Graph Sparsification II: Rank one updates, Interlacing, and Barriers

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Previous Lecture

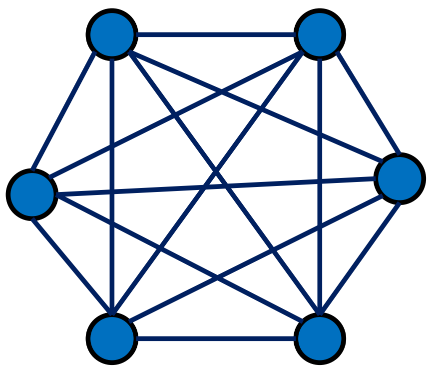
Definition. $H = (V, F, u)$ is a κ −approximation of $G = (V, E, w)$ if:

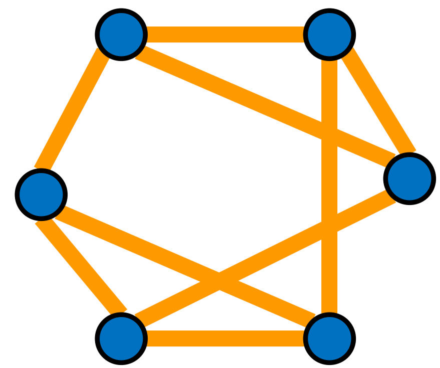
$$L_H \preccurlyeq L_G \preccurlyeq \kappa \cdot L_H$$

Theorem. Every G has a $(1+\epsilon)$ −approximation H with $O(n \log n/\epsilon^2)$ edges. There is a nearly linear time algorithm which finds it.

There is no log(n) here…

G=K$_n$ **H** = random d-regular x **(n/d)**





$|E_G| = O(n^2)$ $|E_H| = O(dn)$

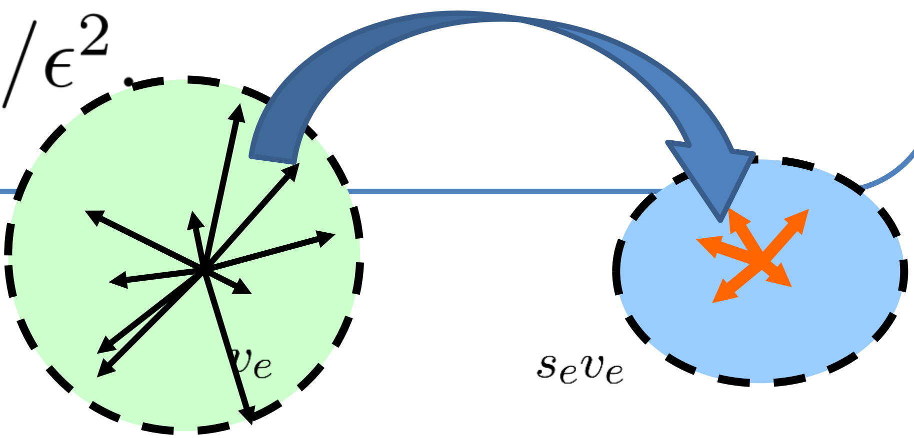
$$d = 2/\epsilon^2$$

$$\forall x, \quad \frac{x^T L_G x}{x^T L_H x} \simeq 1 \pm \epsilon$$

Proof: Approximating the Identity

Given $\sum_{i \leq m} v_i v_i^T = I_n$ there are $s_i \geq 0$ with:

- $(1-\epsilon)I \preceq \sum_i s_i v_i v_i^T \preceq (1+\epsilon)I$
- $\mathrm{supp}(s) \leq n \log n / \epsilon^2$.



Tool: The Matrix Chernoff Bound

Suppose $X_1, \dots, X_k$ are i.i.d. random $n \times n$ **matrices** with

$0 \preccurlyeq X_i \preccurlyeq M \cdot I$ and $\mathbb{E} X_i = I$.

Then

$$\mathbb{P}\left[\left\|\frac{1}{k}\sum_i X_i - I\right\| \geq \epsilon\right] \leq 2\boldsymbol{n} \exp\left(-\frac{k\epsilon^2}{4M}\right)$$

Shows $O\left(\frac{n \log n}{\epsilon^2}\right)$ samples suffice in $\boldsymbol{R}^n$.

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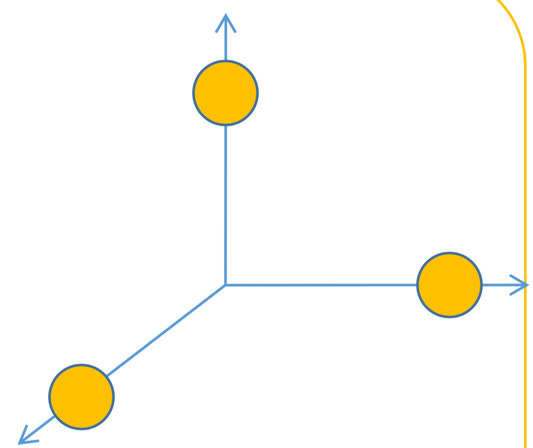
then $O\left(\frac{n \log n}{\epsilon^2}\right)$ samples suffice in $\boldsymbol{R}^n$.

Tight example:

$$X = \sqrt{n} e_i e_i^T \qquad \text{w. prob. } 1/n$$

$$\mathbf{E}X = (1/n) \sum_{i \le n} n e_i e_i^T = I$$

$$\Sigma_q(i,i) = \text{num. of balls in bin } i$$



Tool: The Matrix Chernoff Bound

Suppo ... matrices with

... $X_i = I$.

then $O\Big($ ... ϵ^2 ... $\Big)$ ... e in $\boldsymbol{R}^n$.

Simple greedy algorithm gets $O(n)$:
place next ball in emptiest bin

Tight example:

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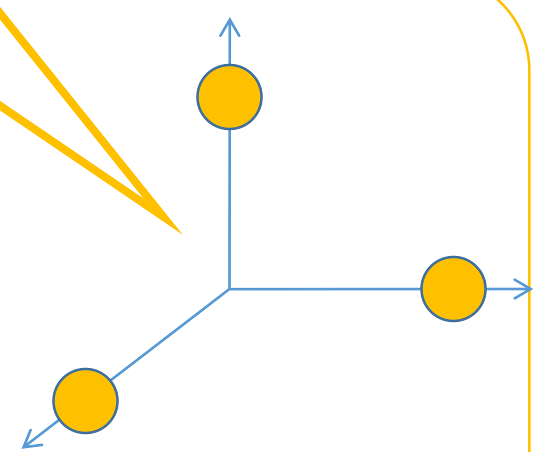
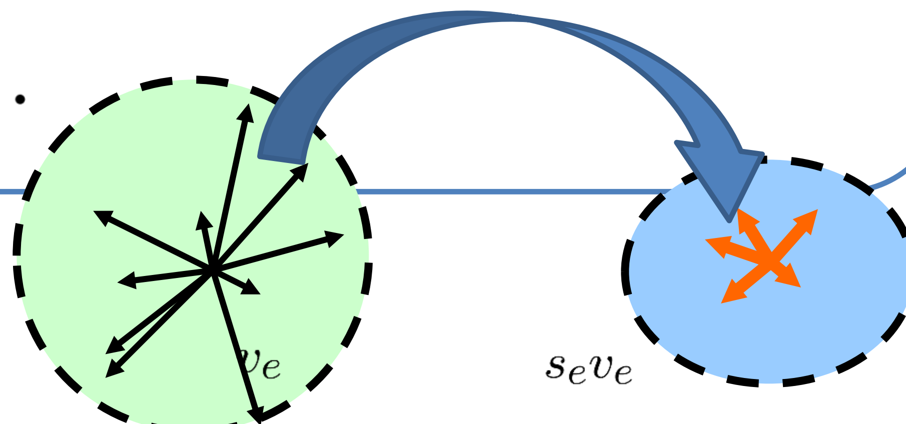


This Lecture [Batson-Spielman-S'09]

Spectral Sparsification Theorem:

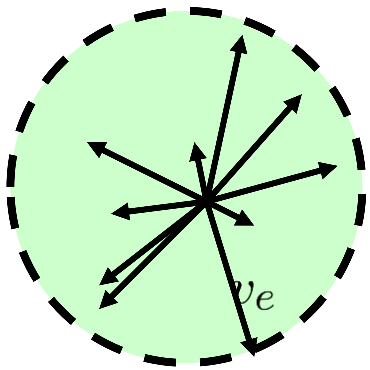
Given $\sum_{i \le m} v_i v_i^T = I_n$ there are $s_i \geq 0$ with:

- $(1-\epsilon)I \preceq \sum_i s_i v_i v_i^T \preceq (1+\epsilon)I$
- $\mathrm{supp}(s) \leq 4n/\epsilon^2$.



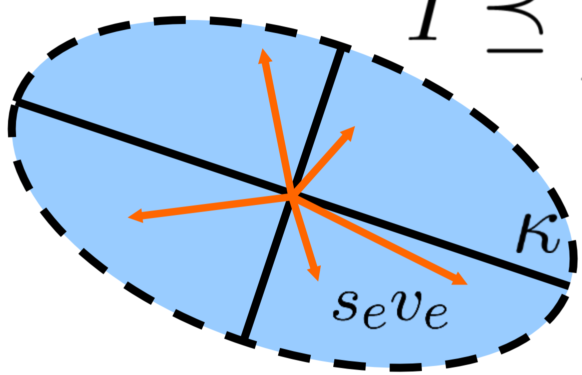
uses greedy approach.

This is the goal

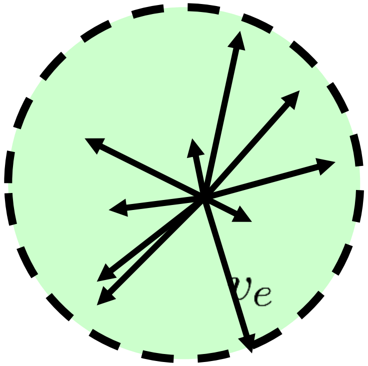


$$I = \sum_{i \leq m} v_i v_i^T$$

$$I \preceq \sum_i s_e v_i v_i^T \preceq \kappa I$$

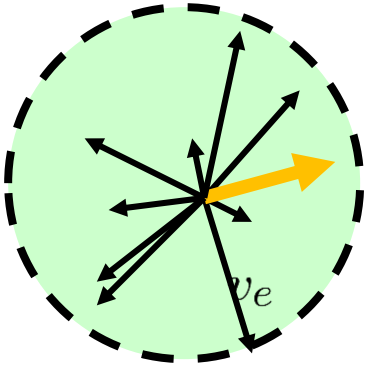


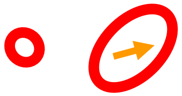
Plan: Choose vectors one at a time



$$A = 0$$

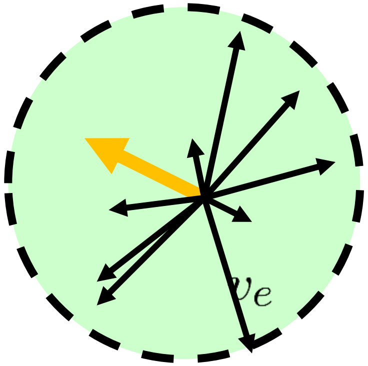
Plan: Choose vectors one at a time

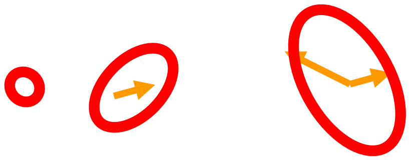




$$A = s_{e_1} v_{e_1} v_{e_1}^T$$

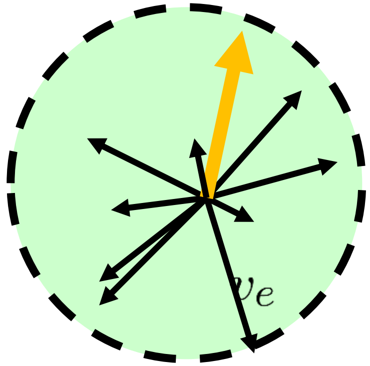
Plan: Choose vectors one at a time

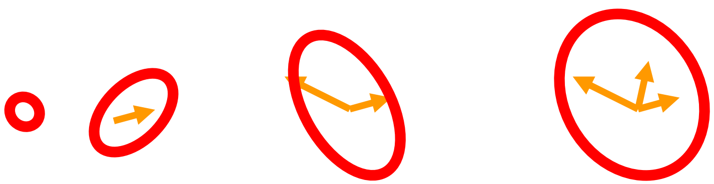




$$A = s_{e_1} v_{e_1} v_{e_1}^T + s_{e_2} v_{e_2} v_{e_2}^T$$

Plan: Choose vectors one at a time





$$A = s_{e_1} v_{e_1} v_{e_1}^T + s_{e_2} v_{e_2} v_{e_2}^T + s_{e_3} v_{e_3} v_{e_3}^T$$

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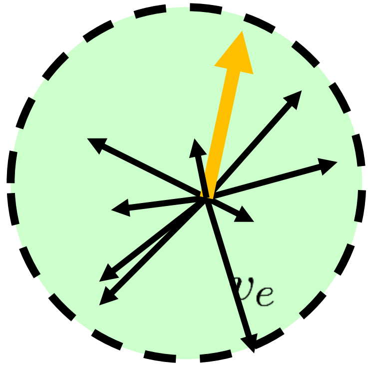

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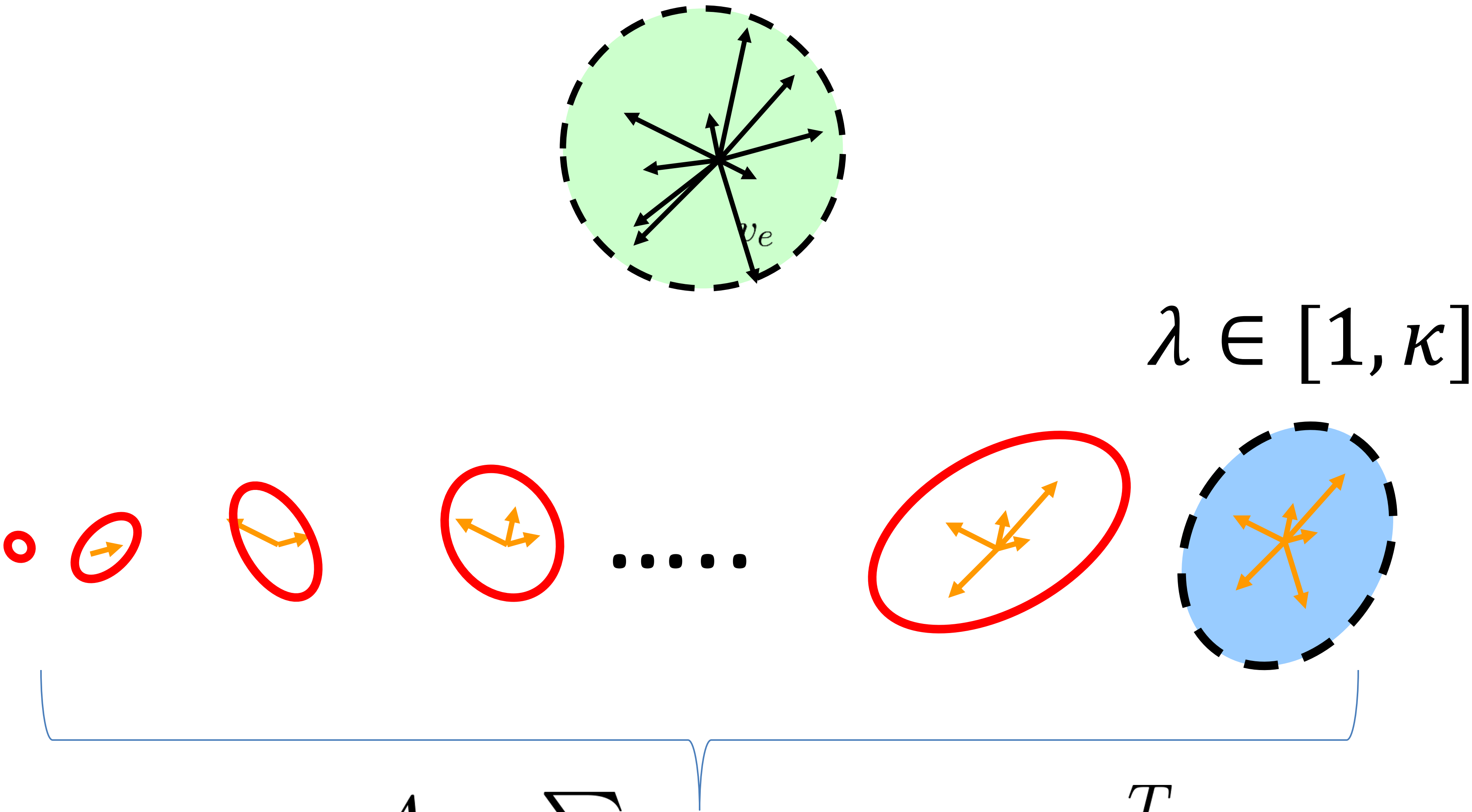
Plan: Choose vectors one at a time

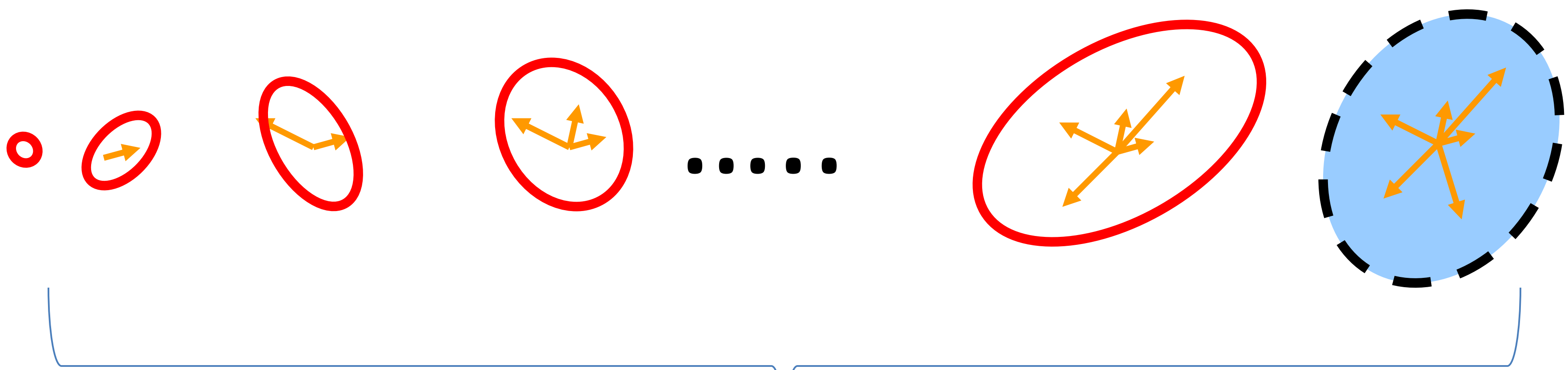


$$A = \sum_{j \leq O(n)} s_{e_j} v_{e_j} v_{e_j}^T$$

Plan: Choose vectors one at a time

Basic Question: What does a rank one update do to the eigenvalues?

