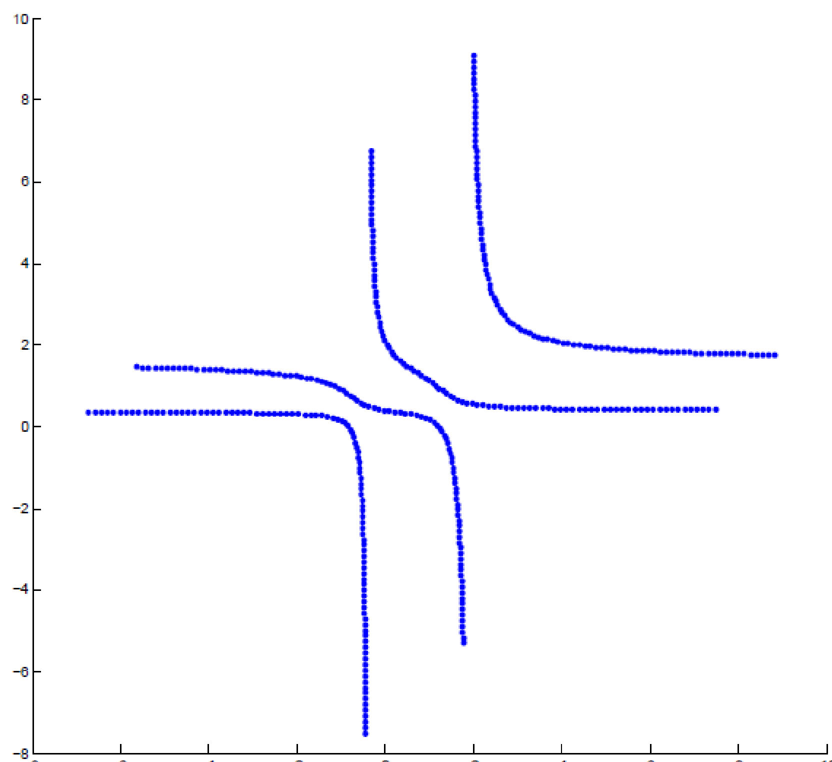

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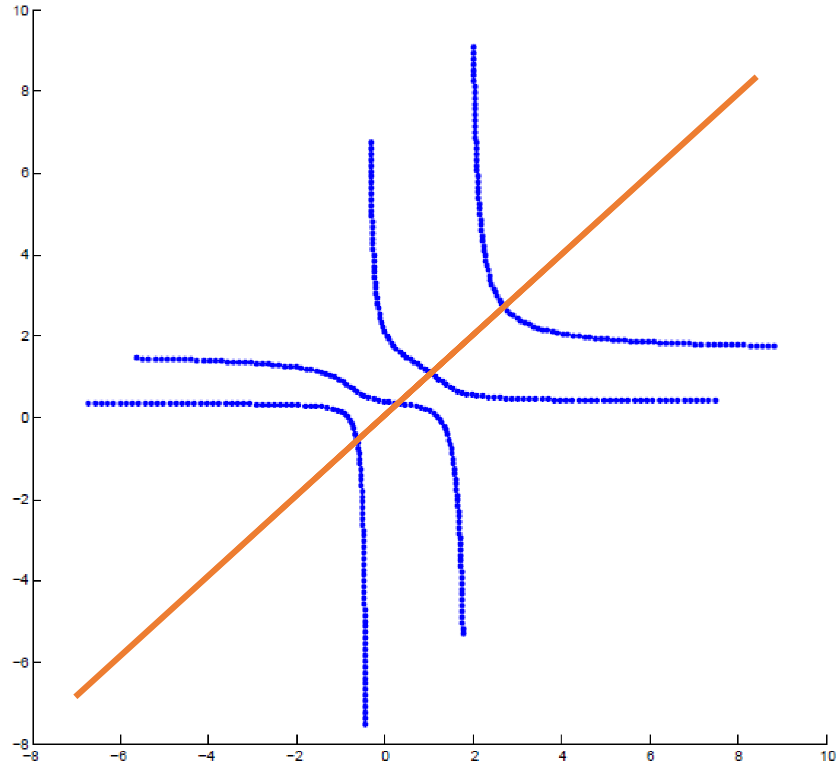

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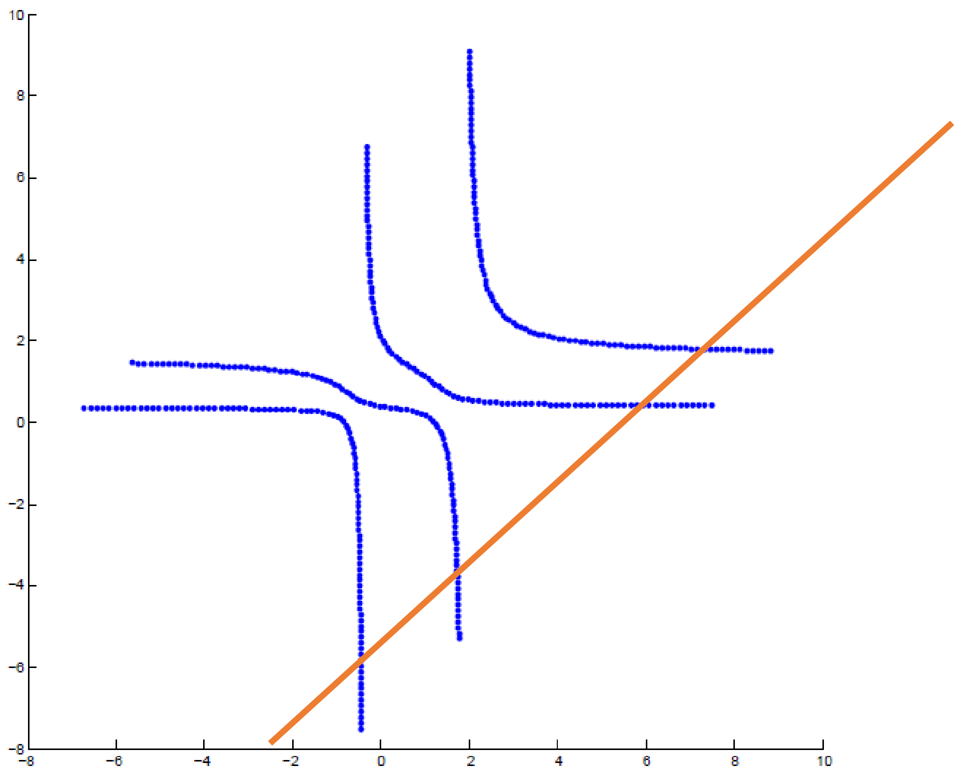

