Hamiltonian Complexity in the Thermodynamic Limit.

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(joint work with Dorit Aharonov)

- Just posted to the arXiv.
- Similar paper also appearing from Toby Cubitt + James Watson.

Local Hamiltonian

Input: Hamiltonian $H = \sum_a h_a$

N d-dimensional particles.

Each h_a is hermetian & acts non-trivially on $\leq \ell$ particles.

$a, b.$ s.t. $b - a \geq 1/\text{poly}(N)$.

Output: Accept if $\lambda_0(H) \leq a$.
Reject if $\lambda_0(H) \geq b$.

[Kitaev]: Local-Hamiltonian is QMA-complete.

Quantum Merlin-Arthur (QMA).

Problem P partitions $\{0,1\}^*$ into : $\left.\begin{array}{l}\text{Accept}\\ \text{Reject}\end{array}\right\}$ Valid
Invalid

$P \in$ QMA if $\exists$ poly-time quantum verifier V

$x \in \text{Accept} \implies \exists\ |\psi\rangle\ \text{Prob}[V(x, |\psi\rangle) \text{ accepts}] \geq 2/3$

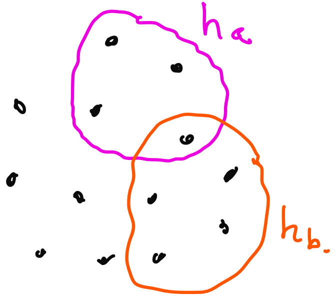
$x \in \text{Reject} \implies \forall\ |\psi\rangle\ \text{Prob}[V(x, |\psi\rangle) \text{ rejects}] \geq 2/3$

$|\psi\rangle$: quantum state on poly$(|x|)$ qubits.

$x \in \text{Invalid} \implies$ No promises!

Kitaev's construction:

- non-translationally invariant.
- not-geometrically local.
- 5-local.



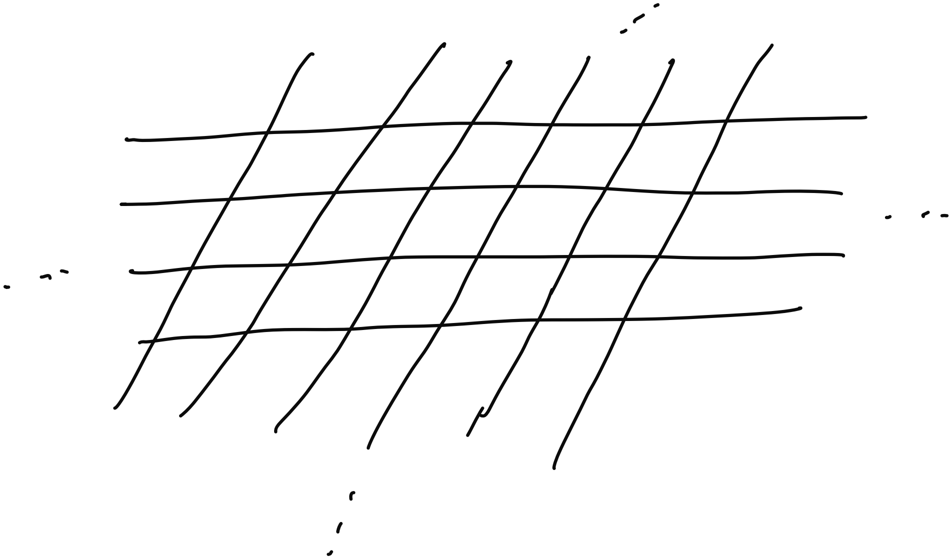
Local Hamiltonian remains QMA-complete, even if....

- 2 local - qubits on a 2D grid. [Terhal, Oliveira]
- 2-local 1D [Aharonov, Gottesman, I., Kempe]
- Translationally-Invariant on a line. [Gottesman, I.]

Finite:

Compute ground energy as a function of N = system size.

Thermodynamic Limit



What are the properties (energy, corr. length) of the ground state as $N \to \infty$?

Translational Invariance: Each dimension (of grid) has its own term (e.g. h_{row} + h_{col})

Thermodynamic Limit

- Given h (or h_{row} + h_{col} for 2D)

  It is undecidable to determine the spectral gap of h — applied in the Therm. Limit.