$\lambda \in [1, k]$

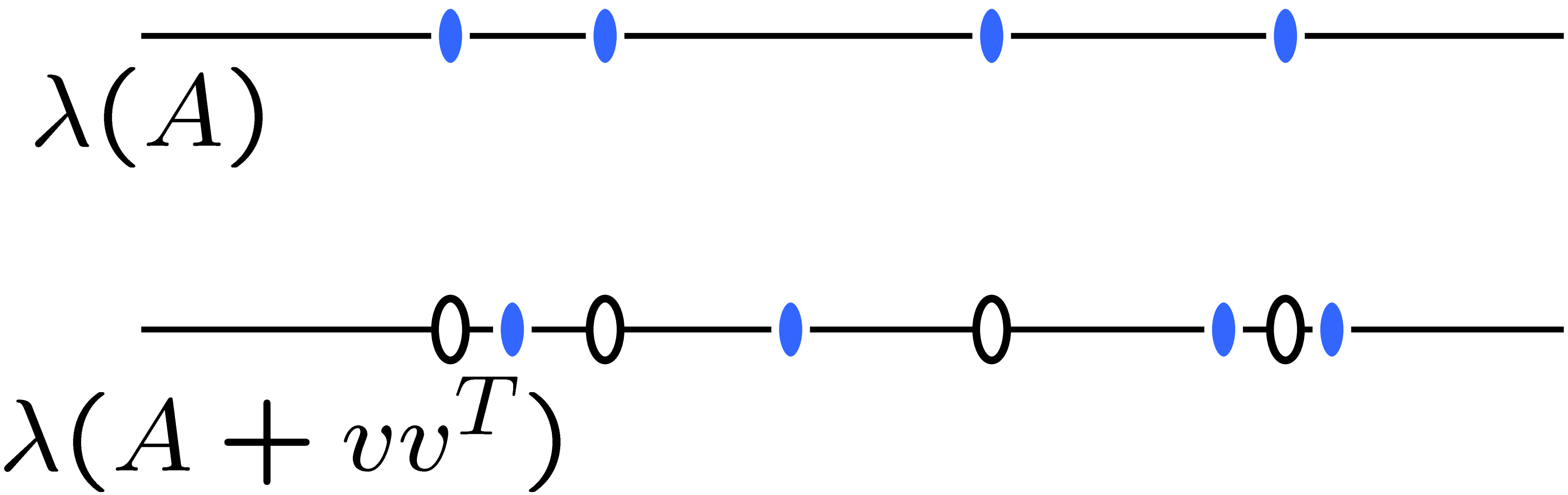


$$A = \sum_{j \leq O(n)} s_{e_j} v_{e_j} v_{e_j}^T$$

What happens when you add a vector?

$\lambda(A)$

Interlacing (Cauchy, 1800s)



The Characteristic Polynomial

Characteristic Polynomial:

$$p_A(x) = \det(xI - A)$$

$$p_A(x) = \prod_i (x - \lambda_i)$$

where $\lambda_1, \ldots, \lambda_n$ = eigs(A).

Proof of Interlacing I

Proof of Interlacing II

Proof of Interlacing III

The Characteristic Polynomial

Characteristic Polynomial:

$$p_A(x) = \det(xI - A)$$

Matrix-Determinant Lemma:

$$p_{A+vv^T} = p_A \left(1 + \sum_i \frac{\langle v, u_i \rangle^2}{\lambda_i - x}\right)$$

The Characteristic Polynomial

Characteristic Polynomial:

$$p_A(x) = \det(xI - A)$$

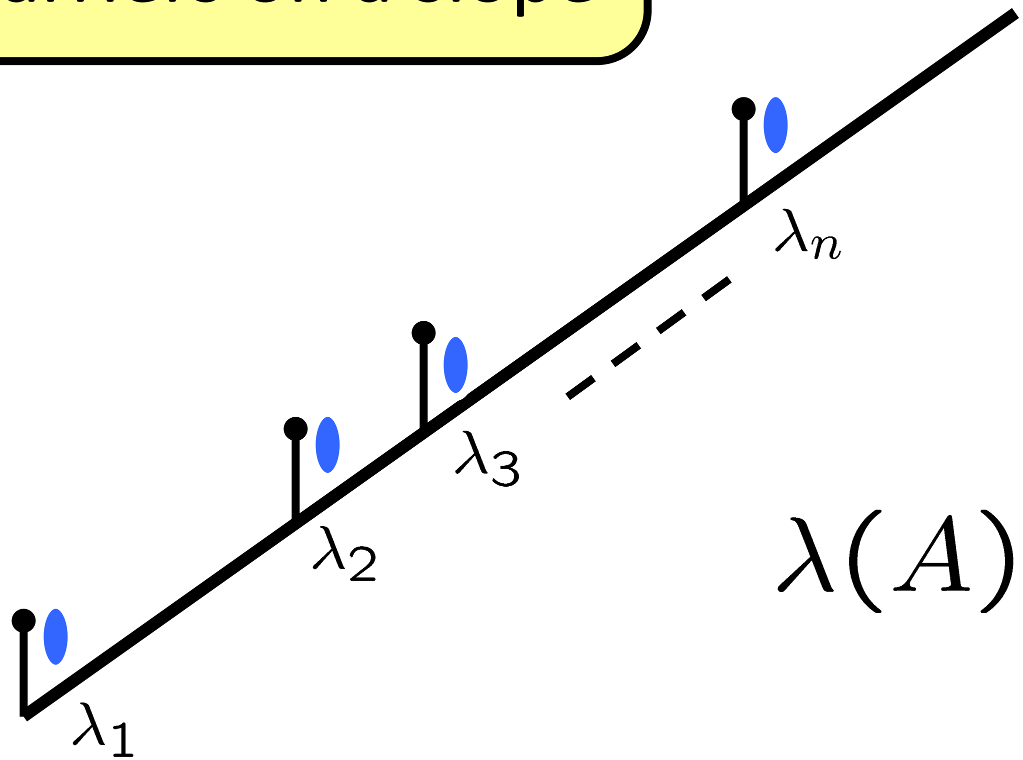
$\lambda(A + vv^T)$
are zeros of this.

Matrix-Determinant Le

$$p_{A+vv^T} = p_A \left(1 + \sum_i \frac{\langle v, u_i \rangle^2}{\lambda_i - x}\right)$$

Physical model of interlacing

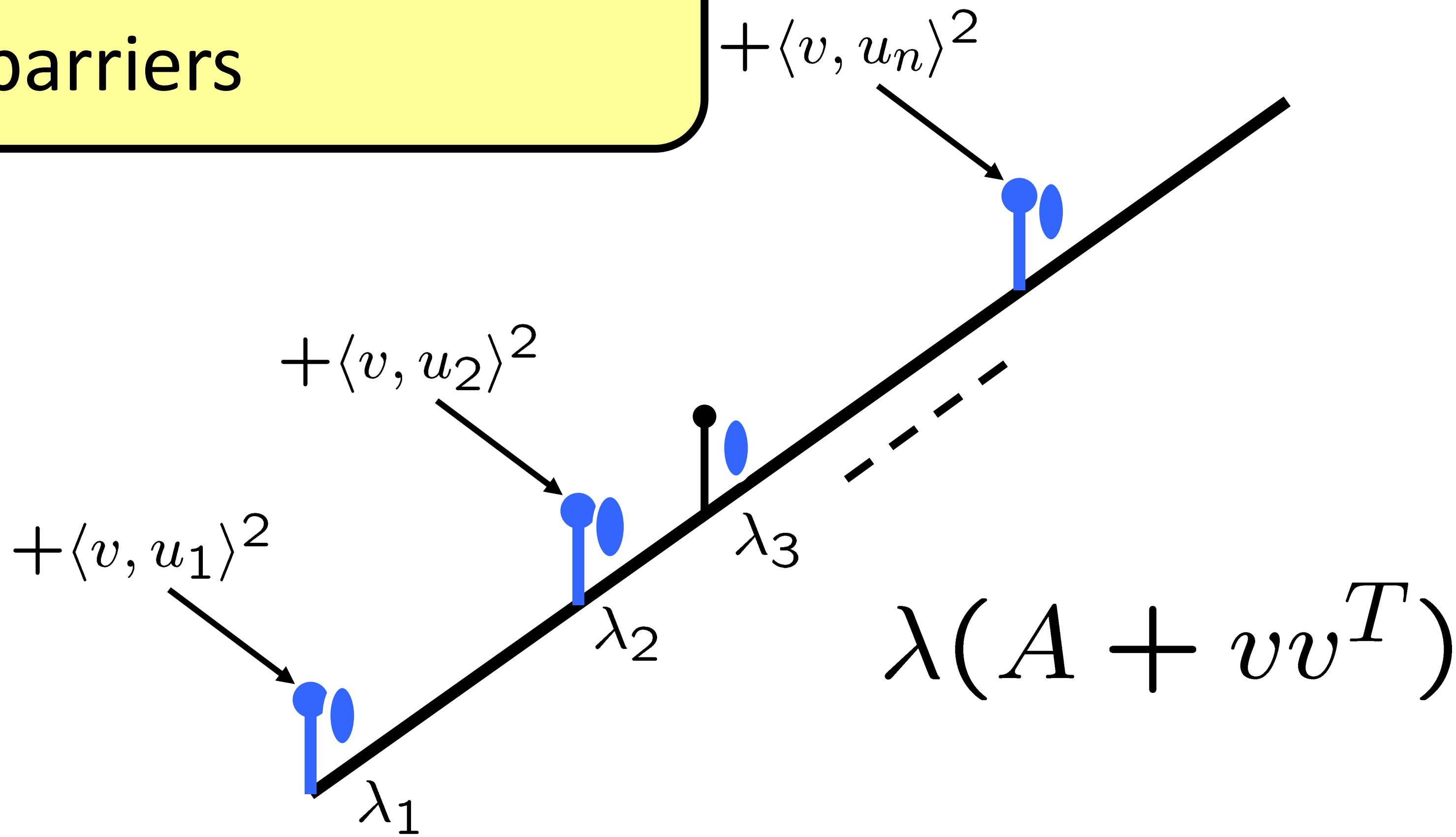
λ_i = positive unit charges resting at barriers on a slope



$$\lambda(A)$$

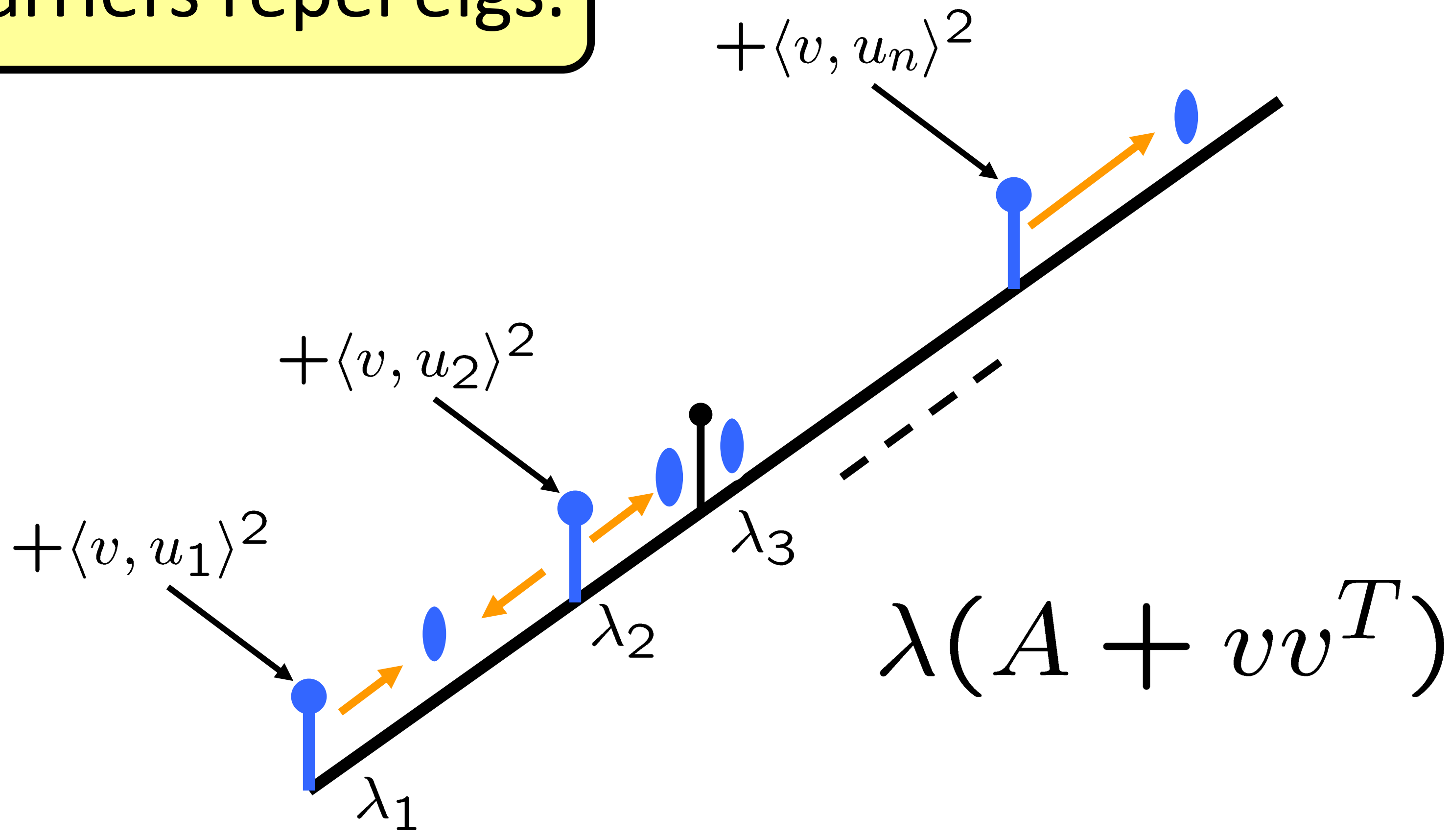
Physical model of interlacing

$\langle v, u_i \rangle^2$ =charges added to barriers



Physical model of interlacing

Barriers repel eigs.