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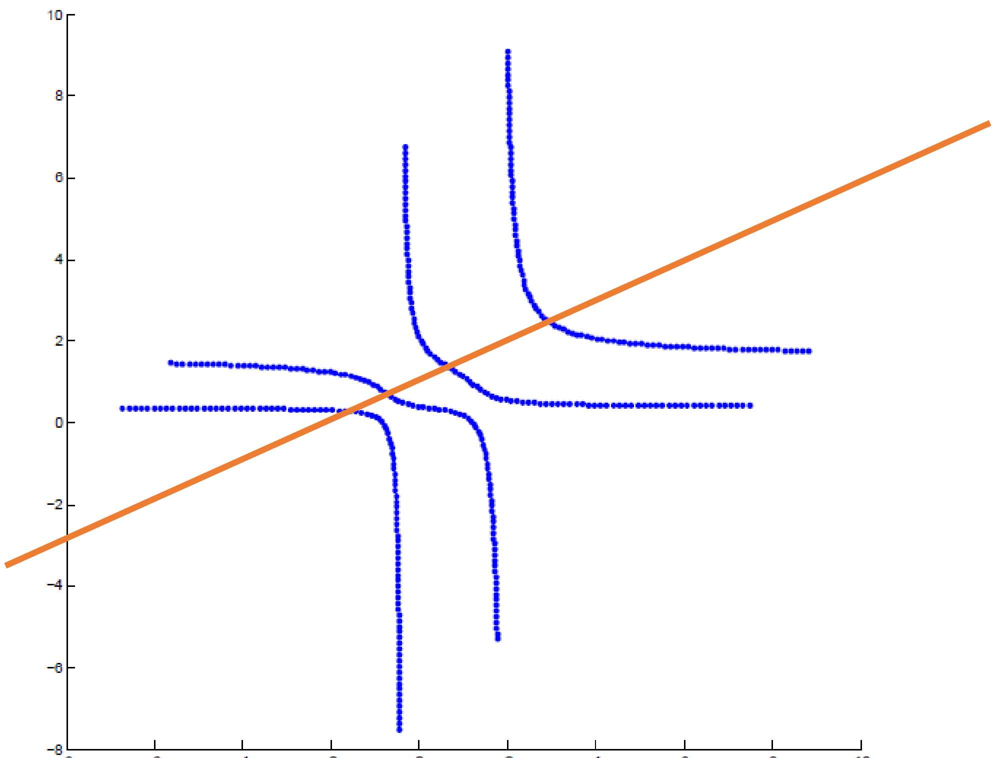

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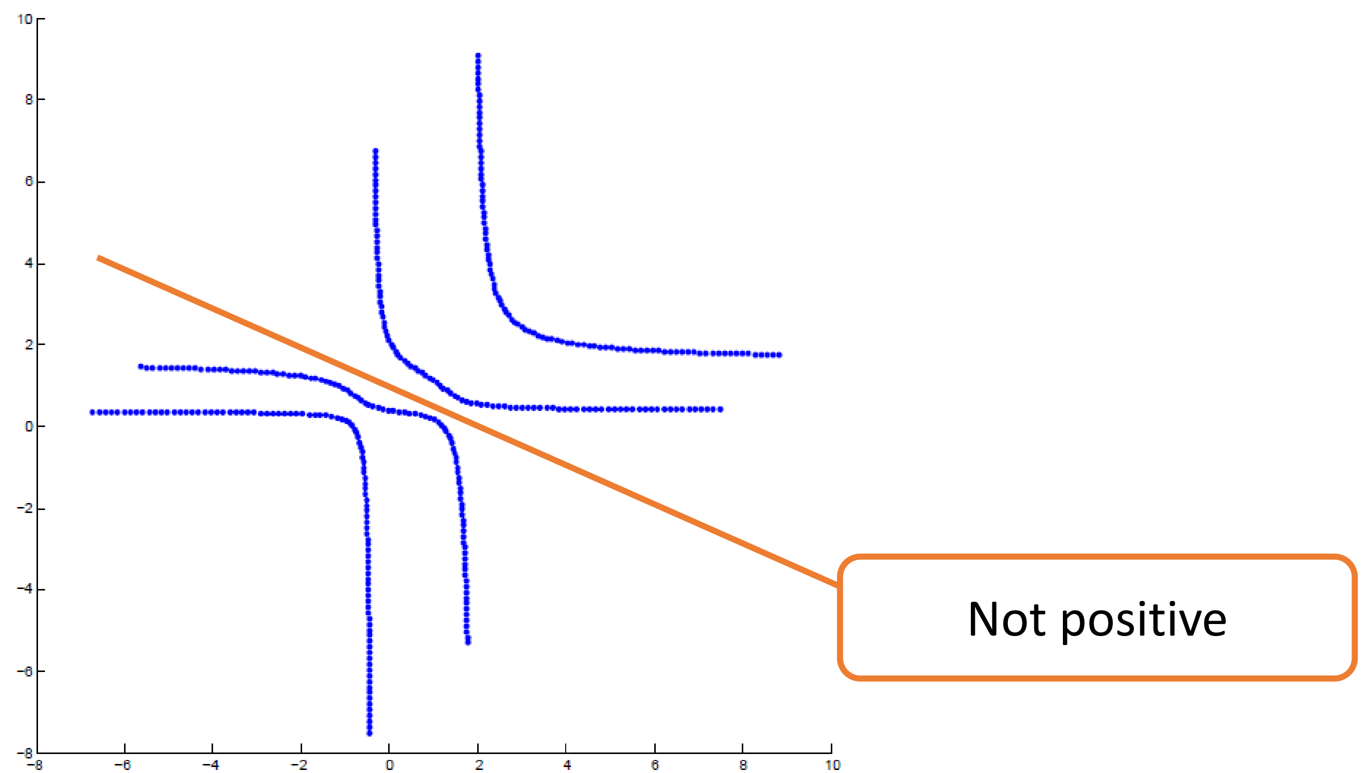

