Noise vs Imprecision in the Hardness of Random Quantum Circuits

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Based on "**On the complexity and verification of quantum random circuit sampling**" with A. Bouland, C. Nirkhe, U. Vazirani (Nature Physics, arXiv: 1803.04402)

And "**Noise and the frontier of quantum supremacy**" with A. Bouland, Z. Landau, Y. Liu (Talk at QIP 2021, arXiv: 2102.01738)

THE UNIVERSITY OF CHICAGO

Berkeley UNIVERSITY OF CALIFORNIA

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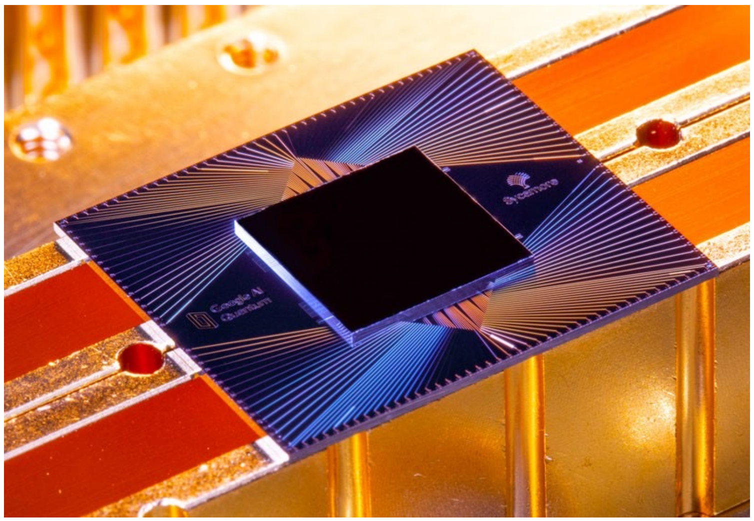
The first "Quantum supremacy" claims have now been made…



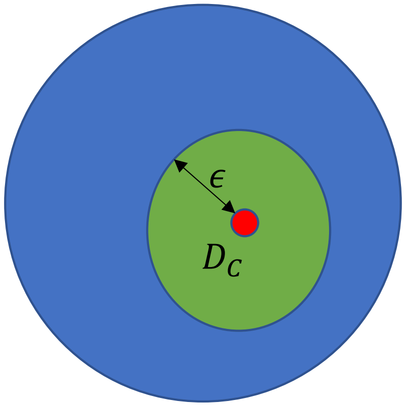
Random Circuit Sampling (Google "Sycamore") in late 2019

Gaussian BosonSampling (USTC "Jiuzhang") in late 2020

This talk: the latest complexity theoretic evidence to believe these experiments might be solving hard problems for classical computers

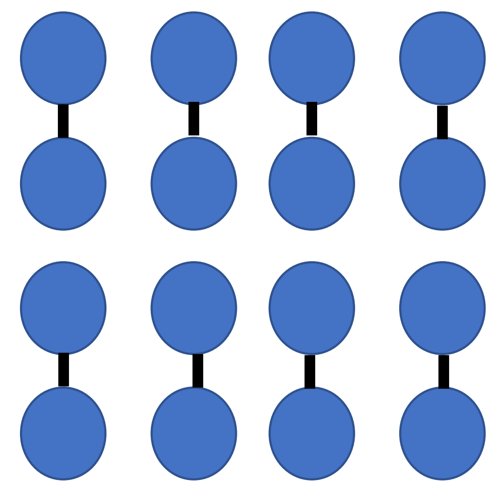
These experiments have a similar theoretical framework

- They both solve "random quantum circuit sampling"
  - i.e., the hard problem is to sample from the output distribution of a randomly chosen quantum circuit
- **Initial theory goal:** prove impossibility of an efficient *"classical Sampler"* algorithm that:
  - takes as input a random circuit C with output distribution D_C over $\{0,1\}^n$
  - outputs a sample from *any* distribution X so that:
    - $|X - D_C|_{TV} \leq \epsilon$ with high probability over choice of circuit C



Google's RCS proposal [Boixo et. al. 2017]