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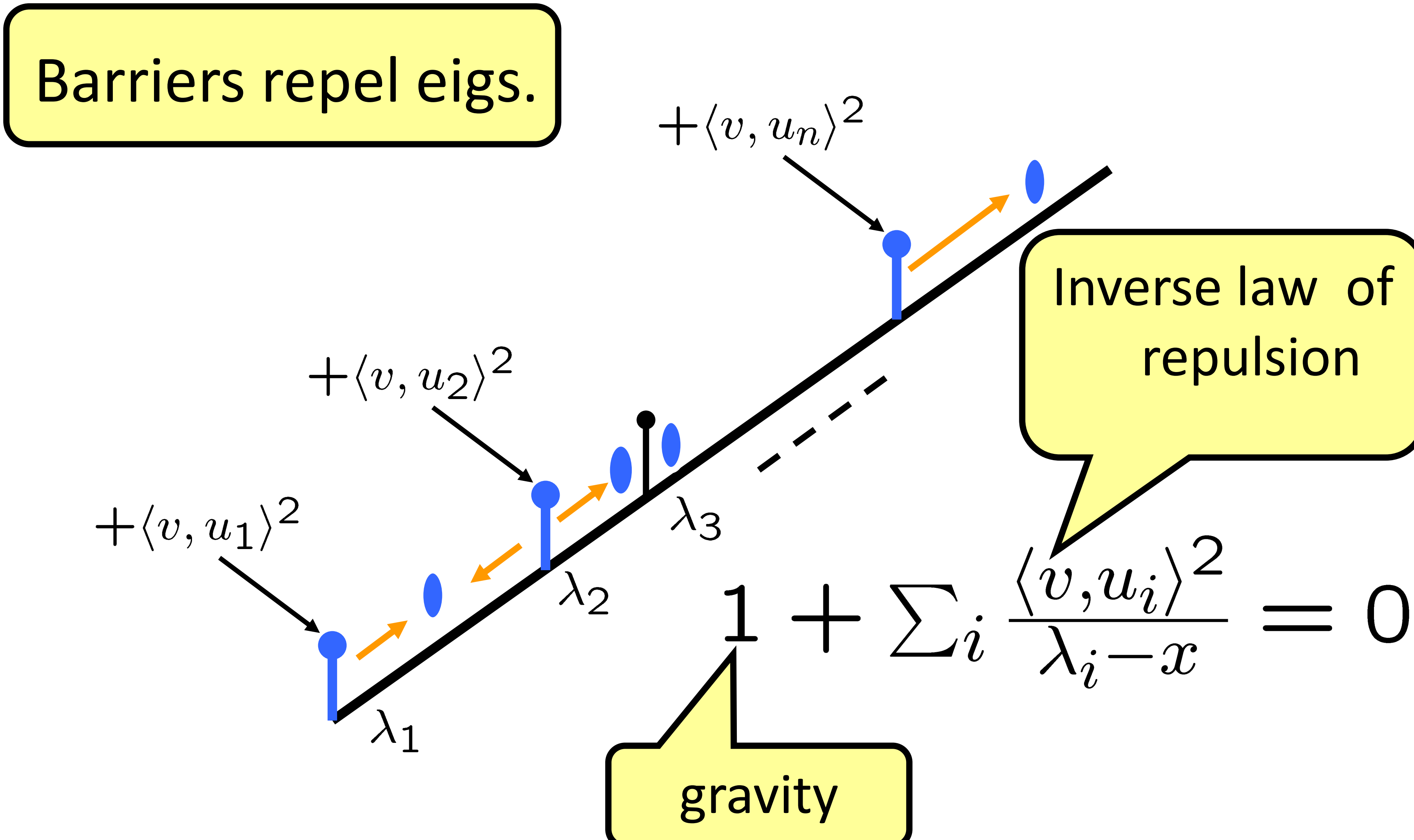

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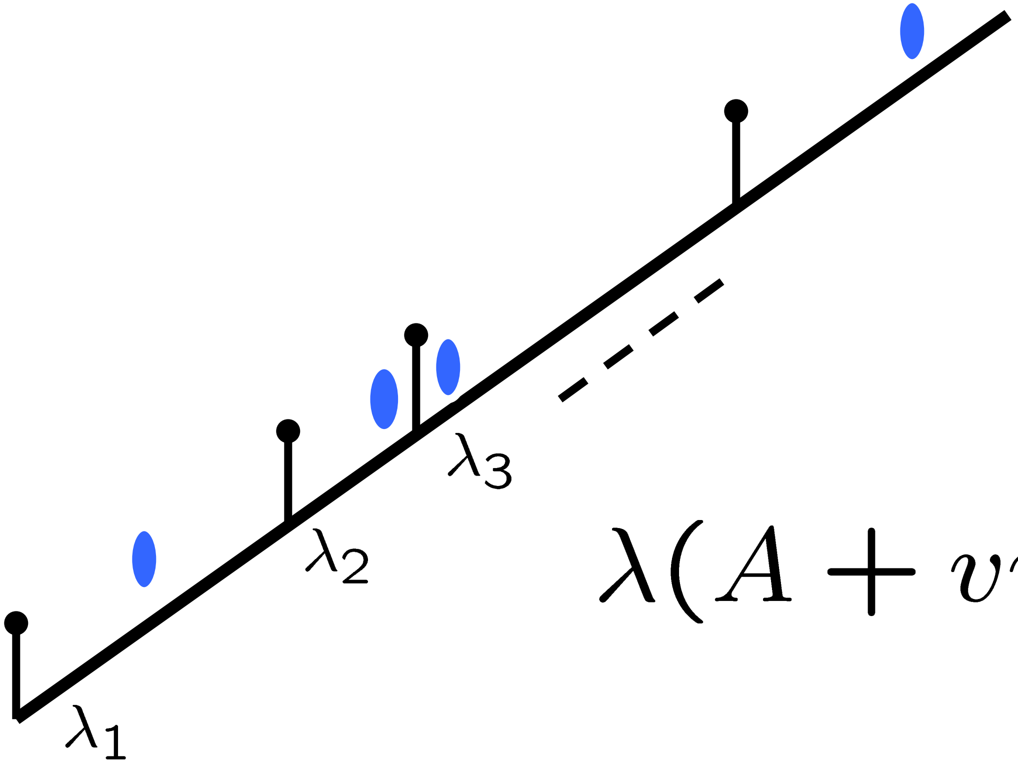
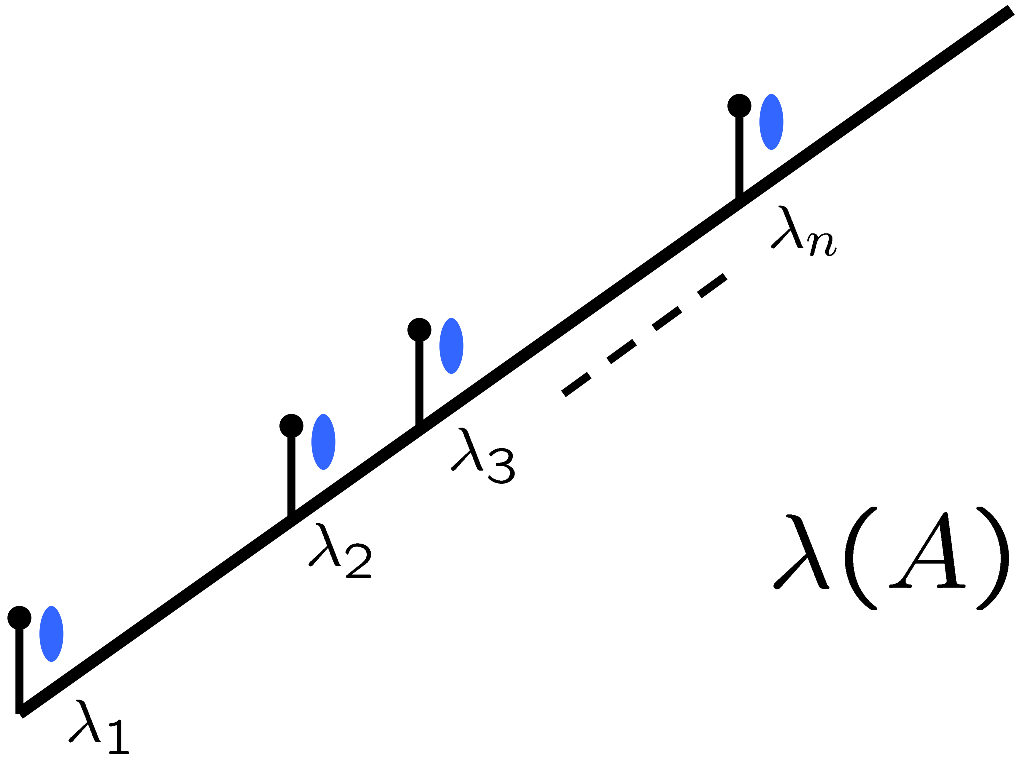
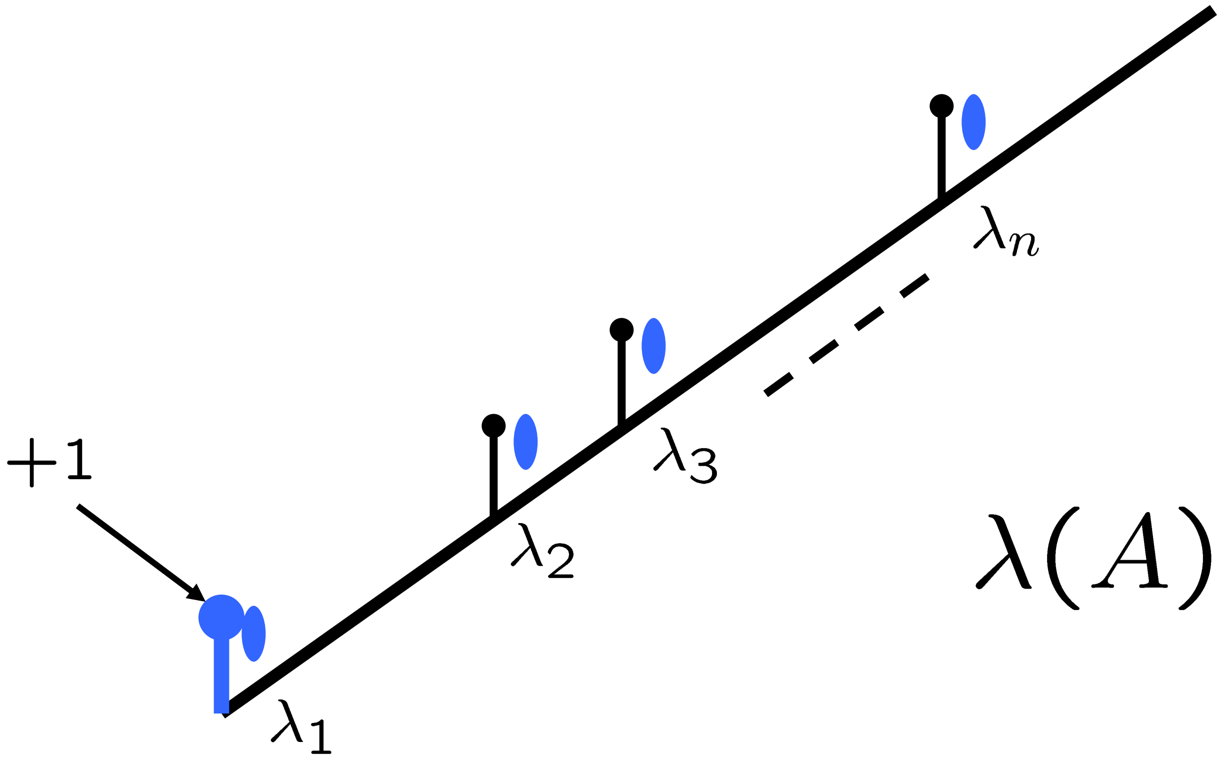


$$\lambda(A + vv^T)$$

Examples



Ex1: All weight on $\boldsymbol{u_1}$



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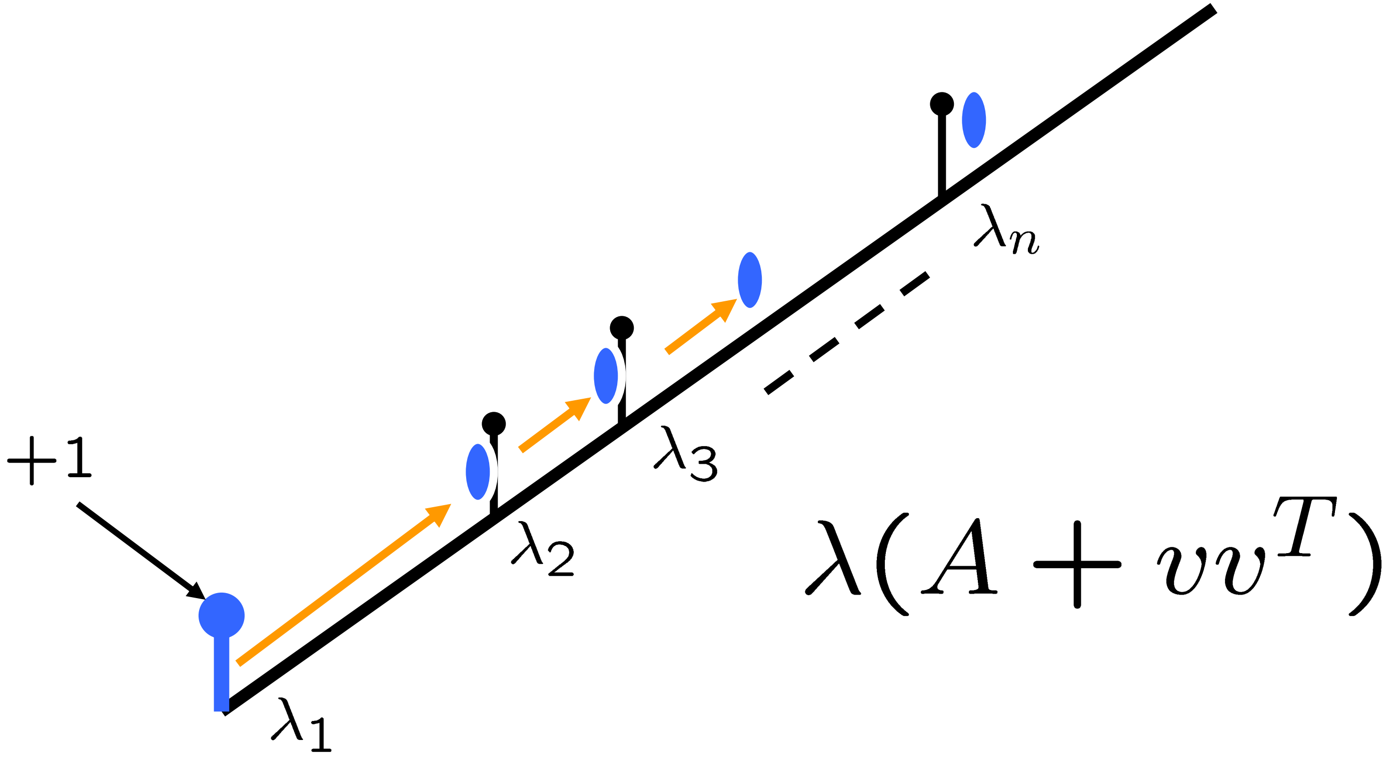