A Useful Real Stable Poly

Borcea-Brändén ‘08:

For PSD matrices $A_1, \ldots, A_k$

$$\det\left(\sum_i z_i A_i\right)$$

is real stable

A Useful Real Stable Poly

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For PSD matrices $A_1, \ldots, A_k$

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Proof: Every positive univariate restriction is the characteristic polynomial of a symmetric matrix.

$$\det\left(\sum_i x_i A_i + t \sum_i y_i A_i\right) = \det(tI + S)$$

Excellent Closure Properties

Definition: $p \in \mathbb{R}[z_1, \ldots, z_n]$
is *real stable* if $\text{imag}(z_i) > 0$ for all i
Implies $p(z_1, \ldots, z_n) \neq 0$.

If $p \in \mathbb{R}[z_1, \ldots, z_n]$ is real stable, then so is

1. $p(\alpha, z_2, \ldots, z_n)$ for any $\alpha \in \mathbb{R}$

2. $(1 - \partial_{z_i}) p(z_1, \ldots z_n)$ [Lieb-Sokal'81]

A Useful Real Stable Poly

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For PSD matrices $A_1, \ldots, A_k$

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Plan: apply closure properties to this
to show that $\mathbb{E}\det\left(xI - \sum_i v_i v_i^T\right)$ is real stable.

Central Identity

Suppose $v_1, \ldots, v_m$ are **independent** random vectors with $A_i := \mathbb{E} v_i v_i^T$. Then

$$\mathbb{E}\det\left(xI - \sum_i v_i v_i^T\right)$$
$$= \prod_{i=1}^{m}\left(1 - \frac{\partial}{\partial z_i}\right)\det\left(xI + \sum_i z_i A_i\right)\Bigg|_{z_1=\cdots=z_m=0}$$

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Key Principle: random rank one updates $\equiv (1 - \partial_z)$ operators.

Proof of Master Real-Rootedness Theorem

Suppose $v_1, \dots, v_m$ are **independent** random vectors with $A_i \coloneqq \mathbb{E} v_i v_i^T$. Then

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Real Stable
Borcea-Branden
'08

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Real Stable: Closure under $(1-\partial)$

Proof of Master Real-Rootedness Theorem

Suppose $v_1, \ldots, v_m$ are **independent** random vectors with $A_i \coloneqq \mathbb{E} v_i v_i^T$. Then

$$\mathbb{E} \det\left(xI - \sum_i v_i v_i^T\right)$$

$$= \prod_{i=1}^{m} \left(1 - \frac{\partial}{\partial z_i}\right) \det\left(xI + \sum_i z_i A_i\right)\Bigg|_{z_1 = \cdots = z_m = 0}$$

Real stable: closure under $z_i = 0$

Proof of Master Real-Rootedness Theorem

Suppose $v_1, \ldots, v_m$ are **independent** random vectors with $A_i := \mathbb{E} v_i v_i^T$. Then

$$\mathbb{E}\det\left(xI - \sum_i v_i v_i^T\right)$$

Real Stable ⇒ Real Rooted!

$$= \prod_{i=1}^{m}\left(1 - \frac{\partial}{\partial z_i}\right)\det\left(xI + \sum_i z_i A_i\right)\Bigg|_{z_1=\cdots=z_m=0}$$

□

The Whole Proof

$\mathbb{E}\det\left(xI - \sum_i v_i v_i^T\right)$ is real-rooted for all indep. v_i.

The Whole Proof

rank one structure naturally reveals interlacing.

$\mathbb{E}\chi_{A_s}(d-x)$ is real-rooted for all product distributions on signings.

$\mathbb{E}\det\left(xI - \sum_i v_i v_i^T\right)$ is real-rooted for all indep. v_i.

The Whole Proof

$\mathbb{E}\chi_{A_s}(x)$ is real-rooted for all product distributions on signings.

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The Whole Proof

$\{\chi_{A_s}(x)\}_{x\in\{\pm1\}^m}$ is an interlacing family

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The Whole Proof

$\exists s$ *such that* $\lambda_{max}(\chi_{A_s}) \leq \lambda_{max}(\mathbb{E}\chi_{A_s})$

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3-Step Proof Strategy

1. Show that some poly does as well as the $\mathbb{E}$. $\checkmark$

$\exists s$ *such that* $\lambda_{max}(\chi_{A_s}) \leq \lambda_{max}(\mathbb{E}\chi_{A_s})$

2. Calculate the expected polynomial. $\checkmark$

$$\mathbb{E}\chi_{A_s}(x) = \mu_G(x)$$

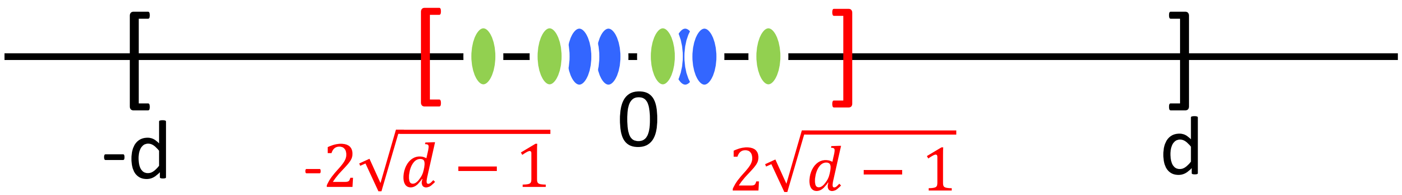
3. Bound the largest root of the expected poly. $\checkmark$