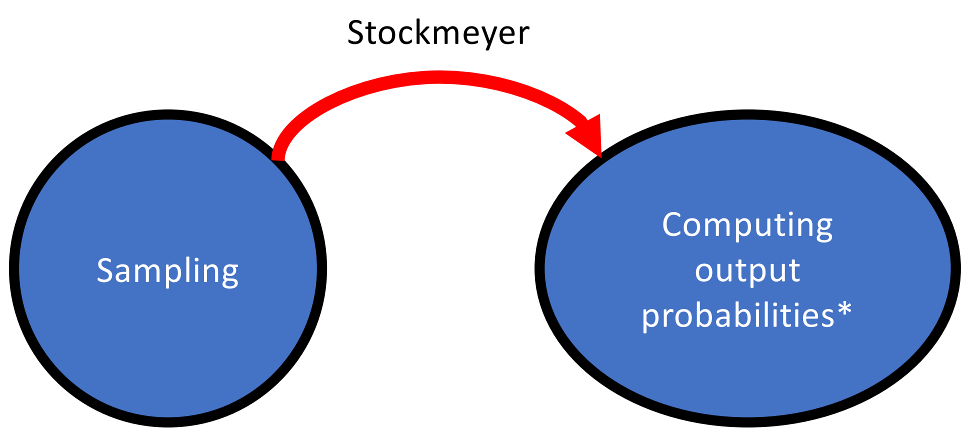
- Generate a quantum circuit C on n qubits on a 2D lattice, with $d \sim \sqrt{n}$ layers of (Haar) random nearest-neighbor gates
  - In practice use a discrete approximation to the Haar random distribution
- Start with $|0^n\rangle$ input state, apply random quantum circuit and measure in computational basis



(single layer of Haar random two qubit gates applied on 2D grid of qubits)

Proof first step: from sampling to computing

- By well-known reductions [Stockmeyer ’85], [Aaronson & Arkhipov ’11] it suffices to prove that ***estimating*** the output probability of a ***random*** quantum circuit is **#P**-hard

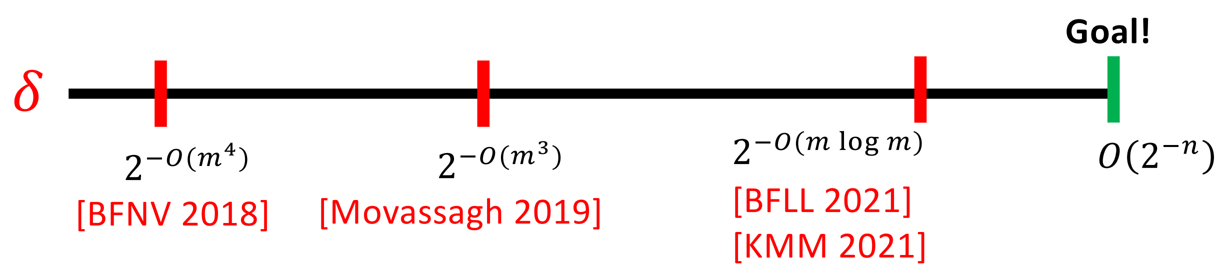


Formal statement of q. supremacy conjecture

- **Definition:** Let the "output probability", $p_0(C) = |\langle 0^n | C | 0^n \rangle|^2$
- Then consider the $\delta - \boldsymbol{Random\ Circuit\ Estimation}$ problem:

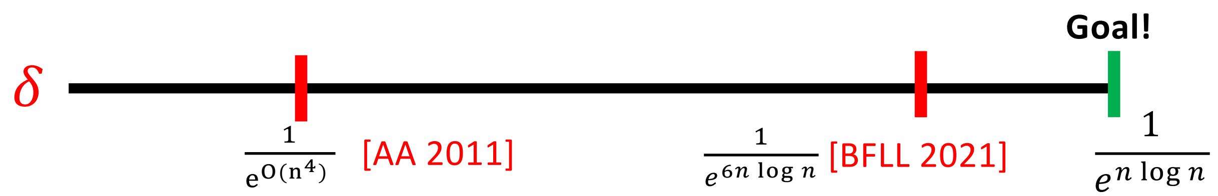
> Given as input circuit C, output q so that $|q - p_0(C)| \leq \boldsymbol{\delta}$ with probability 2/3 over C