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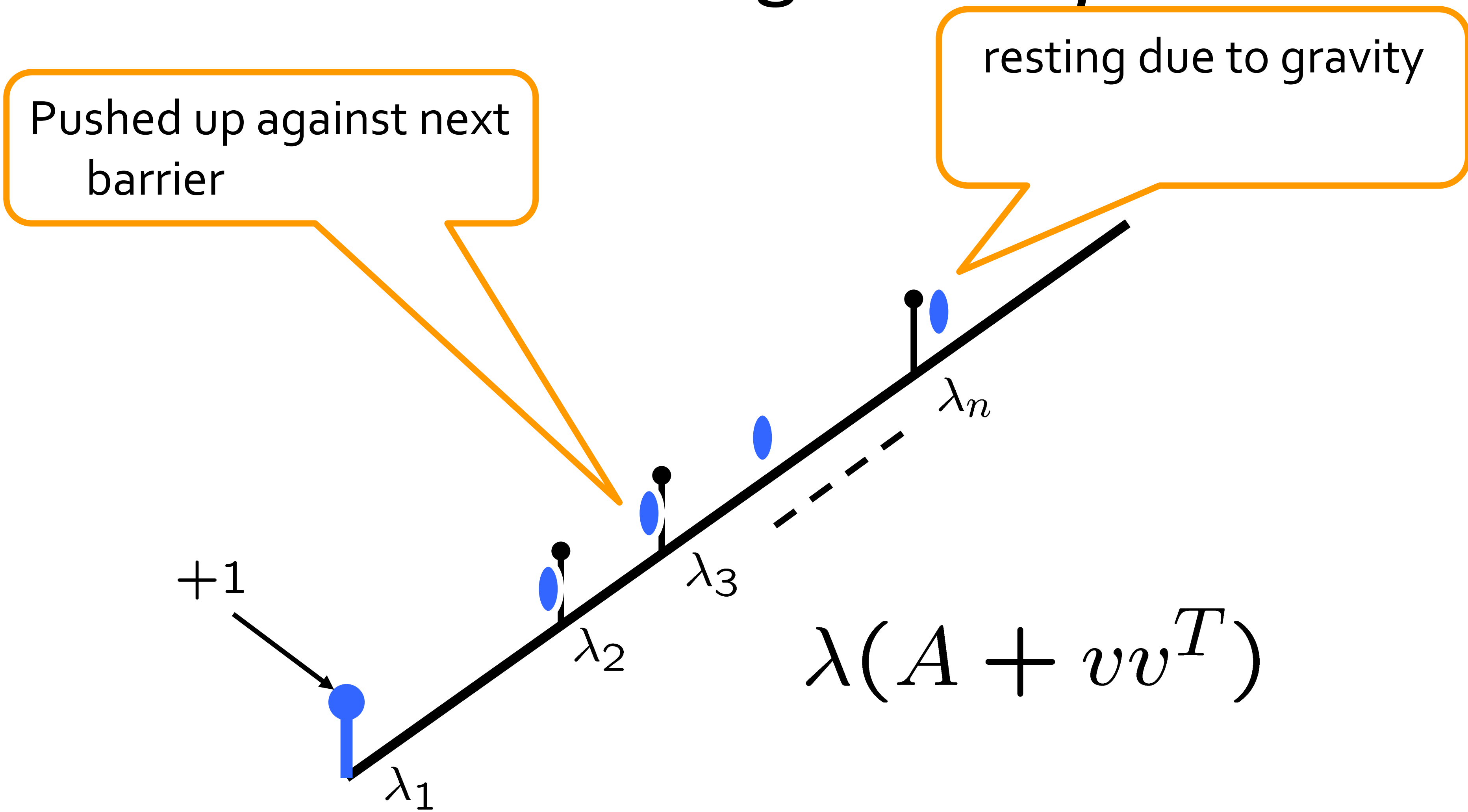
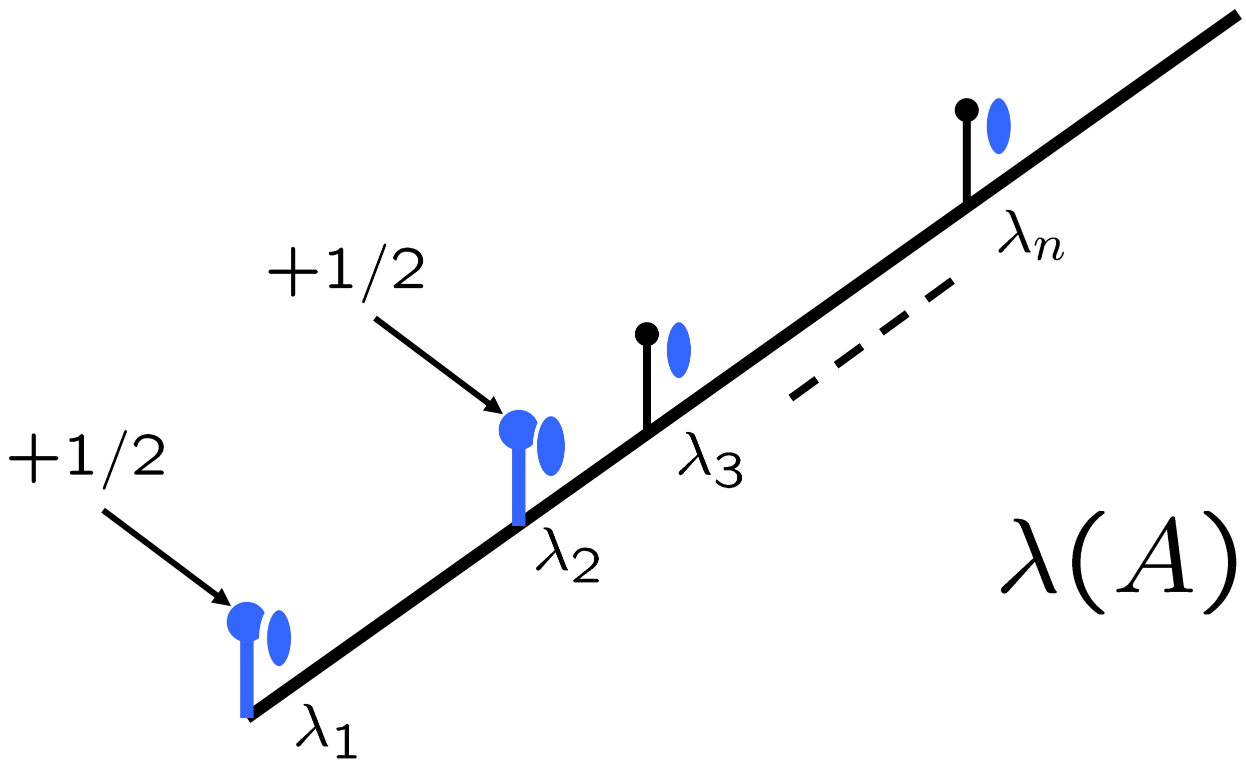
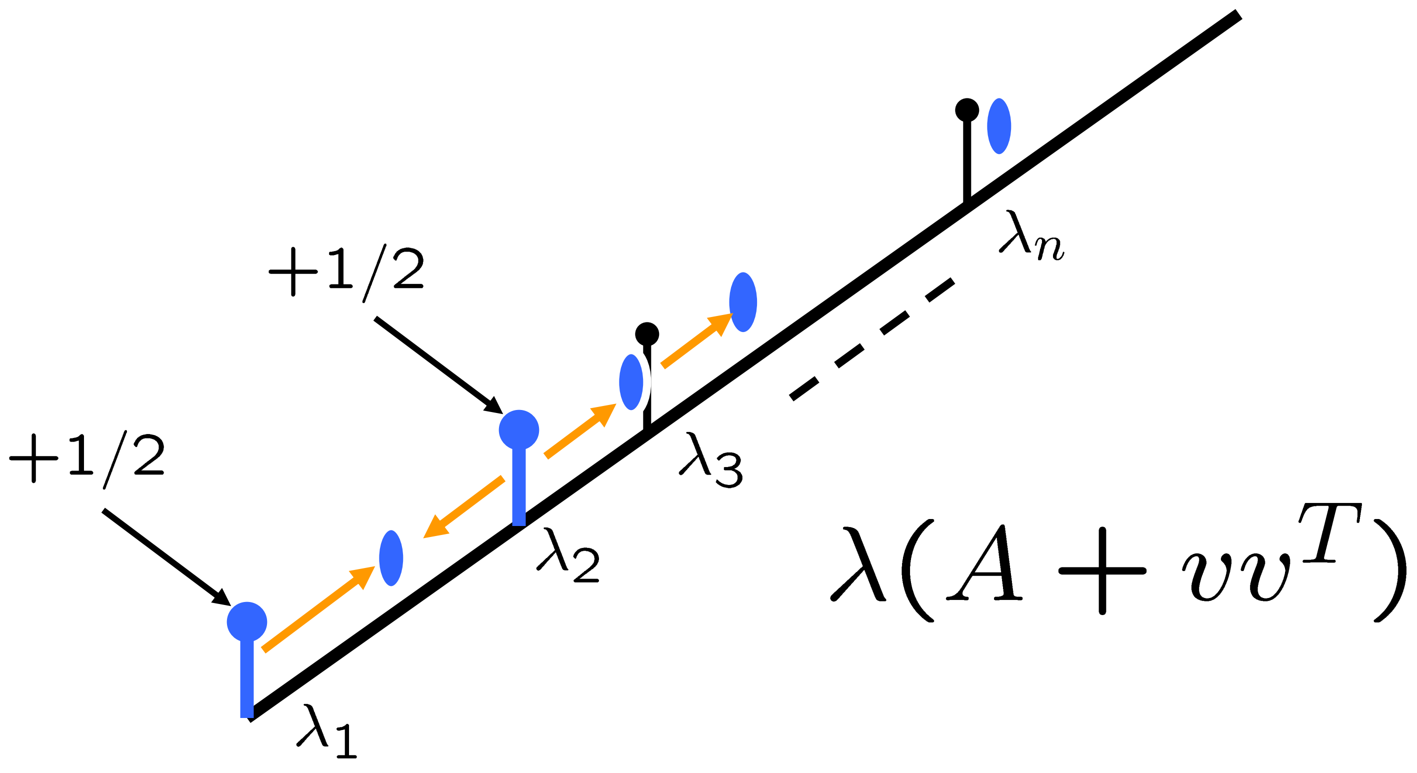


Ex2: Equal weight on u_1, u_2



Ex2: Equal weight on $\boldsymbol{u_1}$,$\boldsymbol{u_2}$



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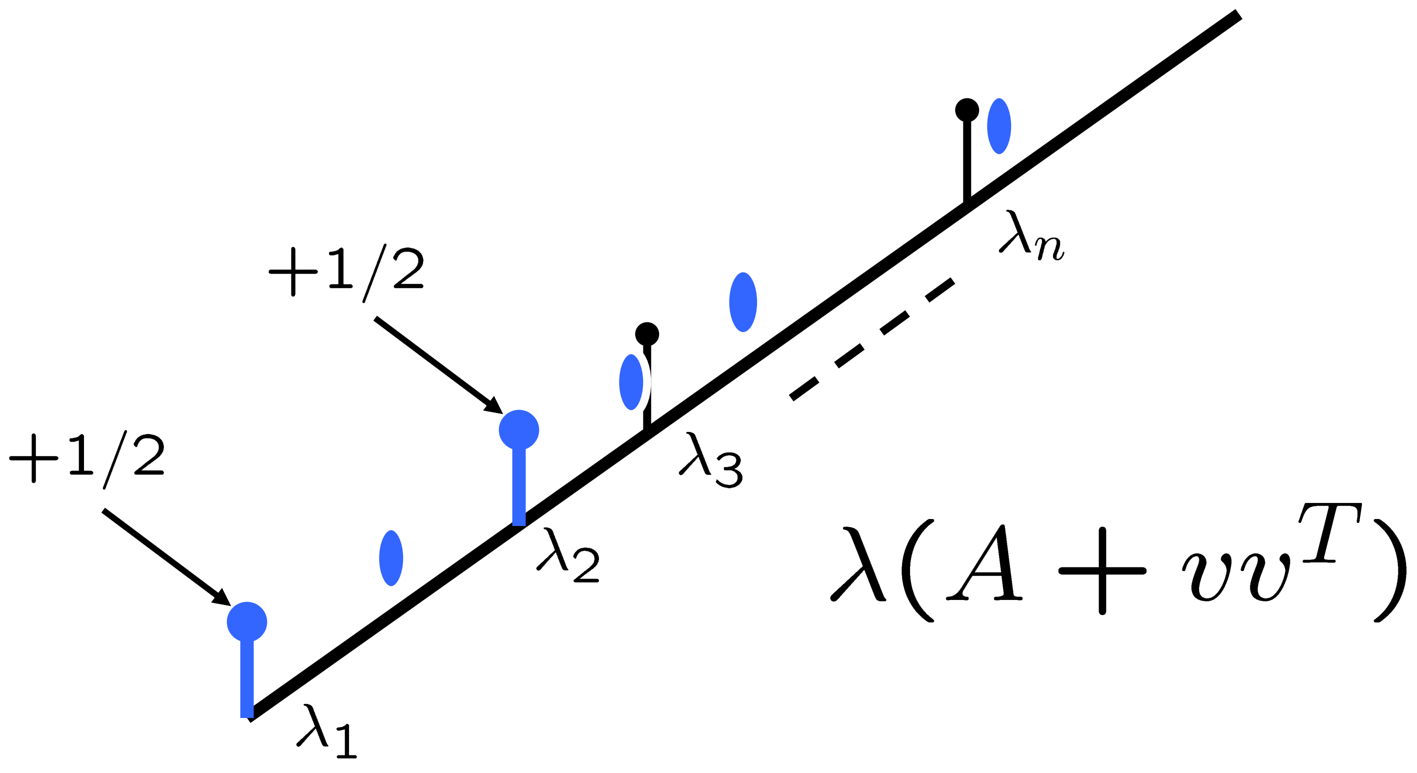
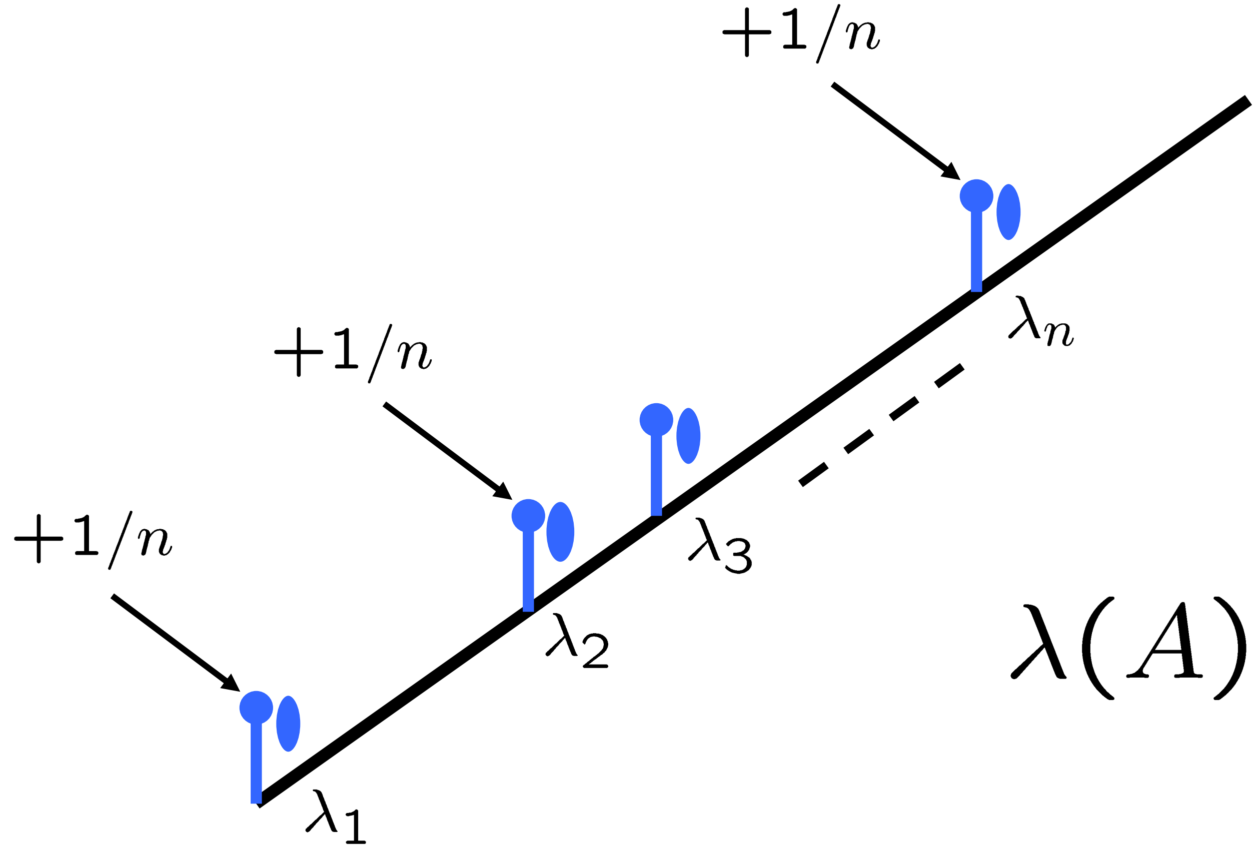


Ex3: Equal weight on all $\boldsymbol{u_1}$, $\boldsymbol{u_2}$, ...$\boldsymbol{u_n}$