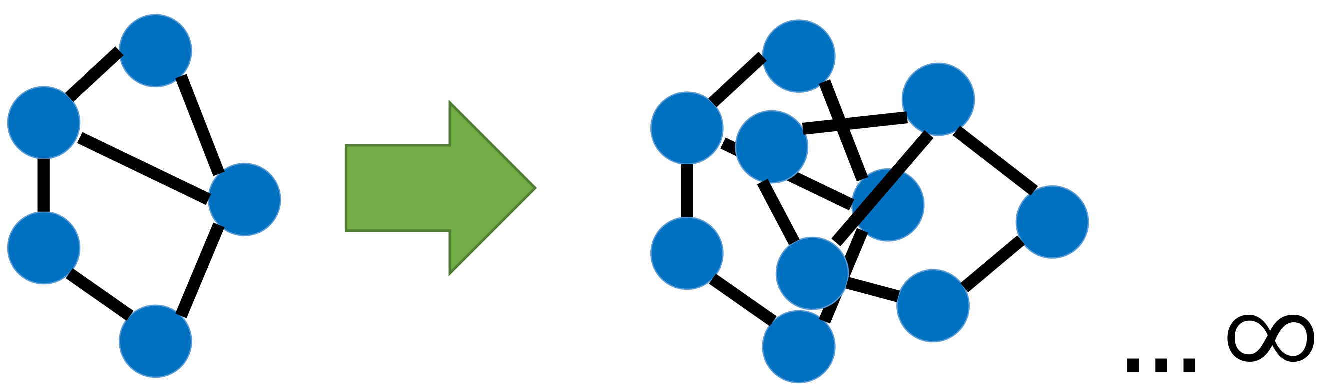
$$\lambda_{max}(\mu_G(x)) \leq 2\sqrt{d-1}$$

☐

Infinite Sequences of Bipartite Ramanujan Graphs

Find an operation which doubles the size of a graph without blowing up its eigenvalues.





Main Theme

Reduced the existence of a good matrix to:

1. Proving real-rootedness of an expected polynomial.

2. Bounding roots of the expected polynomial.

Main Theme

Reduced the existence of a good matrix to:

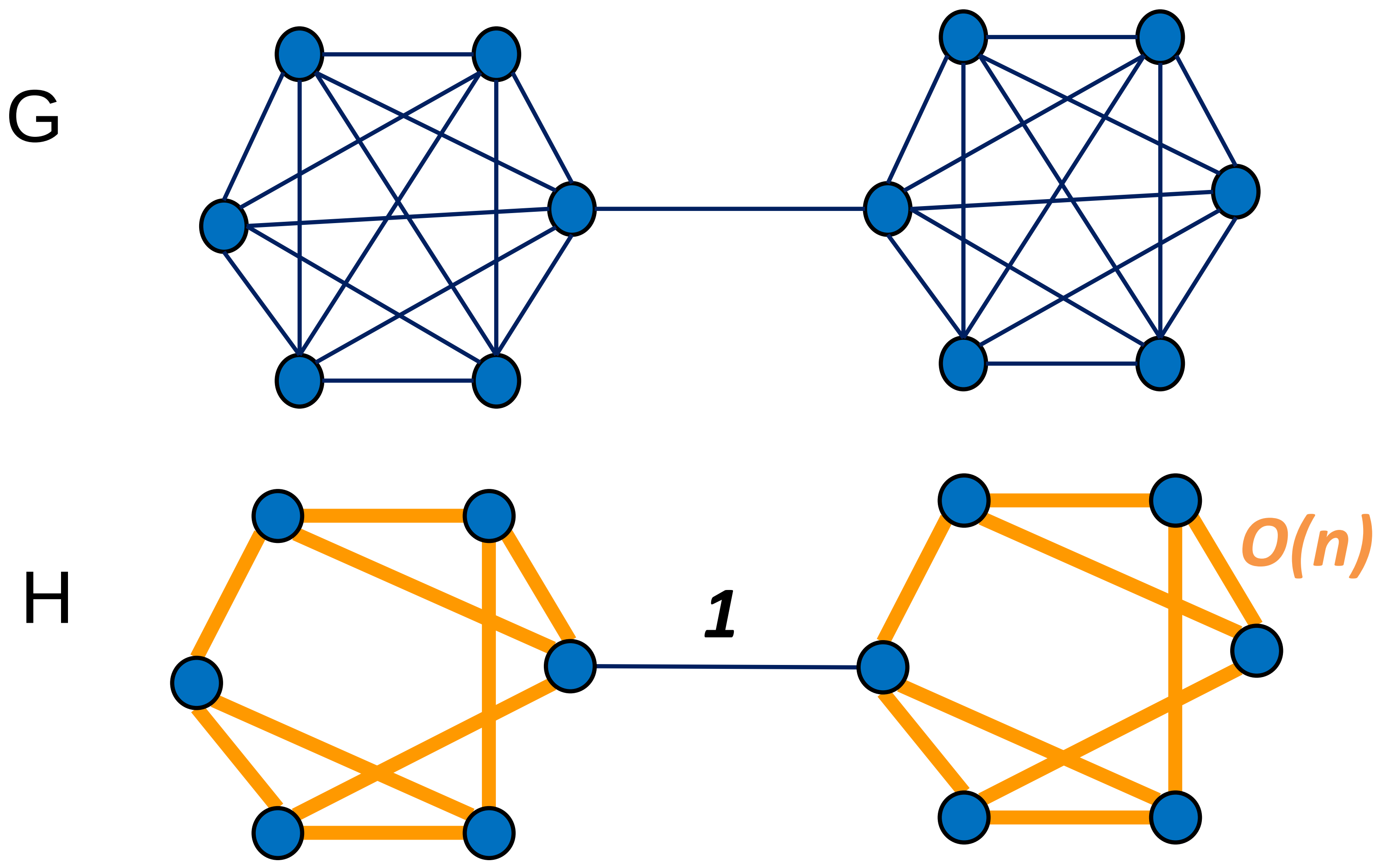
1. Proving real-rootedness of an expected polynomial.
   (rank-1 structure + real stability)