  Cubitt, Perez-Garcia, Wolf (2D)

  Bausch, Cubitt, Lucia, Perez-Garcia (1D)

- Uncomputability of Phase Diagrams

  Bausch, Cubitt, Watson.

Thermodynamic Limit

Polynomials p, q, r.

Input: N, and Hamiltonian term h.

(entries of h are integer multiples of $1/r(N)$)

Output: Is the ground energy density in the thermodynamic limit $\leq 1/p(N)$ OR $\geq 1/p(N) + 1/q(N)$

$\Rightarrow$ QMA-EXP-complete [Gottesman, I.]

Does it make sense to study G.E.D. in the therm. limit as a function of the Hamiltonian?

Different Varieties of Translational Invariance

Infinite Systems.

Infinite family of Hamiltonians

- Input : $(d, \cancel{N}, h)$

particle dimension

system size (in binary)

Hamiltonian term

- Problem Parameters $(d, \underbrace{h_1, h_2}, p)$

Hamiltonian terms

poly

Input: N.

$$h = h_1 + \frac{1}{p(N)} h_2$$

- Problem Parameters $(d, h) \rightarrow$ 1 Hamiltonian

~~Input : N~~.

Ground Energy Densit = α_0

Translationally Invariant Ground Energy Density (GED)

Hamiltonian terms $h_{row} + h_{col} \longrightarrow H_{2D}(N)$ on $N \times N$ grid.

apply h_{row} (horiz.)
apply h_{col} (vert.)

$$\alpha_0 = \lim_{N \to \infty} \frac{\lambda_0(H_{2D}(N))}{N^2}$$

Function - GED (h_{row}, h_{col})

Input: n (binary number)

Output: α where $|\alpha - \alpha_0| \le 1/2^n$

$$\alpha_0 = .\underbrace{10111001010001001111}_{n}\,01101\ldots$$

Proposed by:
I. Cirac
SIMONS 2017

Theorem : Function-GED is contained in $FEXP^{QMA\text{-}EXP}$

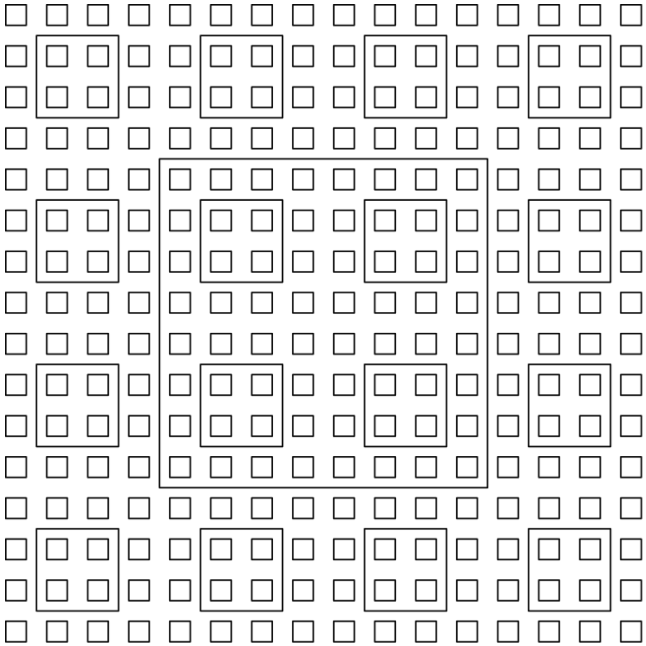
Function-GED is hard for $FEXP^{NEXP}$

$$f \in FEXP^{NEXP} \prec \text{Function-GED}$$

$$\alpha_0 = 10\;111\;00101\;\underbrace{000100111}_{f(x_1)}\;\underbrace{011\;0110}_{f(x_2)}\;\underbrace{111\;00101}_{f(x_3)}\;000100111\;01101 \qquad \cdots\cdots$$

$\Rightarrow$ Encode output of f for different x's in different portions of the (infinite) binary representation of α_0.