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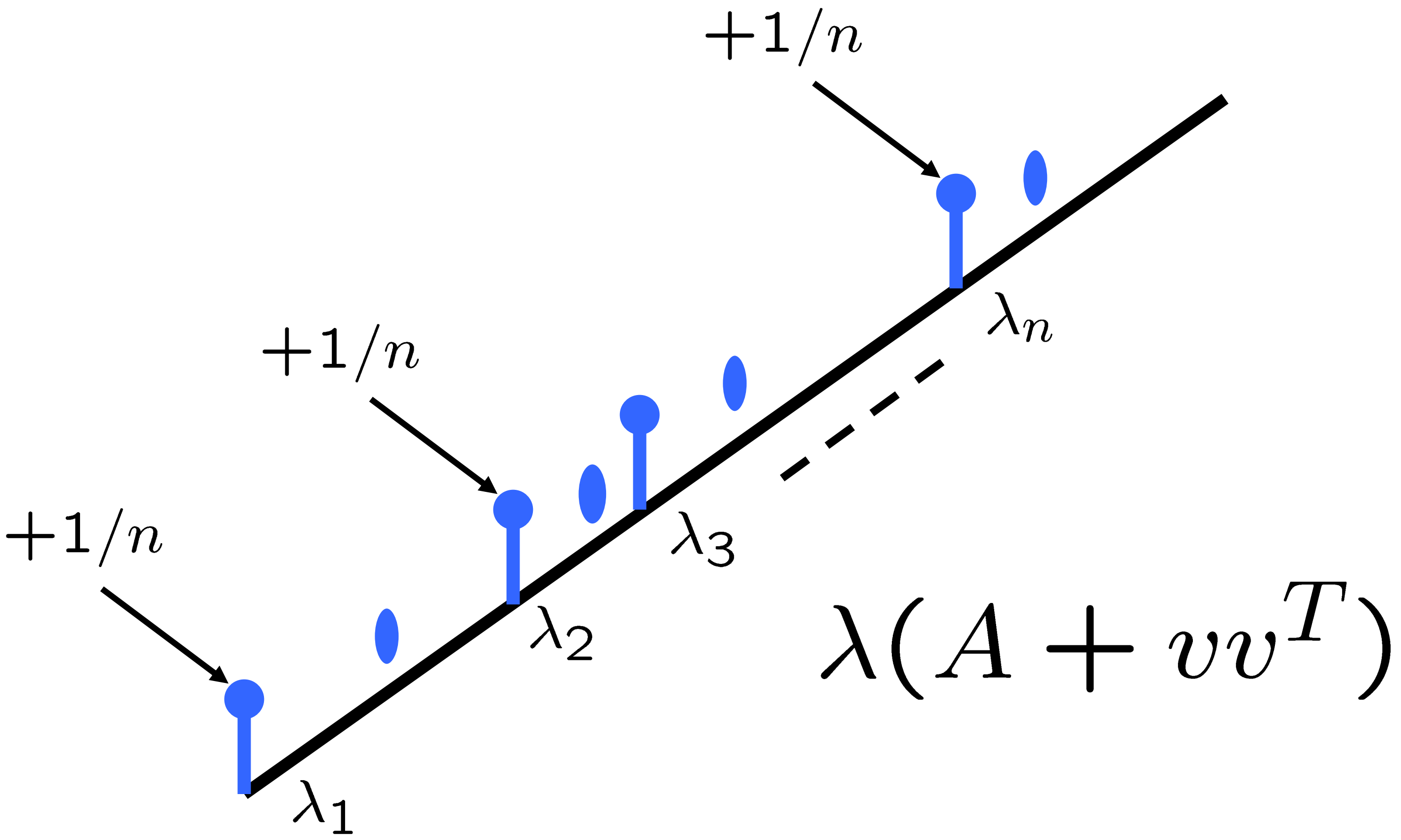


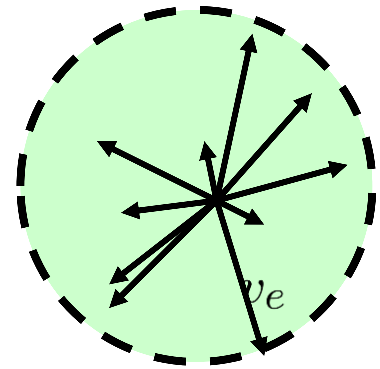
Adding a balanced vector

$$
\begin{aligned}
p_{A+vv^t} &= p_A \left(1 + \sum_i \frac{\langle v, u_i \rangle^2}{\lambda_i - x} \right) \\
&= p_A \left(1 + \sum_i \frac{1}{\lambda_i - x} \right) \\
&= p_A - p'_A
\end{aligned}
$$

Consider a random vector

If

$$\sum_e v_e v_e^T = I$$



For every u_i : $\sum_e \langle v_e, u_i \rangle^2 = 1.$

thus a random vector has the same expected projection in *every* direction ***i*** :

$$\mathbb{E}_e \langle v_e, u_i \rangle^2 = 1/m$$

The Expected Characteristic Poly.

$A^{(0)} = 0$

$p^{(0)} = x^n$

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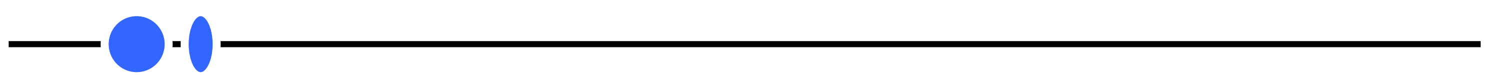
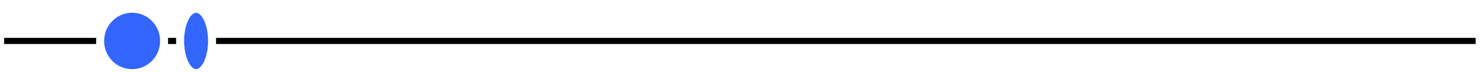


$$A^{(1)} = vv^T$$

$$\mathbb{E}p^{(1)} = p^{(0)} - \frac{1}{m}\frac{\partial}{\partial x}p^{(0)}$$

The Expected Characteristic Poly.

$$\mathbb{E}_e \langle v_e, u_i \rangle^2 = 1/m$$



$$A^{(1)} = vv^T$$

$$\mathbb{E}p^{(1)} = p^{(0)} - \frac{1}{m}\frac{\partial}{\partial x}p^{(0)} = x^n - \frac{n}{m}x^{n-1}$$

The Expected Characteristic Poly.

$$\mathbb{E}_e \langle v_e, u_i \rangle^2 = 1/m$$

$$A^{(2)} = A^{(1)} + vv^T$$

$$\mathbb{E}p^{(2)} = \left(1 - \frac{1}{m}\frac{\partial}{\partial x}\right)p^{(1)} = \left(1 - \frac{1}{m}\frac{\partial}{\partial x}\right)^2 x^n$$

The Expected Characteristic Poly.

$$\mathbb{E}_e \langle v_e, u_i \rangle^2 = 1/m$$

$$A^{(3)} = A^{(2)} + vv^T$$

$$\mathbb{E}p^{(3)} = \left(1 - \frac{1}{m}\frac{\partial}{\partial x}\right)^3 x^n$$

Ideal proof

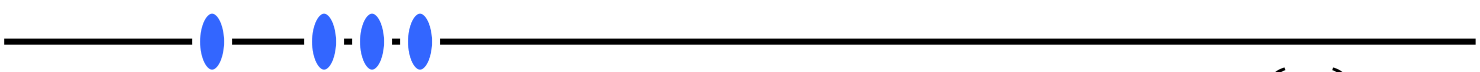
$$\mathbb{E}_e \langle v_e, u_i \rangle^2 = 1/m$$

$$roots(\mathbb{E}p^{(k)})$$

$$\mathbb{E}p^{(k)} = \left(1 - \frac{1}{m}\frac{\partial}{\partial x}\right)^k x^n$$

Ideal proof

$$\mathbb{E}_e \langle v_e, u_i \rangle^2 = 1/m$$



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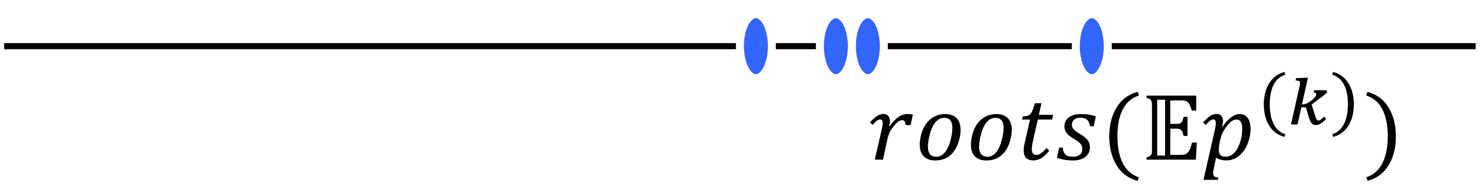

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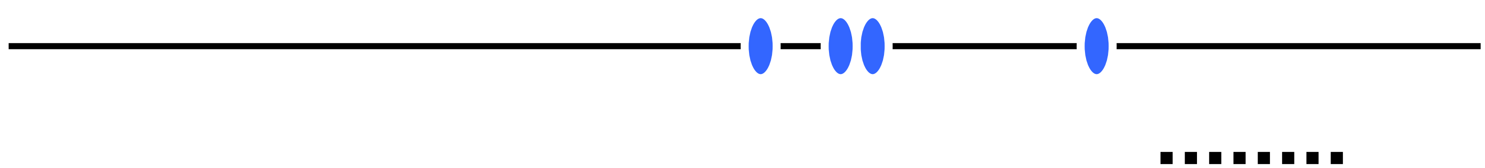


$$\mathbb{E} p^{(k)} = \left(1 - \frac{1}{m} \frac{\partial}{\partial x}\right)^k x^n$$

Ideal proof

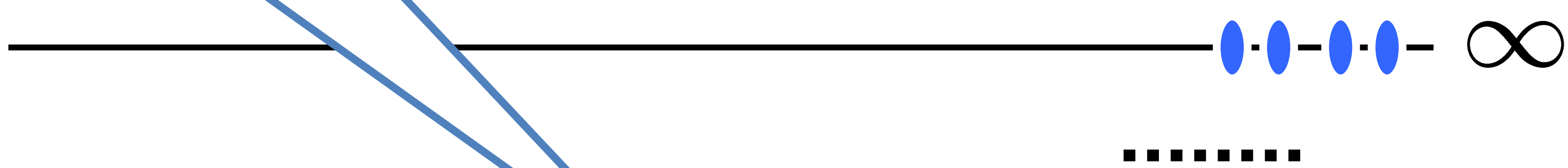
$$\mathbb{E}_e \langle v_e, u_i \rangle^2 = 1/m$$

$$\mathbb{E}p^{(k)} = \left(1 - \frac{1}{m}\frac{\partial}{\partial x}\right)^k x^n$$

Ideal proof

$$\frac{\lambda_n(p^{(k)})}{\lambda_1(p^{(k)})} \leq \kappa?$$

$$\mathbb{E}_e \langle v_e, u_i \rangle^2 = 1/m$$



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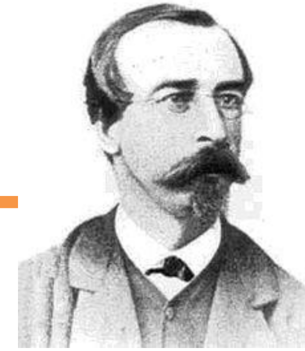
Ideal proof

$$\frac{\lambda_n(p^{(k)})}{\lambda_1(p^{(k)})} \leq \kappa?$$

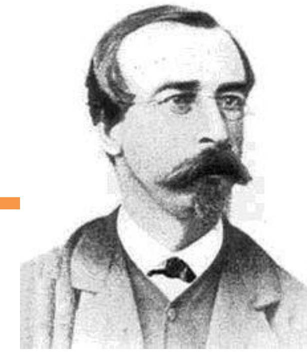


$p^{(k)}$ = Laguerre poly $\mathcal{L}^k$

∞

$$\mathbb{E}p^{(k)} = \left(1 - \frac{1}{m}\frac{\partial}{\partial x}\right)^k x^n$$

Ideal proof



$$\frac{\lambda_n(p^{(k)})}{\lambda_1(p^{(k)})} \leq 1 + \epsilon$$

after $4n/\epsilon^2$ steps.

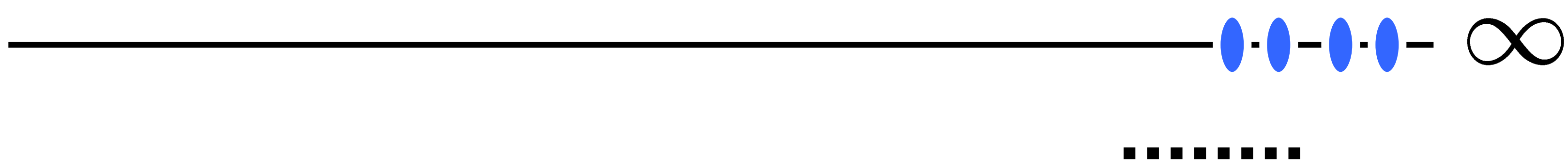
$p^{(k)}$= Laguerre poly $\mathcal{L}^k$

∞

$$\mathbb{E}p^{(k)} = \left(1 - \frac{1}{m}\frac{\partial}{\partial x}\right)^k x^n$$

This is not real

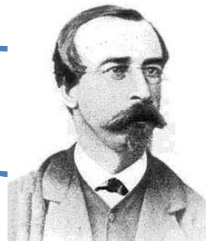
Problem: $roots\left(\mathbb{E}p^{(k)}\right) \neq \mathbb{E}roots\left(p^{(k)}\right)$



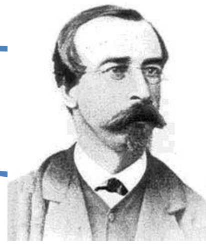
$$\mathbb{E}p^{(k)} = \left(1 - \frac{1}{m}\frac{\partial}{\partial x}\right)^k x^n$$

$$\mathbb{E}p^{(k)} = \left(1 - \frac{1}{m}\frac{\partial}{\partial x}\right)^k x^n$$

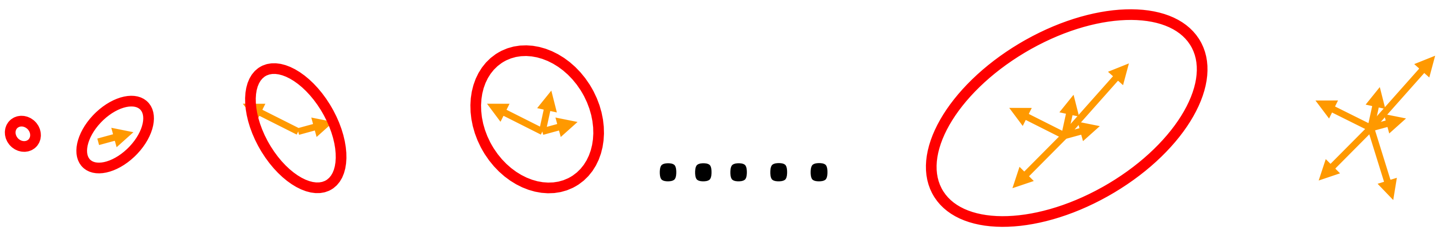
$$4n/\epsilon^2$$

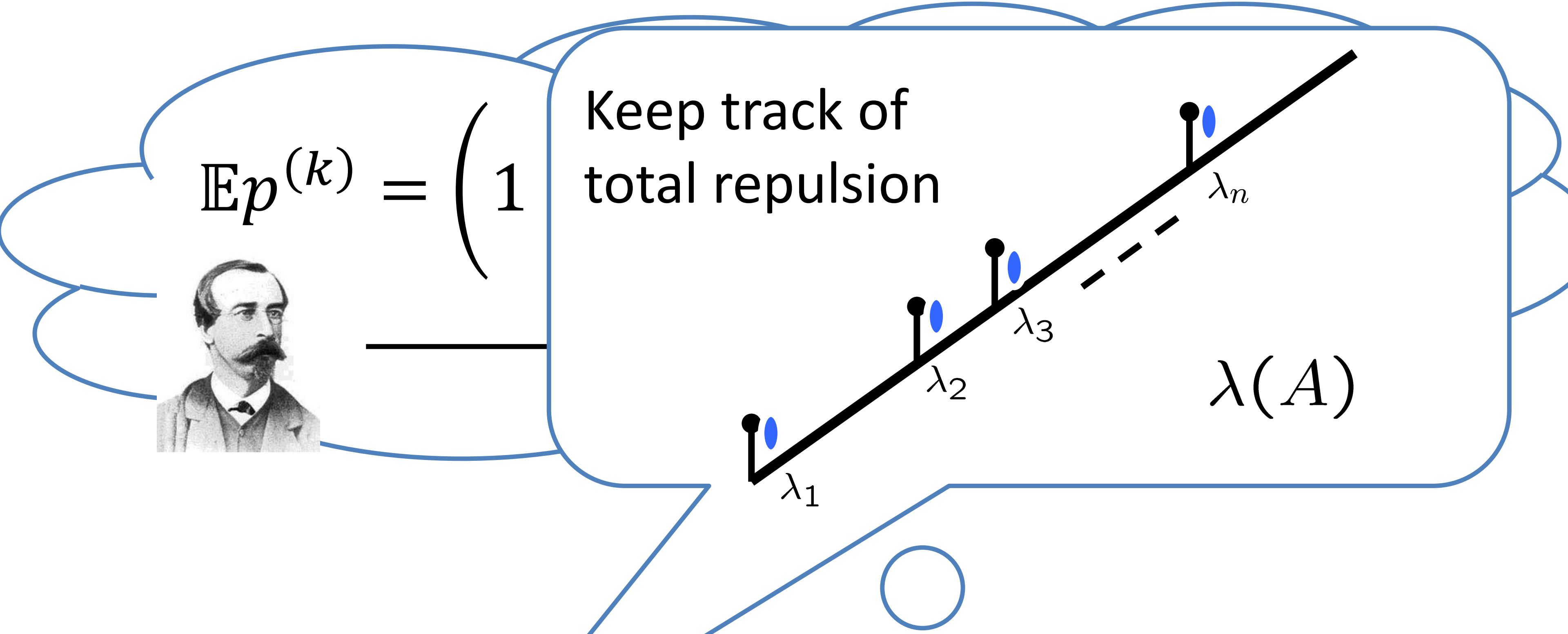


$$\mathbb{E}p^{(k)} = \left(1 - \frac{1}{m}\frac{\partial}{\partial x}\right)^k x^n \qquad 4n/\epsilon^2$$

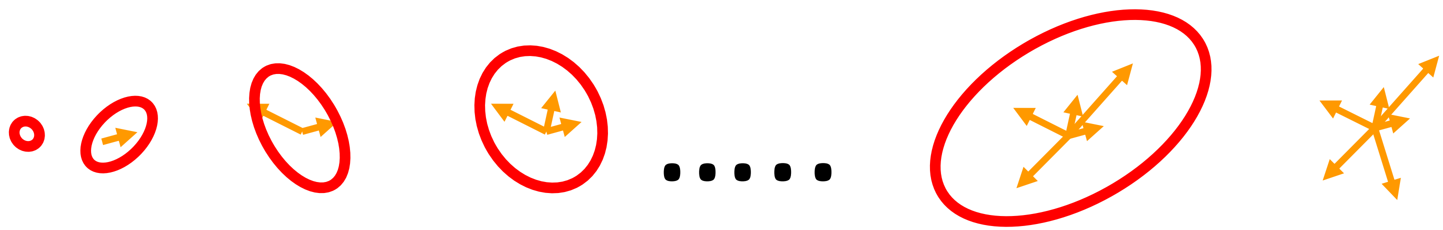


Theorem. If we allow arbitrary **scalings** of the vectors, then there is a **greedy algorithm** which matches what happens in the above dream.