2. Bounding roots of the expected polynomial.
   (matching poly + combinatorics)

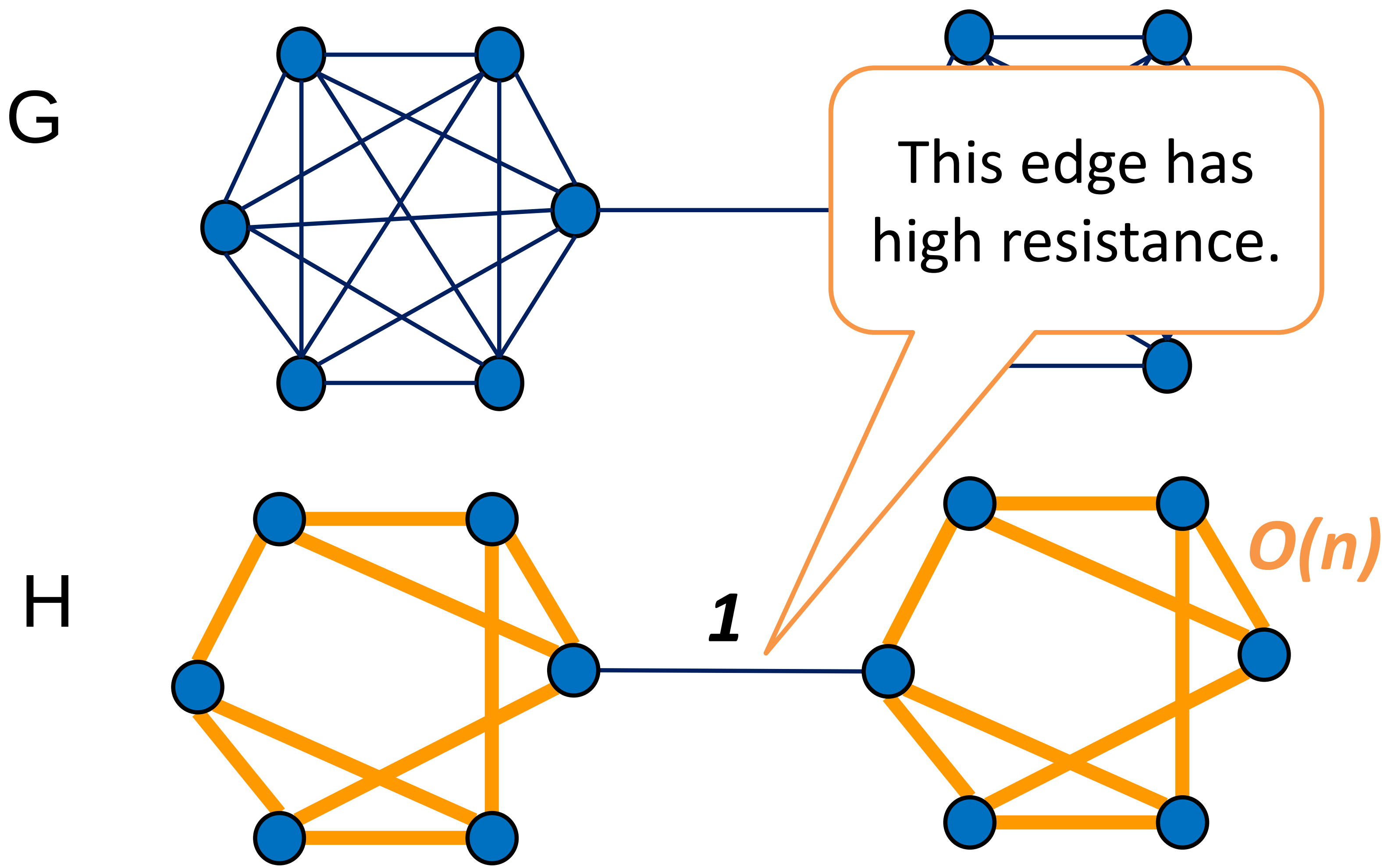
Beyond complete graphs

Unweighted sparsifiers of general graphs?