Robinson Tiles



Finite set of tiling rules

$\Rightarrow$ a periodic structure

for $k \geq 1$.

squares of size 4^k

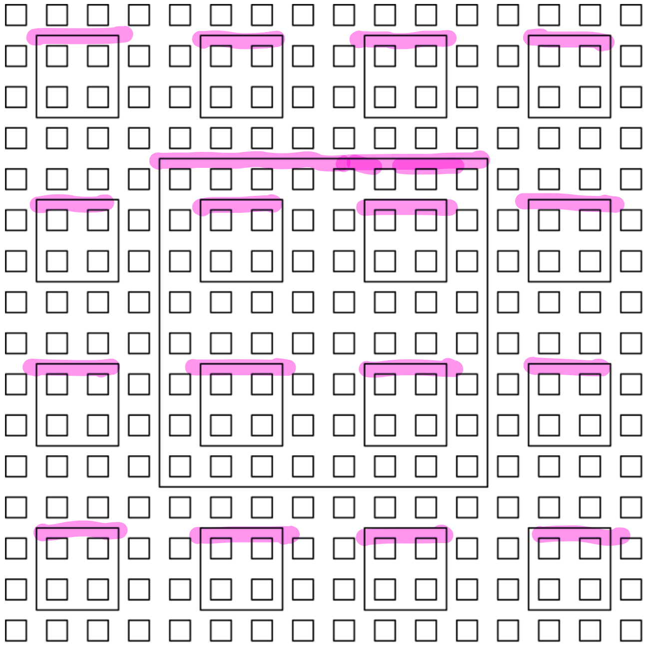
density: $1/4^{2k+1}$

Tiling rules $\equiv$ classical Hamiltonian.

Robinson

Cubitt, García-Perez, Wolf.

Robinson Tiles



Squares of size 4^k

density : $1/4^{2k+1}$

Layer 1: Robinson Tiling Rules

Layer 2: 1D TI Hamiltonian for finite chains

Ground Energy Density:

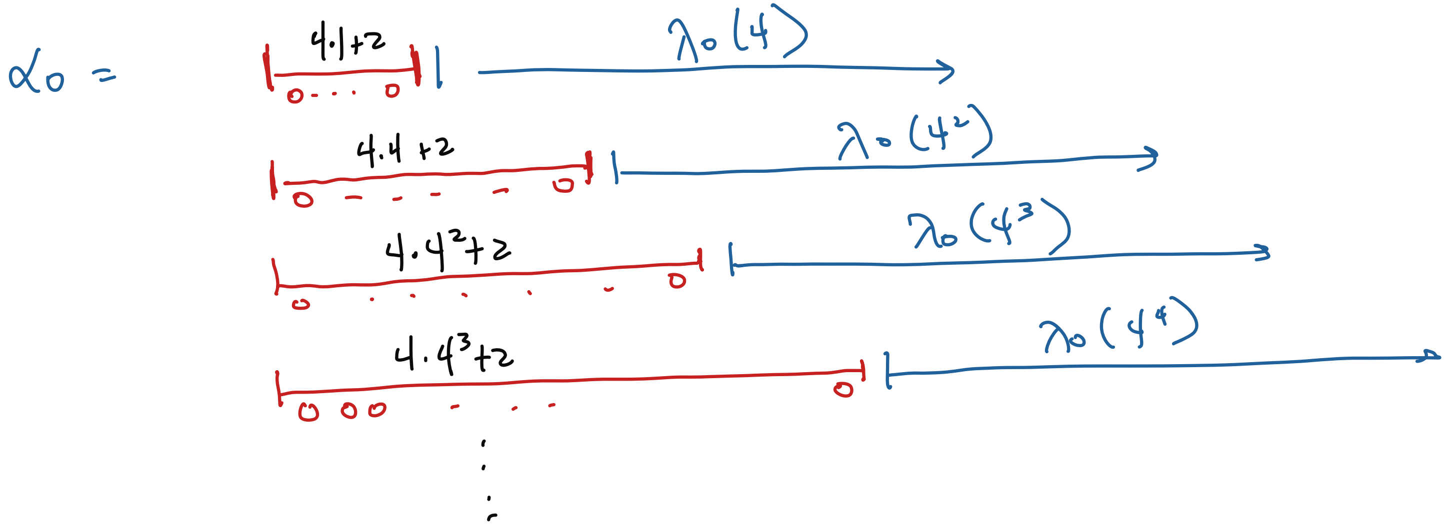
$$\alpha_0 = \sum_{k=1}^{\infty} \frac{\lambda_0(4^k)}{4^{2k+1}}$$

Ground energy for 1D on chain of length 4^k.