Theorem. If we allow arbitrary **scalings** of the vectors, then there is a **greedy algorithm** which matches what happens in the above dream.

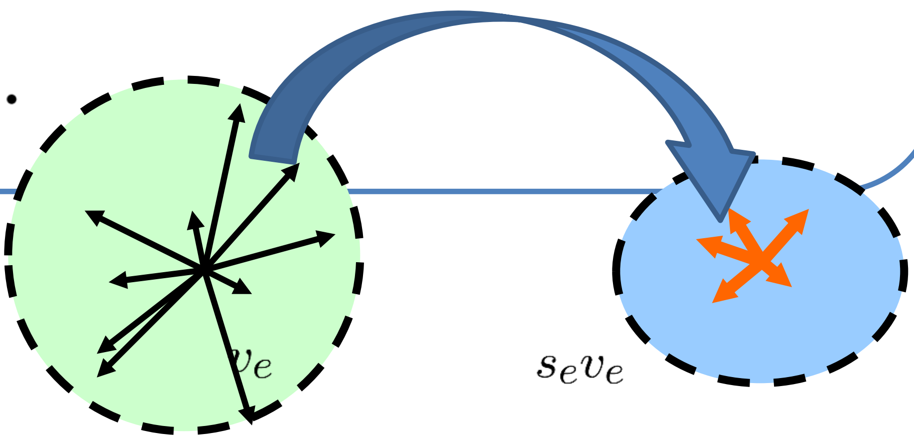


End Result [Batson-Spielman-S'09]

Spectral Sparsification Theorem:

Given $\sum_{i \leq m} v_i v_i^T = I_n$ there are $s_i \geq 0$ with:

- $(1-\epsilon)I \preceq \sum_i s_i v_i v_i^T \preceq (1+\epsilon)I$
- $\mathrm{supp}(s) \leq 4n/\epsilon^2$.

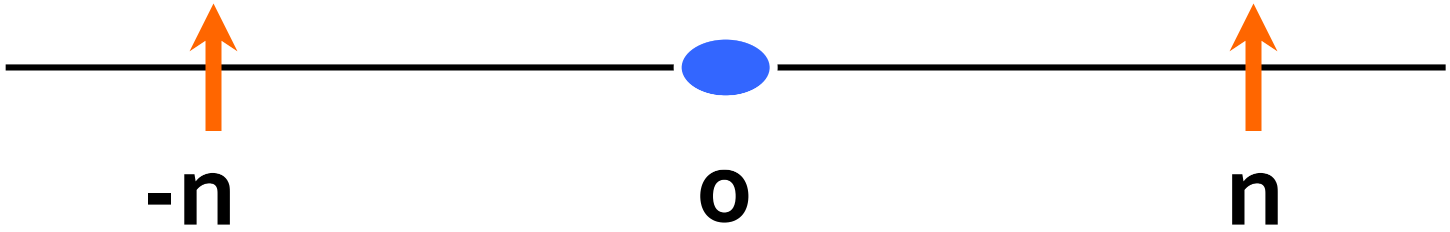


Actual Proof

(for 6n vectors, 13-approx)

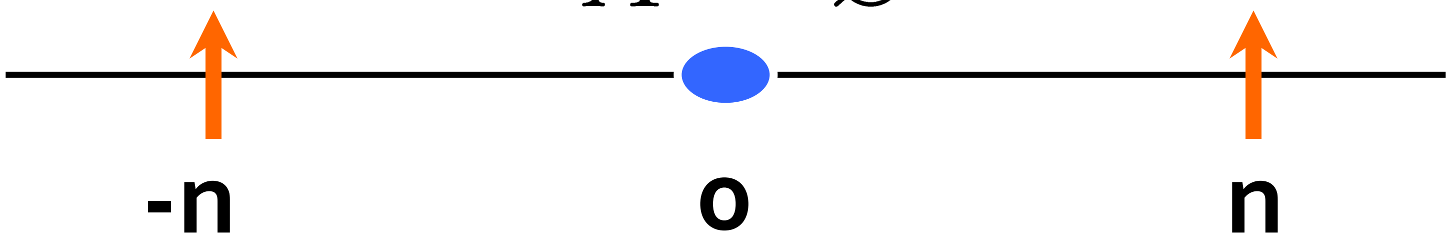
Steady progress by moving barriers

$$A = \varnothing$$



Step 1

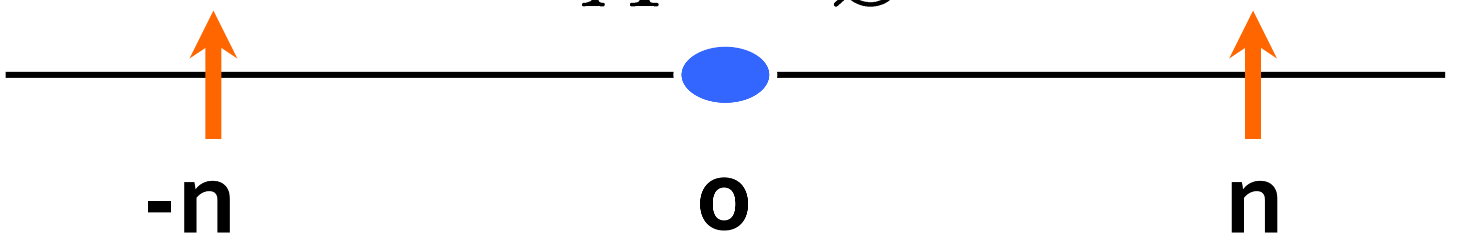
$$A = \varnothing$$



$$+\mathbf{v}\mathbf{v}^T \qquad \mathbf{v} \in \{v_e\}$$

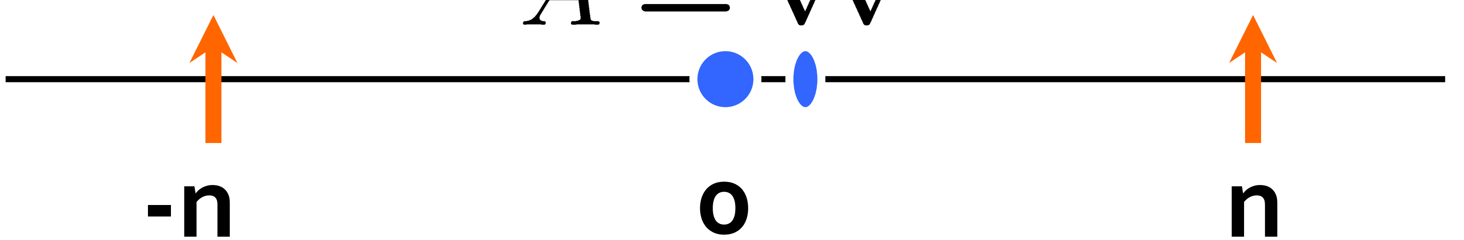
Step 1

$$A = \varnothing$$



$$+\mathbf{v}\mathbf{v}^T \qquad \mathbf{v} \in \{v_e\}$$

$$A = \mathbf{v}\mathbf{v}^T$$



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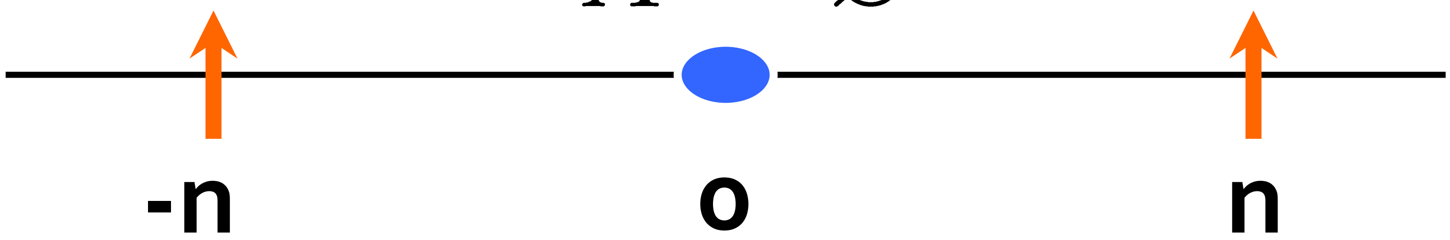
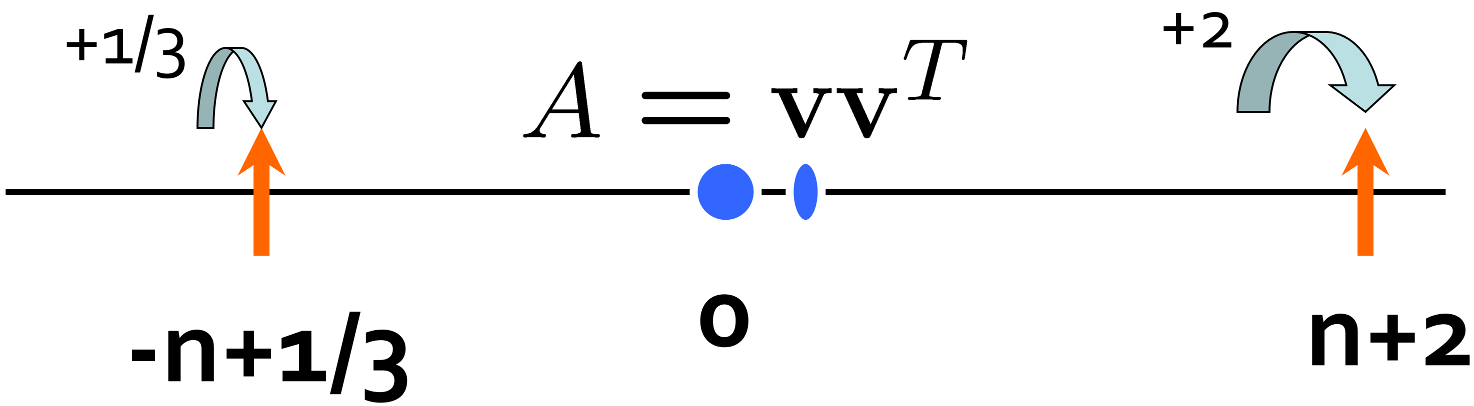
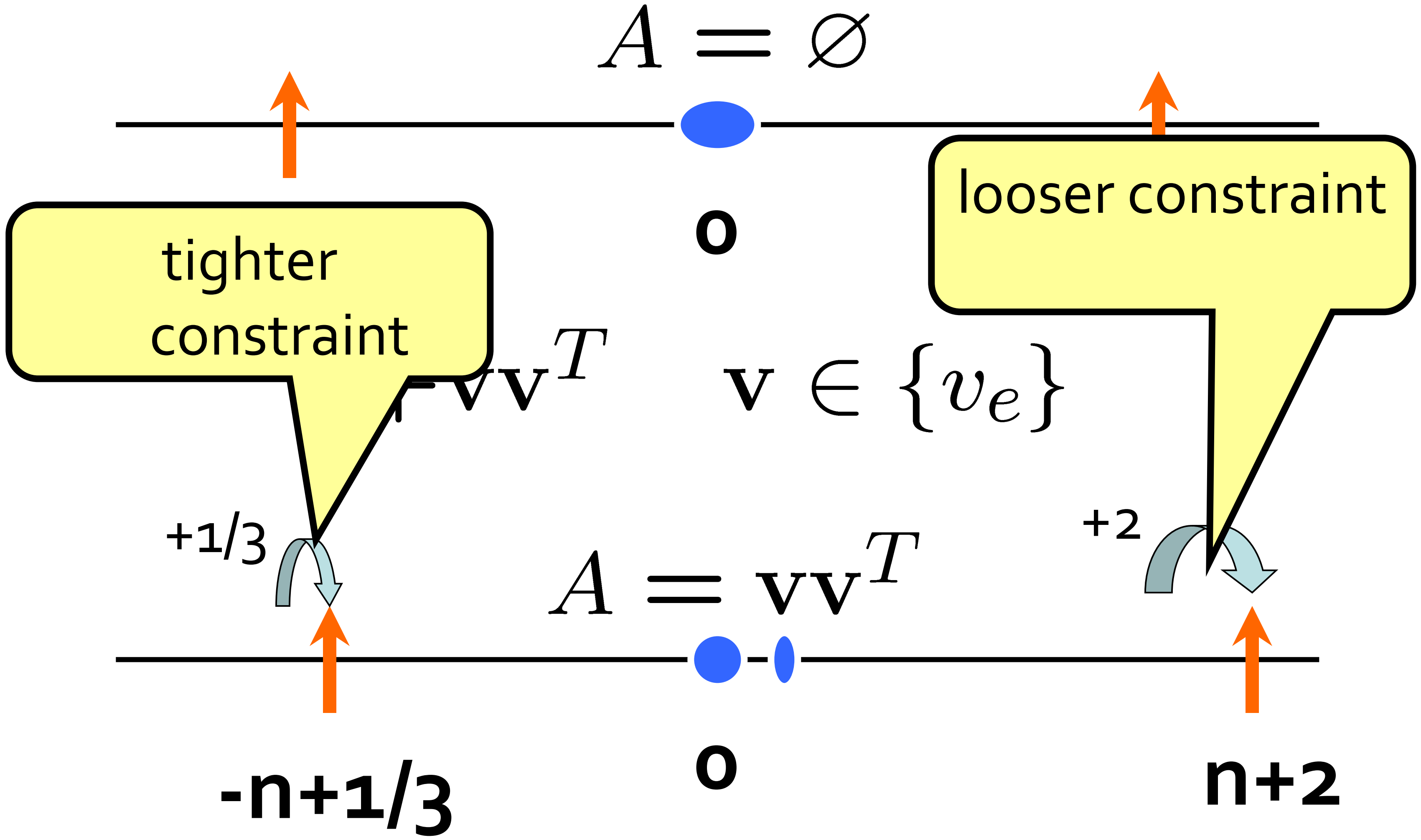


$$+\mathbf{v}\mathbf{v}^T \qquad \mathbf{v} \in \{v_e\}$$

$$A = \mathbf{v}\mathbf{v}^T$$



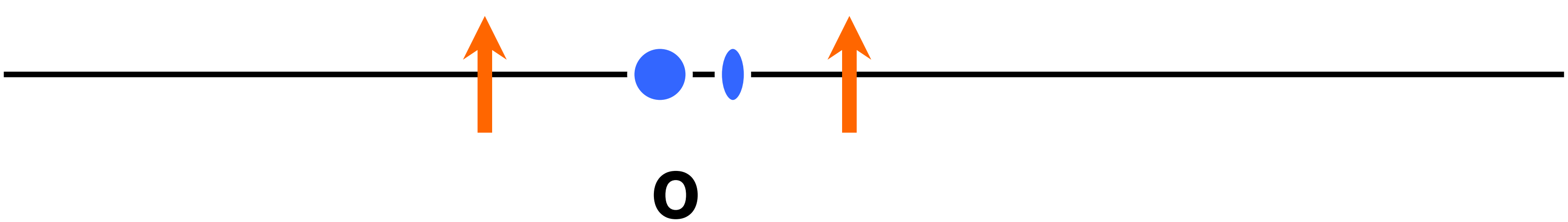
Step 1



$A = \emptyset$
0
tighter constraint
looser constraint
$A = \mathbf{v}\mathbf{v}^T$ $\mathbf{v} \in \{v_e\}$
+1/3
+2
$A = \mathbf{v}\mathbf{v}^T$
-n+1/3
0
n+2