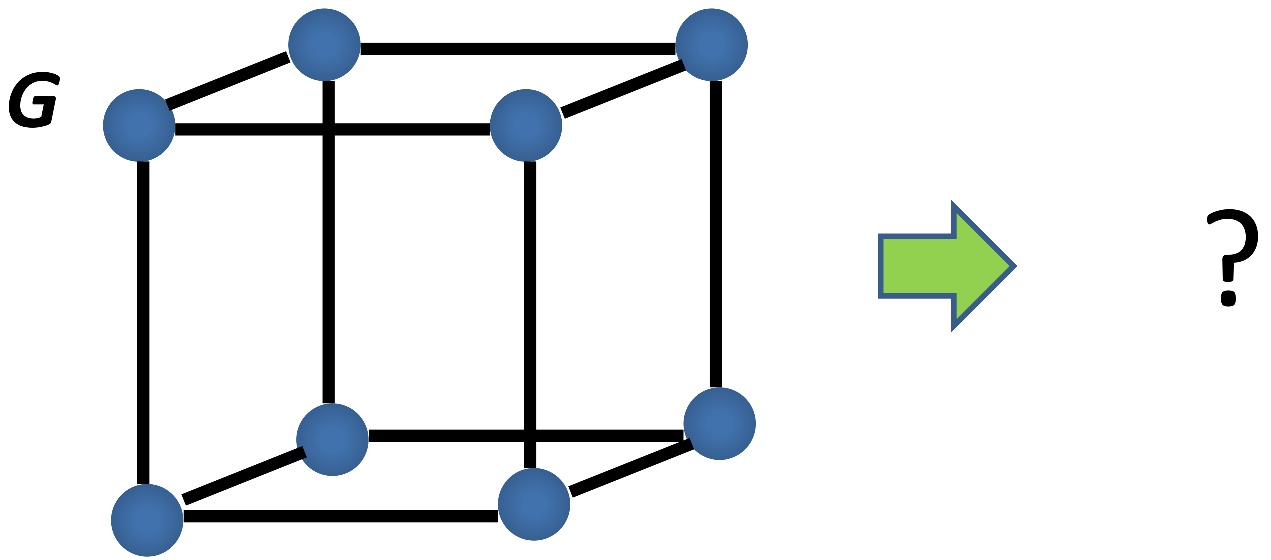
Beyond complete graphs



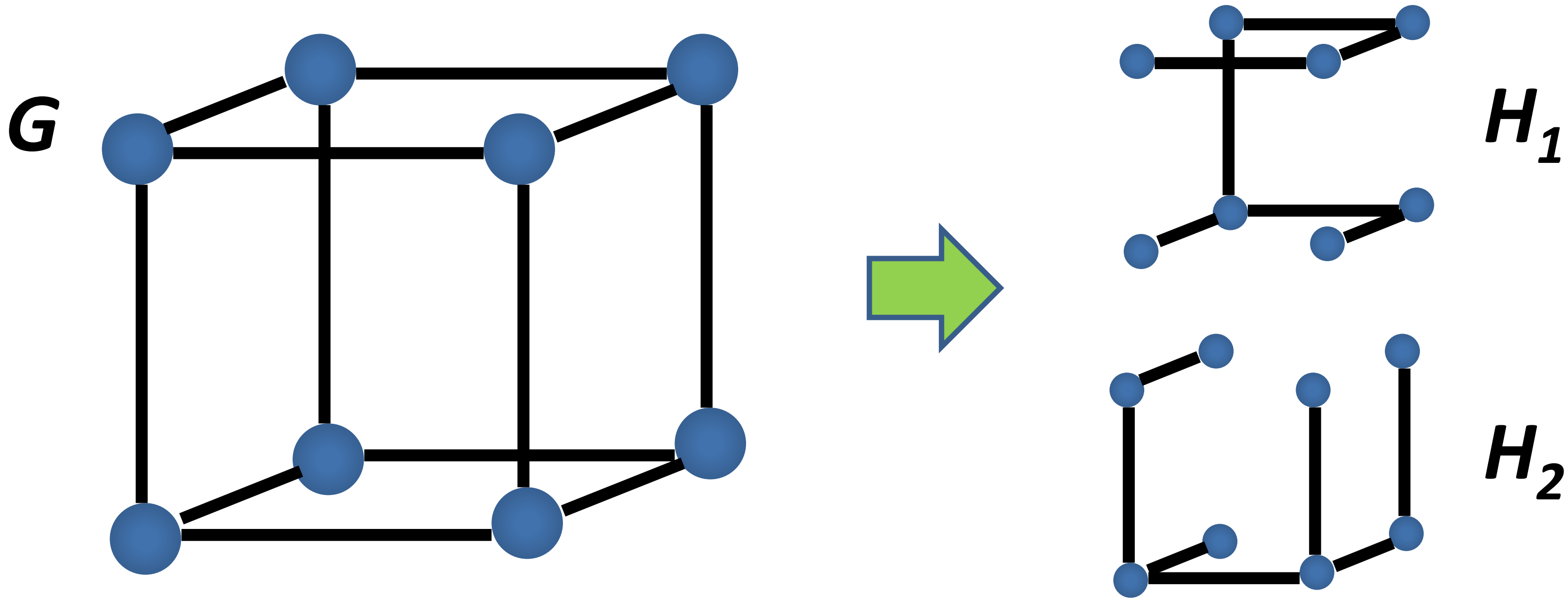
Weights are Required in General



What if all edges are equally important?

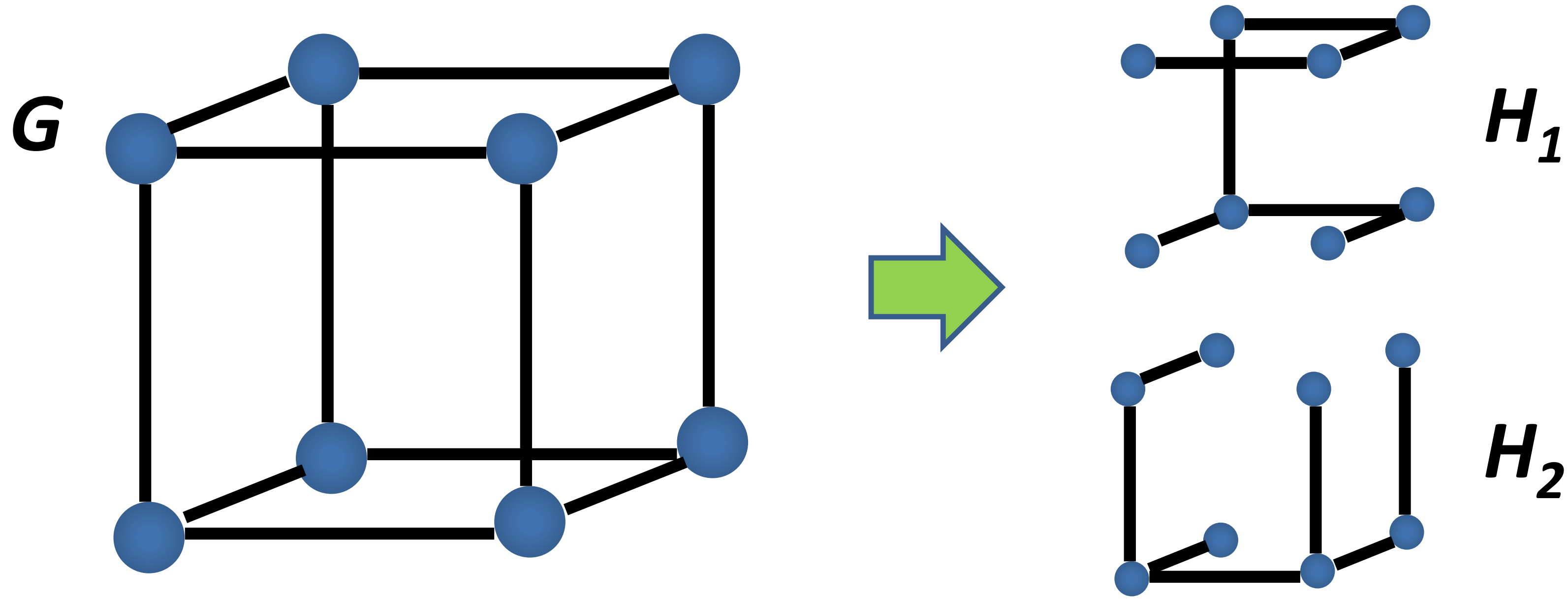


Unweighted Decomposition Thm.