- To prove goal, it suffices to show that the $\delta = O\left(\frac{1}{2^n}\right)$ problem is **#P**-hard
- **Known hardness results** with respect to C on n qubits, size $m = O(n \cdot d)$



Hardness conjecture for BosonSampling

- In the case of BosonSampling, similar arguments take us “even closer” to the goal!
- With respect to BosonSampling with n photons, n^2 modes:



- So we’re only off by a factor of 6 in the exponent!
- So close – yet so far – we’ve hit a barrier – *more on this later!*

Roadmap for the rest of talk

1. Proof of hardness result from [F., with Bouland, Nirkhe & Vazirani '18]
   - To do this, will first discuss Lipton's average-case hardness result for computing the Permanent of a random matrix
   - Then we'll adapt Lipton's result from Permanent to output probability of random quantum circuit
2. We'll prove that these results are "robust to uncorrected depolarizing noise" [F., with Bouland, Landau & Liu'21]
   - And this robustness actually **gives a barrier** against improving our hardness result to larger imprecision δ

Average case hardness for **Permanent** [Lipton '91]

- **Permanent** of $n \times n$ matrix is **#P**-hard in the worst-case [Valiant '79]
  - $Per[X] = \sum_{\sigma \in S_n} \prod_{i=1}^{n} X_{i,\sigma(i)}$
- *Algebraic property*: $Per[X]$ is a degree n polynomial with n^2 variables
- Need compute $Per[X]$ of worst-case matrix X
  - But we only have access to algorithm O that correctly computes *most* permanents over $\mathbb{F}_p$
  - i.e., $\Pr_{Y \in_R \mathbb{F}_p^{n \times n}} [O(Y) = Per[Y]] \geq 1 - \frac{1}{poly(n)}$
- Choose $n + 1$ fixed non-zero points $t_1, t_2 \ldots, t_{n+1} \in \mathbb{F}_p$ and uniformly random matrix R
- Consider line $A(t) = X + tR$
  - *Observation 1 "marginal property"*: for each i, $A(t_i)$ is a random matrix over $\mathbb{F}_p^{n \times n}$
  - *Observation 2*: "univariate polynomial": $Per[A(t)]$ is a degree n polynomial in t
- But now these $n + 1$ points uniquely define the polynomial, so use polynomial extrapolation to evaluate $Per[A(0)] = Per[X]$

[BFNV'18]: Hardness for Random Quantum Circuits

- *Algebraic property*: much like $Per[X]$, output probability of random quantum circuits has polynomial structure
  - Consider circuit $C = C_m C_{m-1} \dots C_1$
  - Polynomial structure comes from Feynman path integral:
    - $\langle 0^n | C | 0^n \rangle = \sum_{y_2, y_3, \dots, y_m \in \{0,1\}^n} \langle 0^n | C_m | y_m \rangle \langle y_m | C_{m-1} | y_{m-1} \rangle \dots \langle y_2 | C_1 | 0^n \rangle$
- This is a polynomial of degree m in the gate entries of the circuit
- So the output probability $p_0(C)$ is a polynomial of degree $2m$

First attempt at adapting Lipton's proof

- Choose and fix $\{H_i\}_{i \in [m]}$ Haar random gates
- Now consider new circuit $C' = C'_m C'_{m-1} \dots C'_1$ so that for each gate $C'_i = C_i H_i$
  - Notice that each gate in C' is completely random – "marginal property"
- **Problem:** no univariate polynomial structure connects worst-case circuit C with the scrambled circuit C' !!