Ground Energy Density:

$$\alpha_0 = \sum_{k=1}^{\infty} \frac{\lambda_0(4^k)}{2^{4k+2}}$$

Ground energy for 1D on chain of length 4^k.



$\lambda(4^k) = 0$ for $k \neq x^2$.

$\alpha_0 =$ $\overset{4\cdot 1+2}{0 \cdots 0}$ Binary rep of $\lambda_0(4)$

$+$ $4\cdot 4+2$ Binary rep of $\lambda_0(4^4)$

$\vdots$

$+$ $4\cdot x^2+2$ Binary rep of $\lambda_0(4^{x^2})$

$+$ $4\cdot(x+1)^2+2$

these bits encode $f(x)$.

$\lambda_0(4^{x^2})$ is the lowest eigenvalue of a $4^{x^2} \times 4^{x^2}$ matrix.

$\Rightarrow$ Exact bounds required: $\left(1 - \cos\frac{\pi}{L}\right)$

Function - Translationally Invariant Hamiltonian.

(Function - TIH).

Problem Parameter: h, c

Input: N.

Output: λ where $|\lambda - \lambda_0(H(N))| \leq 1/N^c$

translationally invariant chain: length L, term h.

Theorem: Function - TIH is in $FP^{QMA\text{-}EXP}$.

Function - TIH is hard for FP^{NEXP}

(Decision version is QMA-EXP-Complete : Gottesman, I.)

	Finite	vs.	Infinite.
Input	N		n
Precision	$\pm 1/N^c$		$\pm 1/2^n$
Complexity	$FP^{NEXP} / FP^{QMA\text{-}EXP}$		$FEXP^{NEXP} / FEXP^{QMA\text{-}EXP}$

$\Rightarrow$ Approximating α_0 to within $\pm 1/N$ requires grid of size poly(N).

Function-TIH closely related to:

APX-SIM [Ambainis 2014]

Approximate the value of a local observable applied to a low energy state of a given local Hamiltonian.

[Gharibian, Yirka] [Gharibian, Piddock, Yirka]

[Bausch, Gharibian, Watson] $\Rightarrow$ translationally invariant.

$\Rightarrow$ oracle complexity classes.

FP^{NP} $\propto$ NP-optimization problem. (Krentel '86)

$f \in FP^{NEXP}$

f is computed by a poly-time TM that makes $m \leq$ poly # queries to an oracle for $L \in NEXP$.

verifier for L is V.

$x \longrightarrow H_x$ so that a good approximation of $\lambda_0(H_x)$ reveals $f(x)$.

$\min_y \mathrm{cost}(x,y) = \mathrm{cost}(x, \tilde{y})$

correct guesses.

$(x,y) \Rightarrow \underbrace{x_1 \, x_2 \cdots x_m}_{\text{inputs to oracle.}}$

If $y_i = 1$ (Guess = yes) $\rightarrow$ run verifier on x_i

if V returns no $\Rightarrow$

$\mathrm{cost} = 2^{m+1}$ (large)

If $y_i = 0$ $\mathrm{cost} = 2^{m-i}$

Since queries are adaptive costs must decrease exponentially $\Rightarrow$ costs depend on input.

cannot encode directly into fixed h.

Circuit-to-Hamiltonian Construction

Propagation Hamiltonian H_{prop} has ground state

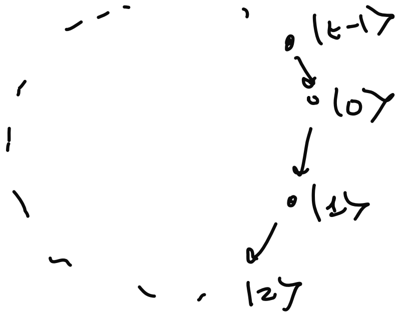
$$\sum_{t=0}^{L-1} \underbrace{|t\rangle}_{\text{clock}} \underbrace{|\phi_t\rangle}_{\text{computation.}}$$

$|\phi_t\rangle$ = state of computation after t-steps.

Typical Construction: $H_{prop} = 1/2 \cdot$ Laplacian of a path

+1 on diagonal, depending on if computation is rejecting.