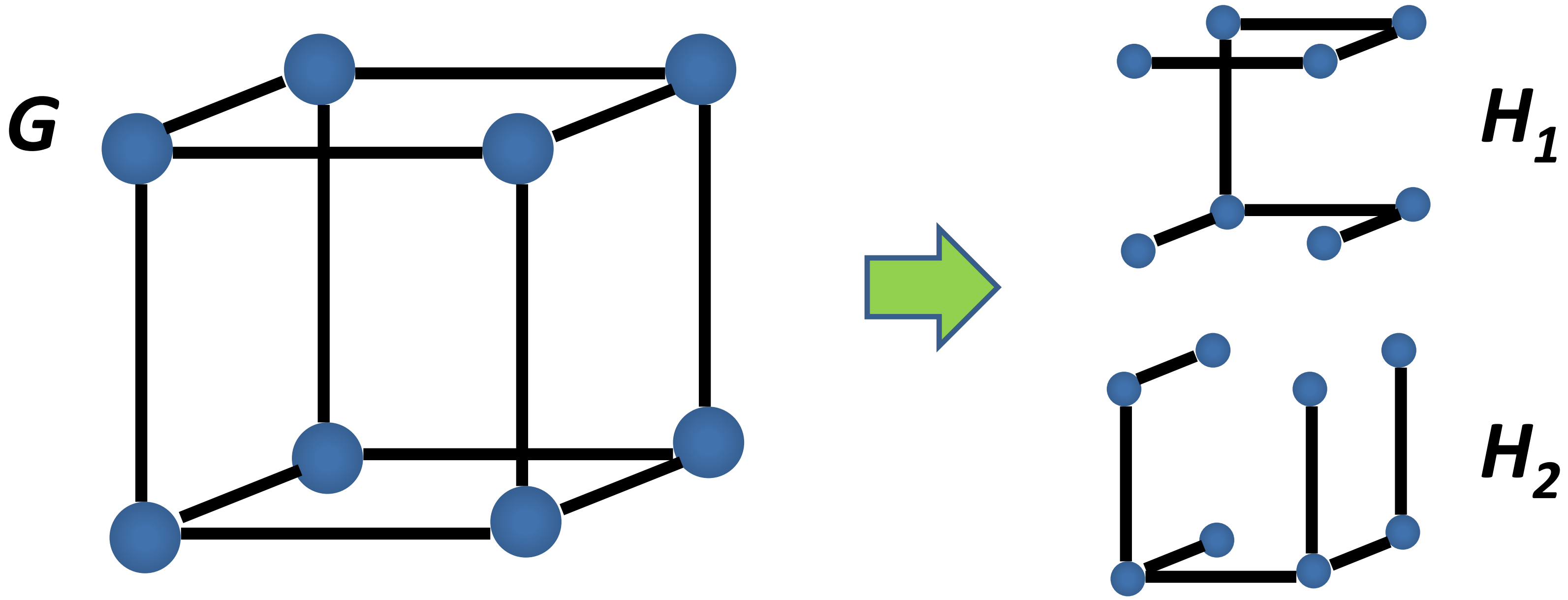
Theorem [MSS'13]: If all edges have resistance $O(n/m)$, there is a partition of **G** into **unweighted** $1+\epsilon$-sparsifiers, each with $O\left(\frac{n}{\epsilon^2}\right)$ edges.

Unweighted Decomposition Thm.



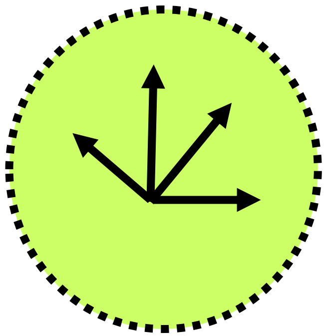
Theorem [MSS'13]: If all edges have resistance $\leq \alpha$, there is a partition of **G** into **unweighted** $O(1)$-sparsifiers, each with $O(m\alpha)$ edges.

Unweighted Decomposition Thm.

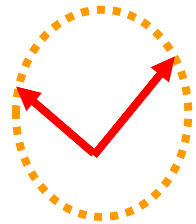


Theorem [MSS'13]: If all edges have resistance α, there is a partition of **G** into two **unweighted** $1 + \alpha$-approximations, each with **half** as many edges.

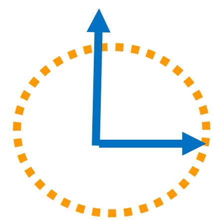
Unweighted Decomposition Thm.



$$\sum_i v_i v_i^T = I$$



$$\sum_i v_i v_i^T \approx \frac{I}{2}$$



$$\sum_i v_i v_i^T \approx \frac{I}{2}$$

Theorem [MSS'13]: Given any vectors $\sum_i v_i v_i^T = I$ and $|v_i| \leq \epsilon$, there is a partition into approximately 1/2-spherical quadratic forms, each $\frac{I}{2} \pm O(\epsilon)$.