Step i+1

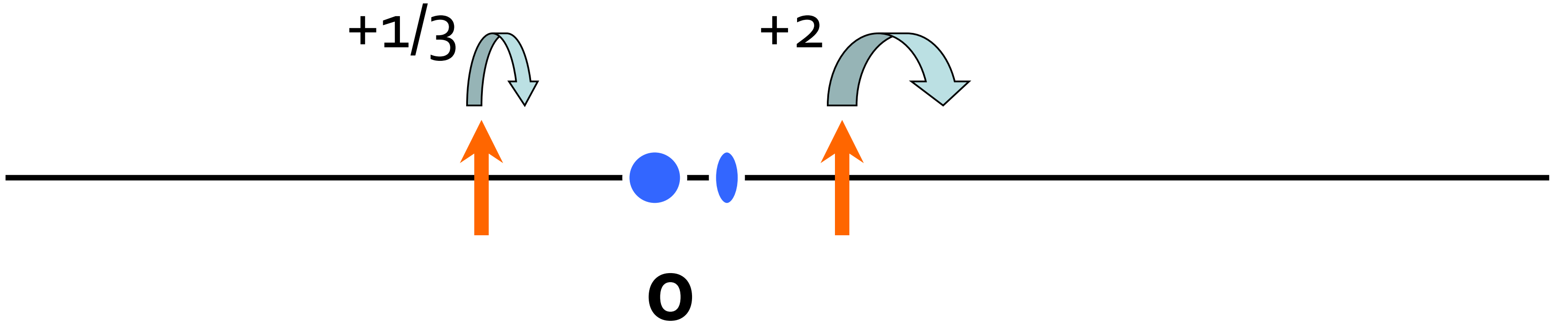
$A^{(i)}$



$$\uparrow \leq \lambda_i \leq \uparrow$$

Step i+1

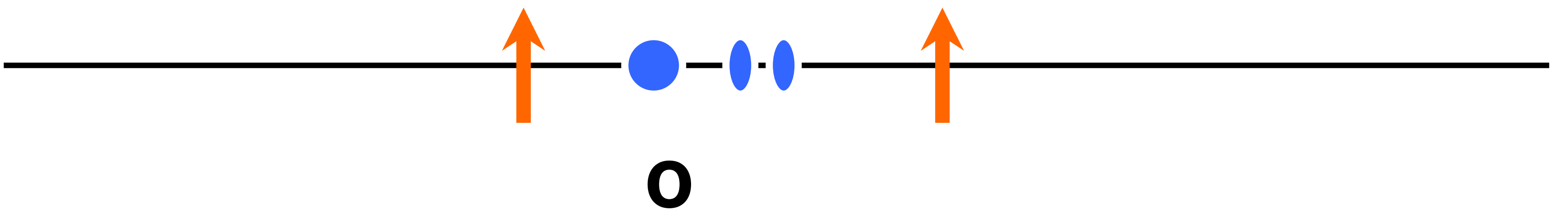
$A^{(i)}$

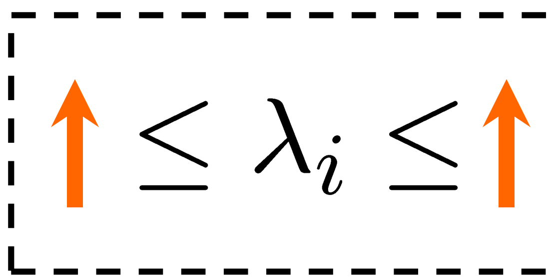


$+\mathbf{v}\mathbf{v}^T$

Step i+1

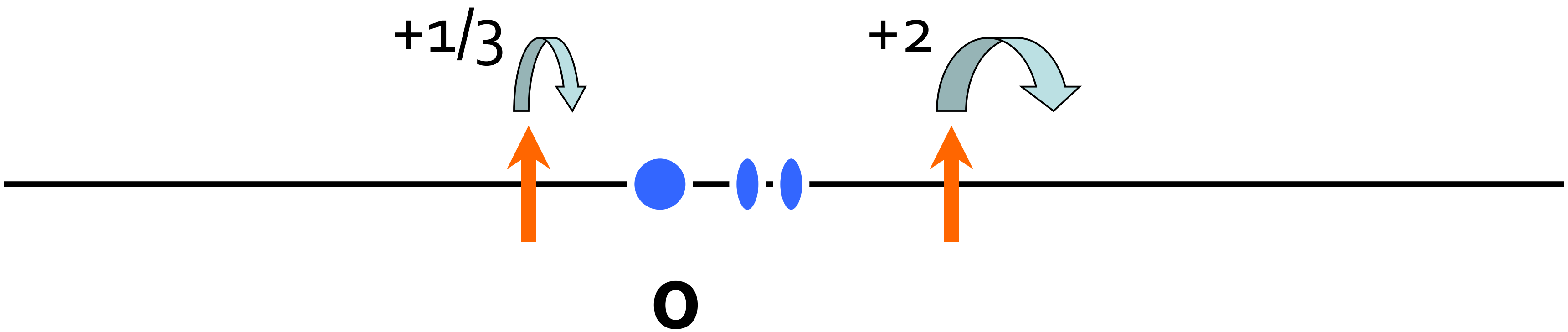
$$A^{(i)}, A^{(i+1)}$$





Step i+1

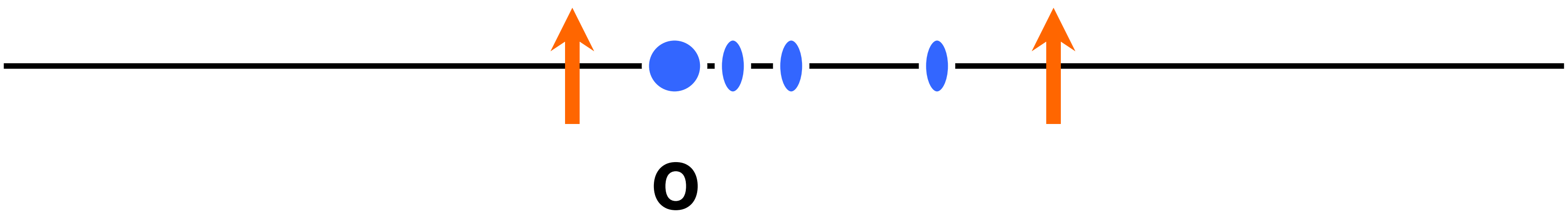
$A^{(i)}, A^{(i+1)}$



$+\mathbf{v}\mathbf{v}^T$

Step i+1

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}$$



$$\uparrow \leq \lambda_i \leq \uparrow$$

Step i+1

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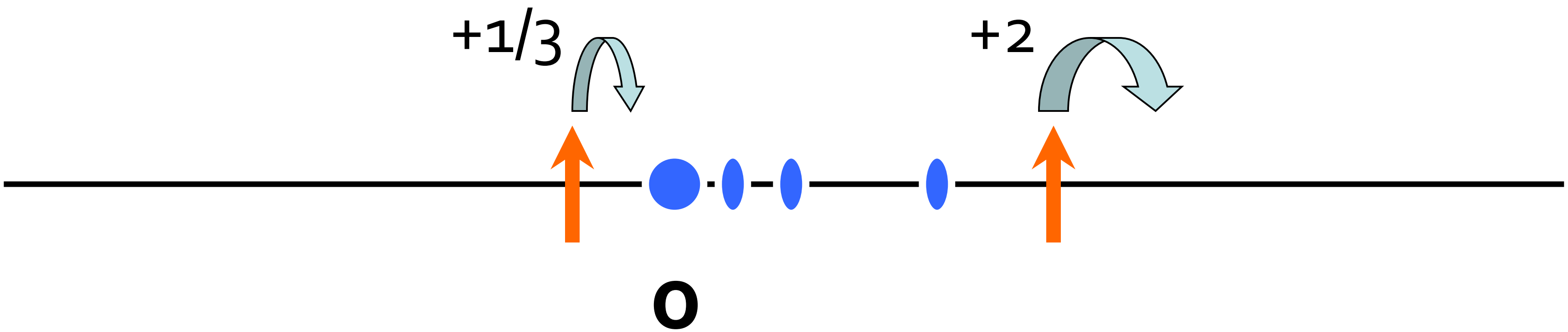
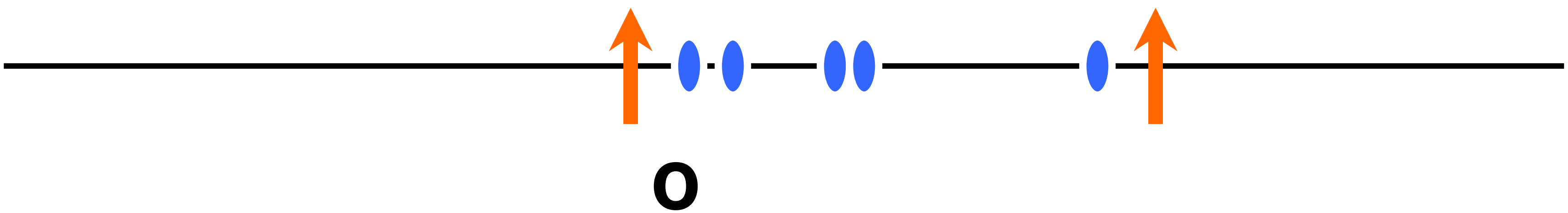
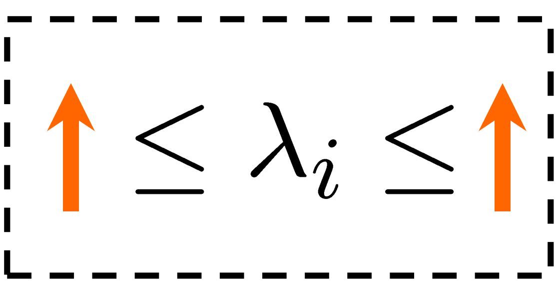


$$+\mathbf{v}\mathbf{v}^T$$

Step i+1

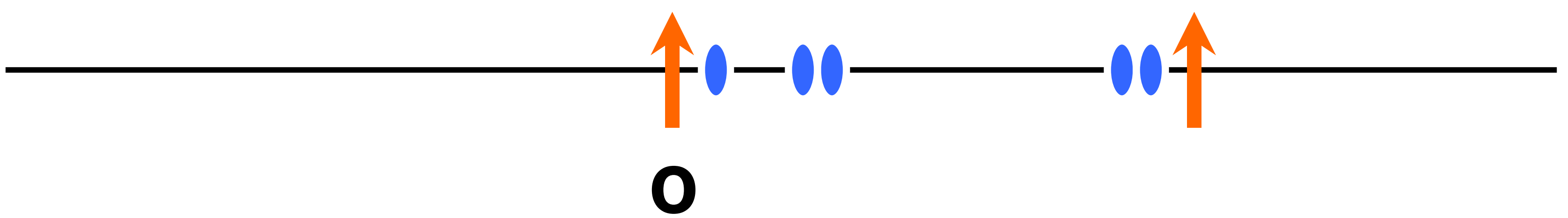
$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}$$

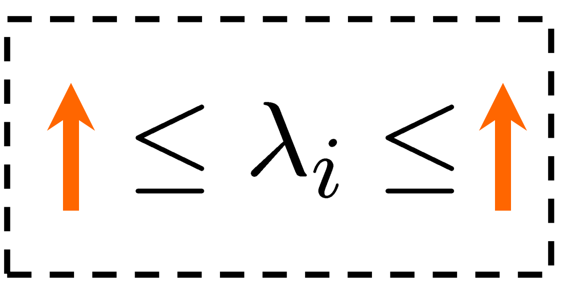




Step i+1

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots$$

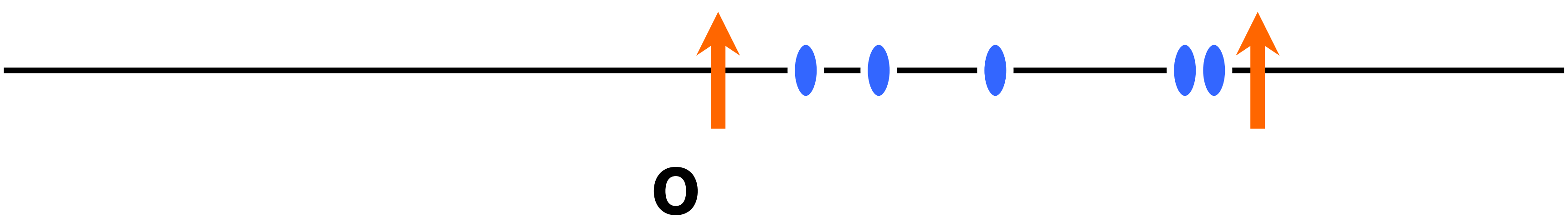




$$\leq \lambda_i \leq$$

Step i+1

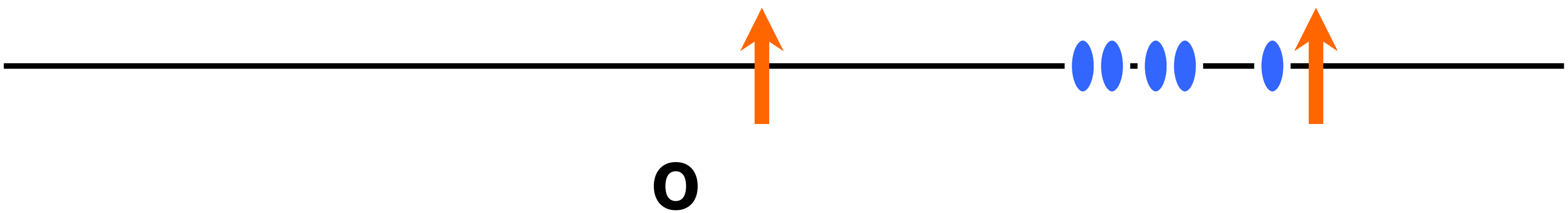
$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots$$

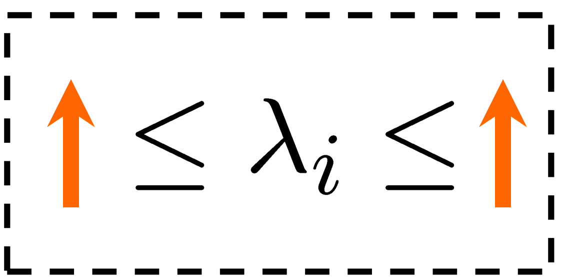


$$\uparrow \le \lambda_i \le \uparrow$$

Step i+1

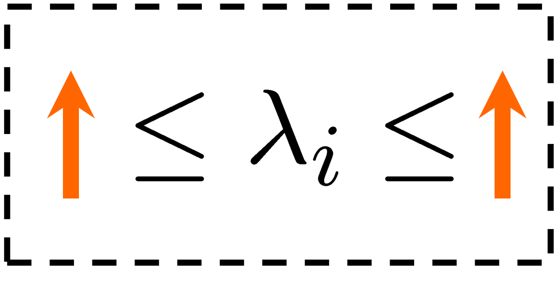
$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots$$





Step 6n

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots, A^{(6n)}$$



Step 6n

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots, A^{(6n)}$$

13-approximation with 6n vectors.

□

Problem

need to show that an appropriate

$$v_e v_e^T$$

always exists.

Problem

need to show that an appropriate

$$v_e v_e^T$$

always exists.

Hope: vectors are well-spread: there must be one which is well-behaved.