Correlating via *quantumness*

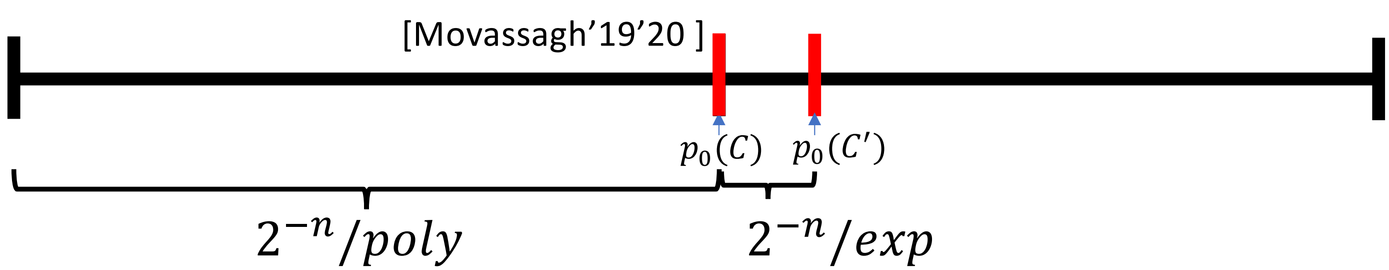
- We need the analogue to Lipton's "*univariate* polynomial structure"
- ***Main idea***: "Implement tiny fraction of H_i^{-1}"
  - i.e., $C_i' = C_i H_i e^{-ih_i\theta}$
  - If $\theta = 1$ the corresponding circuit $C' = C$, and if $\theta \approx small$, each gate is close to Haar random
  - Now take several non-zero but small θ and apply polynomial extrapolation (as per Lipton's proof)

This is still not the “right way” to scramble!

- *Problem*: $e^{-ih_i\theta}$ is not polynomial in θ
- *Solution:* take fixed truncation of Taylor series for $e^{-ih_i\theta}$
  - i.e., each gate of C' is $C_i H_i \sum_{k=0}^{K} \frac{(-ih_i\theta)^k}{k!}$
  - So each gate entry is a polynomial in θ and so is $p_0(C')$
  - Now extrapolate and compute $q(1) = p_0(C)$

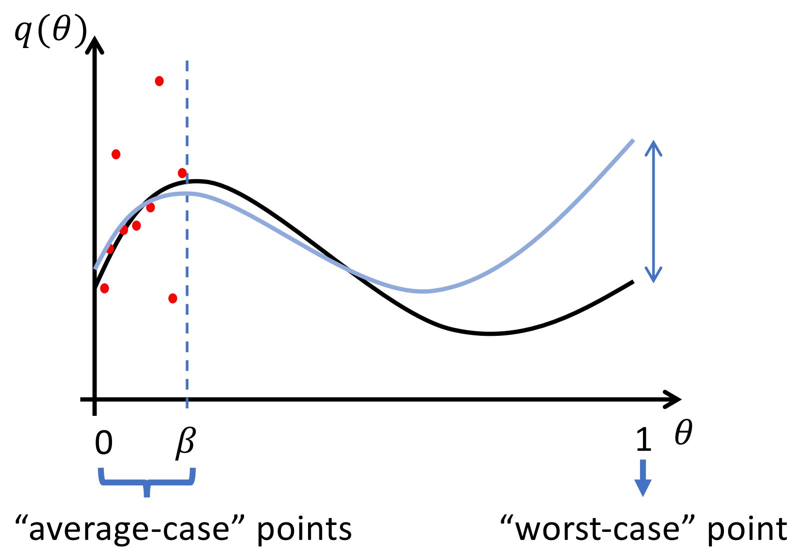
How to motivate the truncations?

- Recall, our goal was **not** to prove the hardness of ***exactly*** computing $p_0(C)$ but rather in computing an ***estimate*** $|y - p_0(C)| \leq O\left(\frac{1}{2^n}\right)$
- Now, our result is proving hardness of computing the output probability of a slightly non-unitary "truncated" circuit $p_0(C')$ which is *extremely close* to $p_0(C)$
- [BFNV'18]: ***Estimating*** $p_0(C')$ is hard **iff *estimating*** $p_0(C)$ is hard
  - **Intuitively, because the "truncation error" is so much smaller than the size of the additive error we are conjecturing is hard.**



On robustness to *imprecision* [BFLL'21]