Proof: Analyze expected charpoly of a random partition:

$$\mathbb{E}\det\left(xI - \sum_i v_i v_i^T\right)\det\left(xI - \sum_i v_i v_i^T\right)$$

$$\sum_i v_i v_i^T = I$$

$$\sum_i v_i v_i^T \approx \frac{I}{2}$$

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Other applications:

Kadison-Singer Problem

Uncertainty principles.

$$\sum_i v_i v_i^T = I$$

$$\sum_i v_i v_i^T \approx \frac{I}{2}$$

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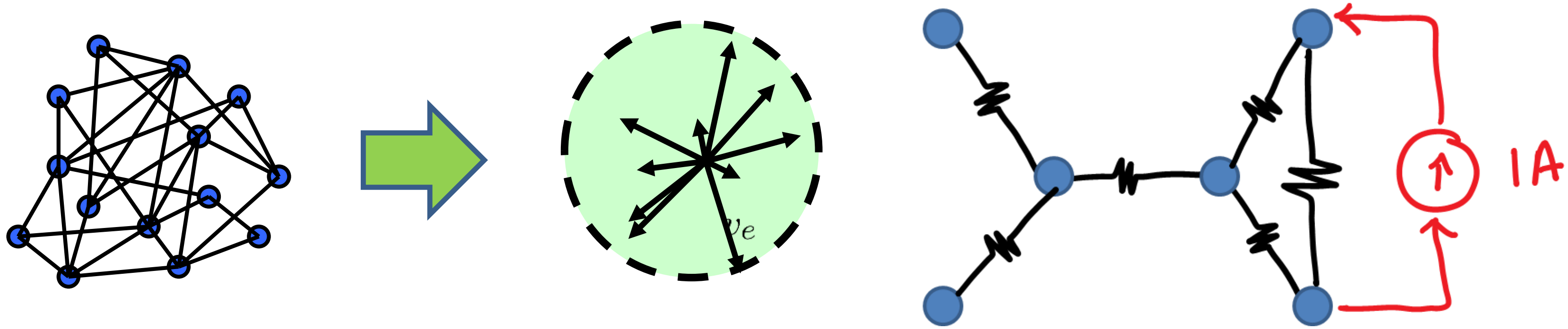
Summary of Algorithms

Result	Edges	Weights	Time
Spielman-S'08	O(nlogn)	Yes	O~(m)



Random Sampling with Effective Resistances

Summary of Algorithms

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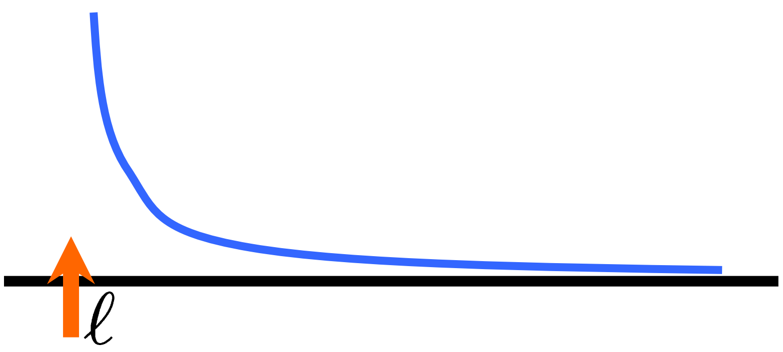
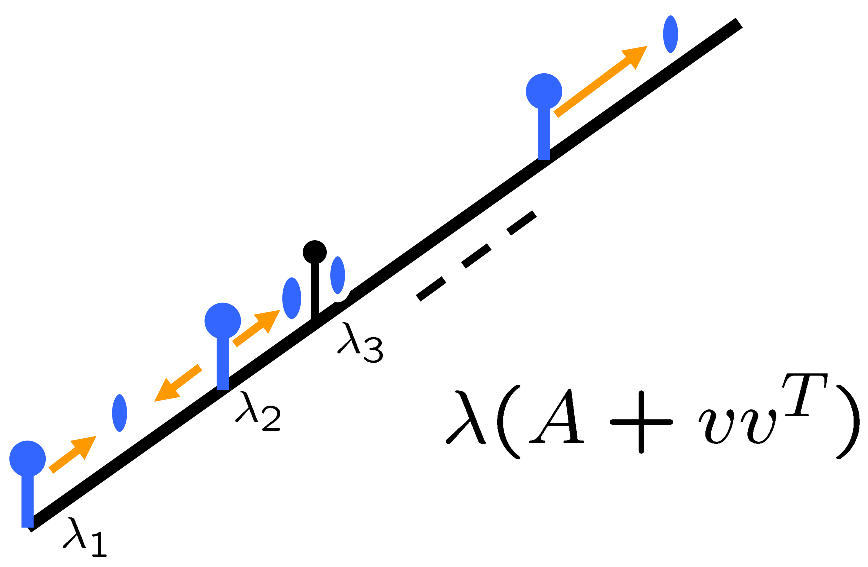
Batson-Spielman-S'09	O(n)	Yes	$O(n^4)$





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Spielman-S'08	O(nlogn)	Yes	O~(m)
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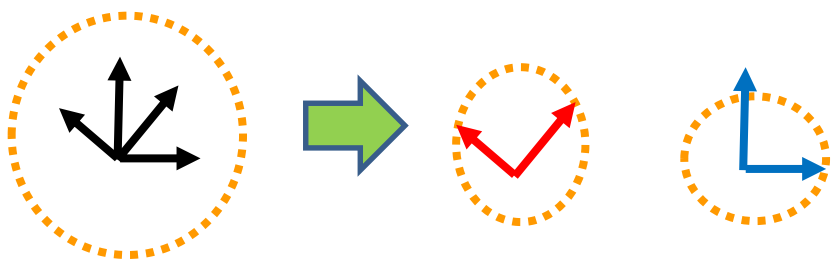
Marcus-Spielman-S'13	O(n)	No	$O(2^n)$



$$\mathbb{E}\det\left(xI - \sum_i v_i v_i^T\right)\det\left(xI - \sum_i v_i v_i^T\right)$$

Open Questions

Non-bipartite graphs

Algorithmic construction
(computing generalized μ_G is hard)

More general uses of interlacing families

Open Questions

Nearly linear time algorithm for $4n/\epsilon^2$ size sparsifiers

Improve to $2n/\epsilon^2$ for graphs?

Fast combinatorial algorithm for approximating resistances

□