Our Construction: cyclical clock:



$$H_{prop} = \begin{bmatrix} 1 & -\frac{1}{2} & & & & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & & & \\ & -\frac{1}{2} & 1 & -\frac{1}{2} & & \\ & & & \ddots & & \\ & & & -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & & & & -\frac{1}{2} & 1 \end{bmatrix} \updownarrow L$$

Every computation has two $+1/2$ penalty terms along the diagonal.

$$\lambda = \left(1 - \cos\frac{\pi}{L+1}\right) \Longrightarrow$$
exactly

Can calibrate λ by varying L.

Computational Process Embedded in the Hamiltonian.

Repeat T times

Regulated by timer that cycles back to time 0.

- Start with blank tape. → penalty term for incorrect start.
- Run binary counter TM for N steps → "x" written on input tape; x : input to checking TM.
- Run "checking" TM for N steps.
- Check tape contents. → penalty term.
- Run "checking" TM for N steps in reverse
- Run binary counter TM for N steps in reverse.

1 iteration takes $p(N)$ steps.

Cycle length $T \cdot p(N)$.

Checking TM Input $(x, \underbrace{y, w}_{\text{witness strings.}})$

- Simulate M on x using y for oracle responses
- Simulate V on x_i for every $y_i = 1$.

  Mark tape w/ "R" if <u>any</u> reject.

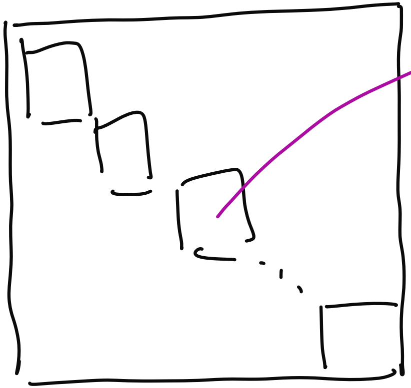
- Compute $\underbrace{T(x,y)}$ in unary.

Implements cost function & encodes $f(x,y)$.

Checking phase introduces a penalty for

- Rejecting computations.
- Timer $T \neq T(x,y)$.

Hamiltonian : Block Diagonal Structure



Each block specified by:

- start configuration.
- witness strings (y, w)
- Timer T : regulates # iterations

Block : $\frac{1}{2}\cdot$ Cycle Laplacian
+ some penalty
terms on diagonal.

"Periodic" costs happen once an iteration.

Penalize bad input.

Incorrect timer length.

Rejecting Verifier computations.

$$\overbrace{\begin{bmatrix} 1 & -\frac{1}{2} & & & & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & & & \\ & -\frac{1}{2} & 1 & -\frac{1}{2} & & \\ & & & \ddots & & \\ & & & -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & & & & -\frac{1}{2} & 1 \end{bmatrix}}^{L = p(N)\cdot T} + \begin{bmatrix} 1 & \overbrace{0 \cdots 0}^{p(N)} & & & & & \\ & & 1 & 0 \cdots & & & \\ & & & & 1 & 0 \cdots & \end{bmatrix}$$

+1 every $p(N)$ steps

$$\lambda \geq \frac{\text{const}}{p(N)^2}$$

Hilbert space divided into blocks for (x,y).

If y has an incorrect "yes" guess.

$\rightarrow \lambda$ for block $\geq \frac{c}{p(N)^2}$ periodic cost.

Otherwise

$$\lambda = \left(1 - \cos\left(\frac{\pi}{p(N)\cdot T(x,y)}\right)\right)$$

$$\sim \Theta\left(\frac{1}{p(N)^2\, T(x,y)^2}\right)$$

$T(x,y)$ maximized for correct y.

Given approx for $\lambda \Rightarrow$ can get $T(x,y) \Rightarrow f(x)$.

FP^{NEXP} vs. $FP^{QMA\text{-}EXP}$.

$NEXP \subseteq QMA\text{-}EXP\text{-}L \subseteq QMA\text{-}EXP$.

quantum witness
quantum verifier
no invalid instances.

Function-TIH is hard for $FP^{QMA\text{-}EXP\text{-}L}$.

Function-GED is hard for $FEXP^{QMA_1\text{-}EXP\text{-}L}$.

Circuit-to-Hamiltonian simulates verifier for oracle language on queries.

Invalid queries $\longrightarrow$ unknown, uncontrolled min energy.

Open Question: Complexity of Function-local Hamiltonian

(even in the non-translationally invariant case)

Thank You!