- See also independent work of [Kondo et. al. '21]!
- **Recall:** I claimed it's **#P**-hard to:
  - Output an estimate $|y - p_0(C)| \leq 2^{-O(m \log m)}$ w.p 2/3 over C
- [*Main technical lemma*] Given:
  - $O(m^2)$ "faulty" evaluation points $\{(\theta_i, y_i)\}$ to a polynomial q(θ) of degree m where:
    1. θ_i are equally spaced in the interval $[0, \beta = 1/\mathrm{m}]$
    2. And we know ***at least*** *2/3* of y_i are δ***-close*** to $\mathrm{q}(\theta_i)$
- Then there's an algorithm (uses **NP** oracle) that outputs z:
  - $|z - q(1)| \leq \delta 2^{O(m \log \beta^{-1})} = \delta 2^{O(m \log m)}$ whp
- **Upshot:** if δ is small enough then we can use these faulty evaluation points to estimate q(1), which is hard!



Understanding hardness of *noisy* random quantum circuits [BFLL'21]

- Without error-correction noise eventually overwhelms
  - e.g., Google's RCS experiment ~0.2% fidelity and 99.8% noise
- How can we model this theoretically for RCS?
- Each random gate C_i is followed by two qubit depolarizing noise channel:
  - $\mathcal{E}_i = (1-\gamma)\rho + \frac{\gamma}{15}\sum_{\alpha,\beta\in\mathcal{P}\times\mathcal{P}-(I,I)}(\sigma_\alpha \otimes \sigma_\beta)\rho(\sigma_\alpha \otimes \sigma_\beta)$
- That is, we can think about choosing a noisy random circuit by:
  - First pick ideal circuit $C = C_m C_{m-1} \ldots C_1$ from the random circuit distribution
  - Then environment chooses operators N, from a distribution $\mathcal{N}$ (i.e., via $\mathcal{E}_i$)
  - We get a sample from output distribution of $N \cdot C$ without learning the noise operators

Similar hardness arguments work with noise!

- By linearity, can write the output probability of the noisy circuit as:
  - $E_{N \sim \mathcal{N}}[|\langle 0^n | N \cdot C | 0^n \rangle|^2] = E_{N \sim \mathcal{N}}[p_0(N \cdot C)]$
- This can be written as a weighted sum of Feynman path integrals:
  - $\sum_N \Pr_{\mathcal{N}}[N] \cdot \left| \sum_{y_1, y_2, \ldots, y_m \in \{0,1\}^n} \langle 0^n | N_m C_m | y_m \rangle \ldots \langle y_2 | N_1 C_1 | 0^n \rangle \right|^2$
  - **Key point:** this is still a polynomial of degree $2m$ in the ideal gate entries
- So by the same arguments as before, we have a worst-to-average case reduction for computing $E_{N \sim \mathcal{N}}[p_{0^n}(N \cdot C)]$ to within $\pm 2^{-O(m \log m)}$
- [Fujii '16] has shown that this quantity is also **#P**-hard to compute in the worst-case if noise rate, γ, is a sufficiently small constant

But there's also a (trivial) classical algorithm!

- **Issue:** uncorrected depolarization noise causes output distribution to rapidly converge to uniform as system size grows
  - And it's clearly not hard to output a probability from the uniform distribution!
- How accurate is this trivial algorithm (i.e., what is the imprecision?)
  - **Conjecture**: for random circuits $\mathbf{2^{-O(m)}}$ [e.g., Boixo, Smelyanskiy, Neven '17]!
  - We can prove this in certain simplified toy models of random circuits [BFLL'21]

The "noise barrier" to improving robustness

- On the one hand we've established that computing output probabilities of ***noisy*** random circuits of size m is **hard** to within imprecision $\mathbf{2^{-O(m \log m)}}$
- On the other hand, we think it's classically **easy** to solve this problem to within imprecision $\mathbf{2^{-O(m)}}$
- **Upshot:** So if we want to dramatically improve this robustness in the ***noiseless*** case we need to invent new proof techniques that are ***not*** robust to noise.

Conclusions

- We improved the imprecision to additive error of computing the output probability of a random quantum circuit of size m to additive error $2^{-O(m \log m)}$
- This hardness remains if the circuit is *noisy*, but in this case we think this same imprecision is *essentially tight*
- We'll likely need very new ideas to get much further!

Thanks!