Bad: Accumulation of Eigenvalues

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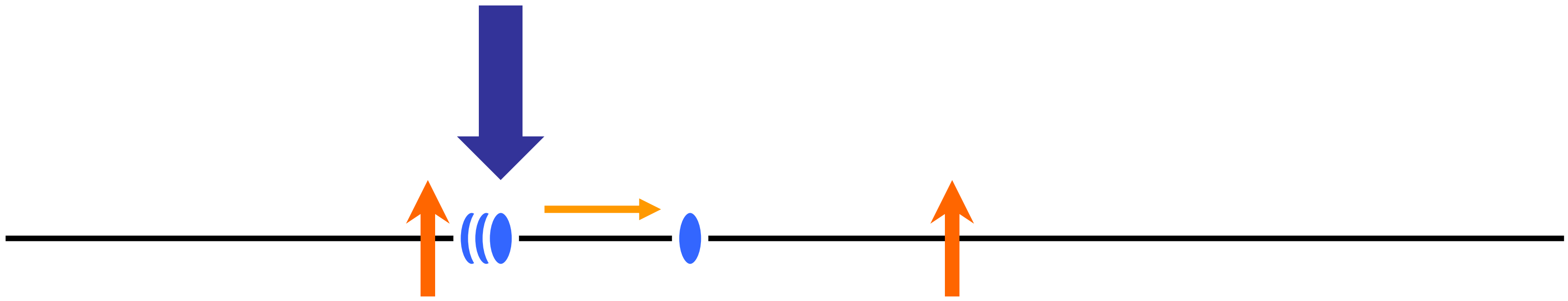
$$+\mathbf{v}\mathbf{v}^T \qquad \mathbf{v} \in \{v_e\}$$



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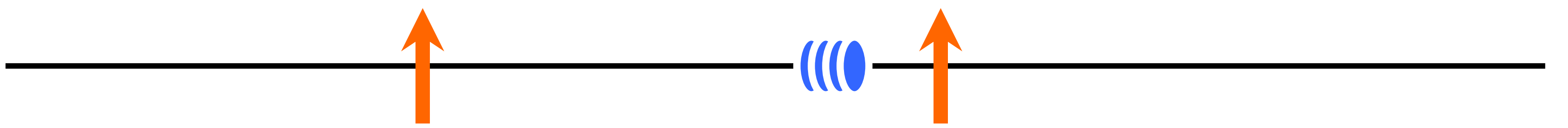

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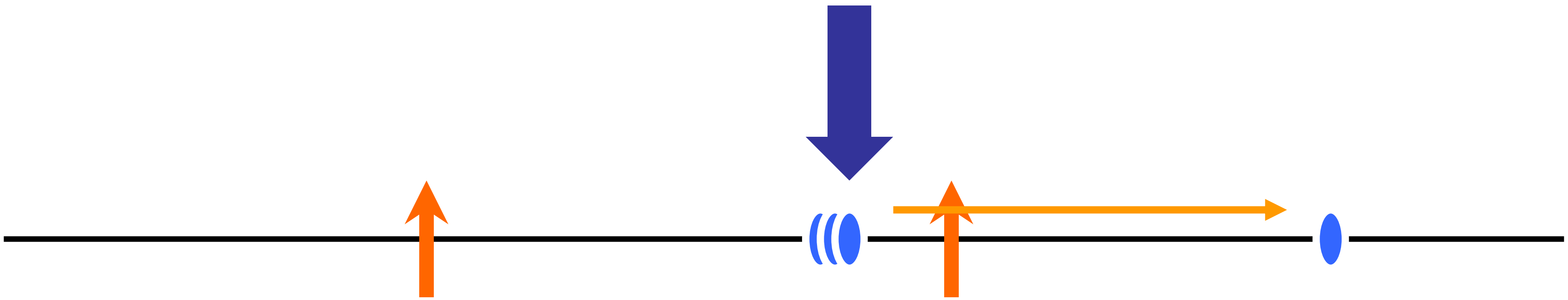

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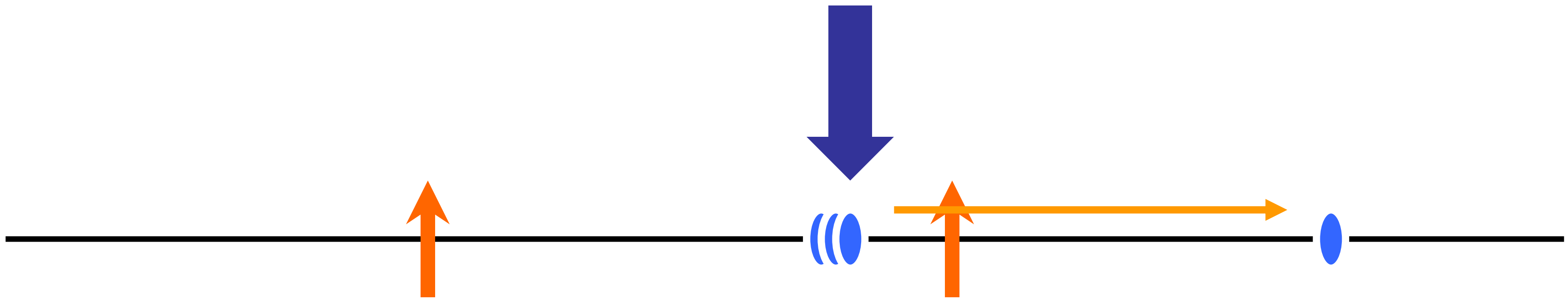
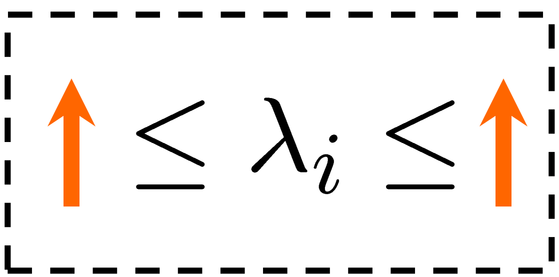
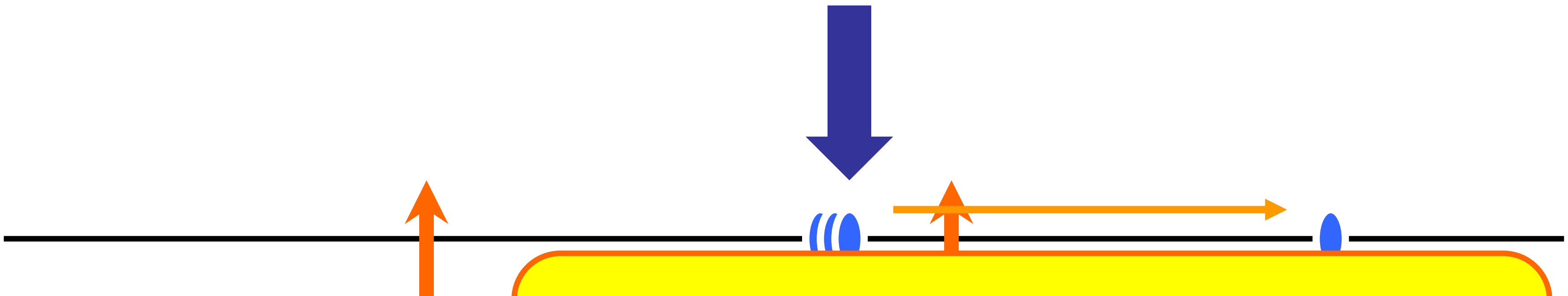




is not strong enough to do the induction.

Bad: Accumulation of Eigenvalues

$$+\mathbf{v}\mathbf{v}^T \quad \mathbf{v} \in \{v_e\}$$



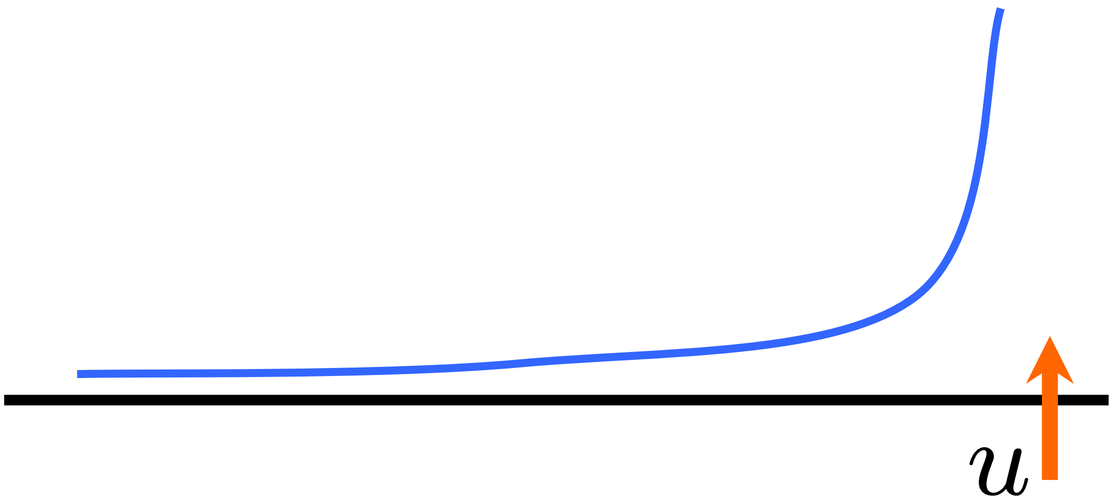
need a better way to measure quality of eigenvalues.

$$\uparrow \leq \lambda_i \leq \uparrow$$

is not strong enough to do the induction.

The Upper Barrier

$$\Phi^u(A) = \mathrm{Tr}(uI - A)^{-1} = \sum_i \frac{1}{u - \lambda_i}$$



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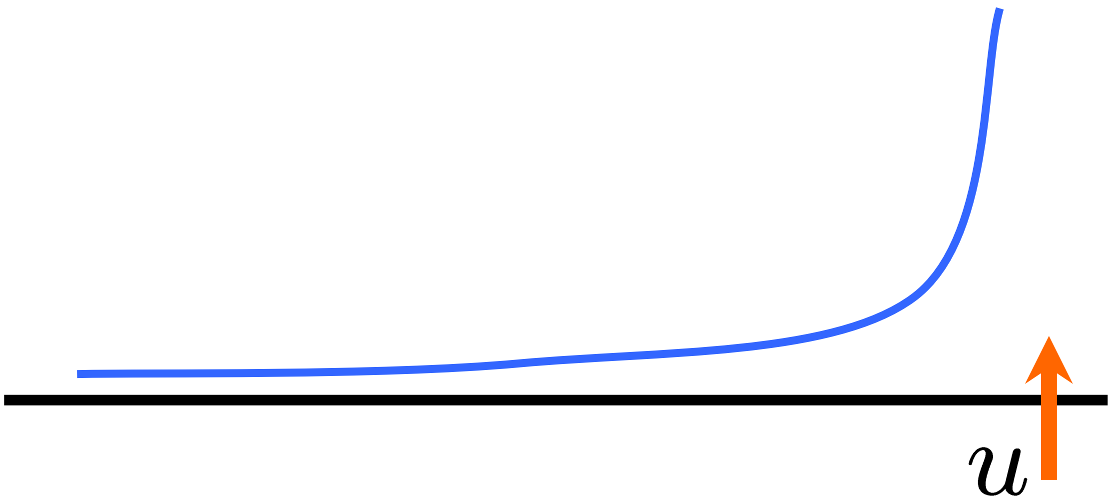


$$\Phi^u(A) \leq 1 \Rightarrow \lambda_{\mathsf{max}}(A) \ll u$$

The Upper Barrier

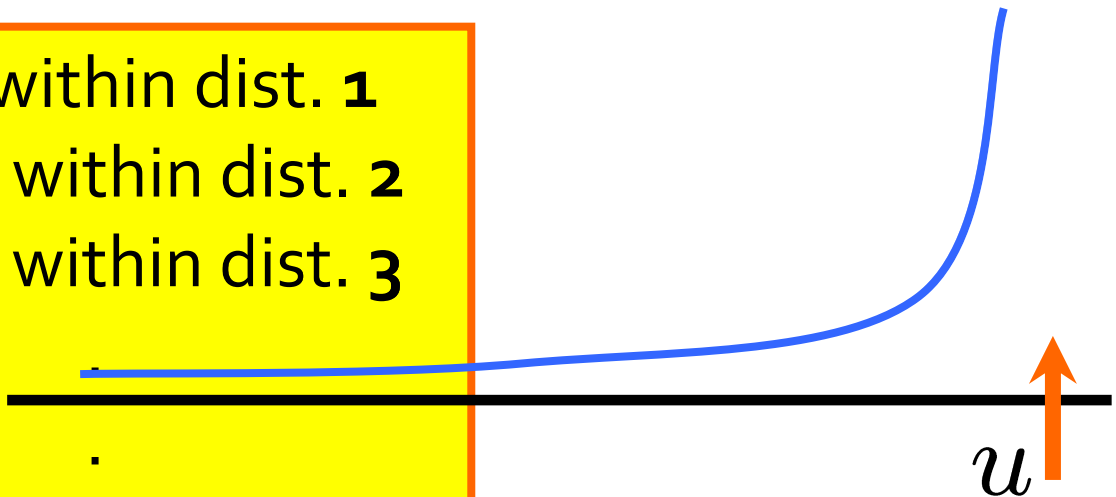
$$\Phi^u(A) = \mathrm{Tr}(uI - A)^{-1} = \sum_i \frac{1}{u - \lambda_i}$$

No λ_i within dist. **1**

No **2** λ_i within dist. **2**

No **3** λ_i within dist. **3**

.



.

No **k** λ_i within dist. **k**

$$\Phi^u(A) \leq 1 \Rightarrow \lambda_{\mathrm{max}}(A) \ll u$$

‘Total repulsion’ in physical model

per Barrier

$$\Phi^u(A) = \mathrm{Tr}(uI - A)^{-1} = \sum_i \frac{1}{u-\lambda_i}$$

No λ_i within dist. **1**

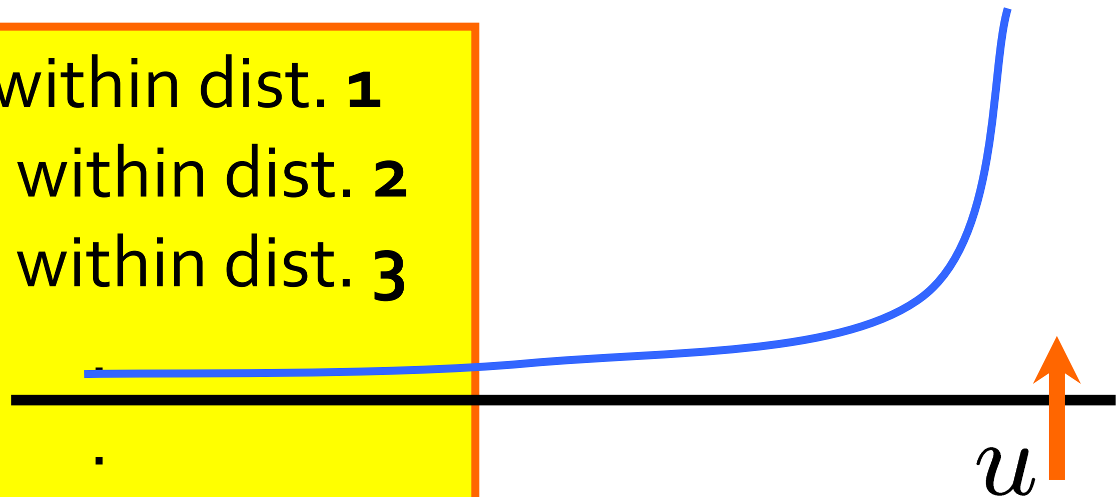
No **2** λ_i within dist. **2**

No **3** λ_i within dist. **3**

.

.

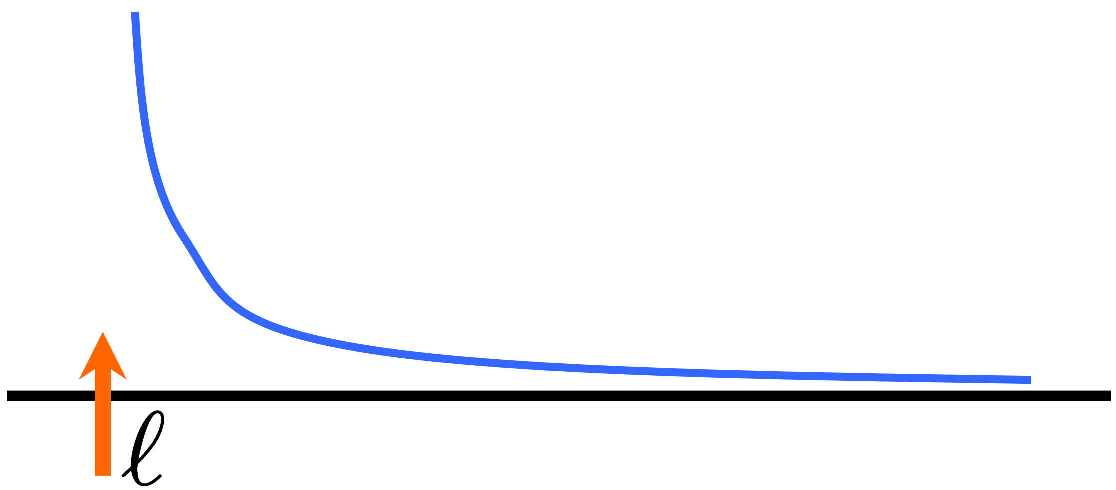
No **k** λ_i within dist. **k**



$$\Phi^u(A) \leq 1 \Rightarrow \lambda_{\mathrm{max}}(A) \ll u$$

The Lower Barrier

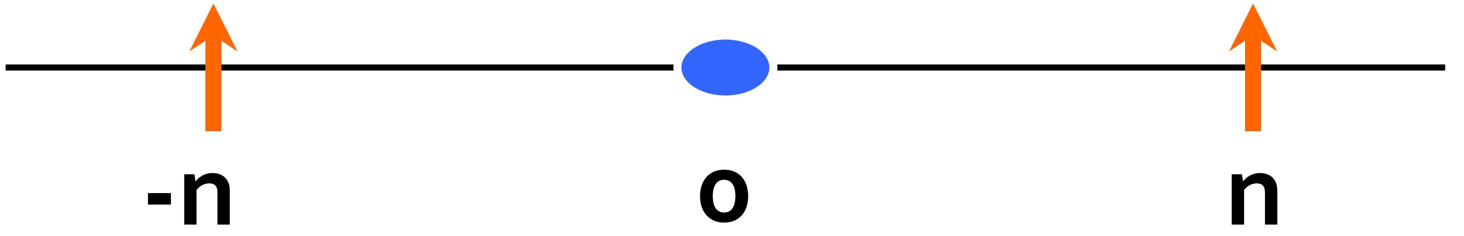
$$\Phi_\ell(A) = \mathrm{Tr}(A - \ell I)^{-1} = \sum_i \frac{1}{\lambda_i - \ell}$$



$$\Phi_\ell(A) \leq 1 \Rightarrow \lambda_{\min}(A) \gg \ell$$

The Beginning

$$A = \varnothing$$



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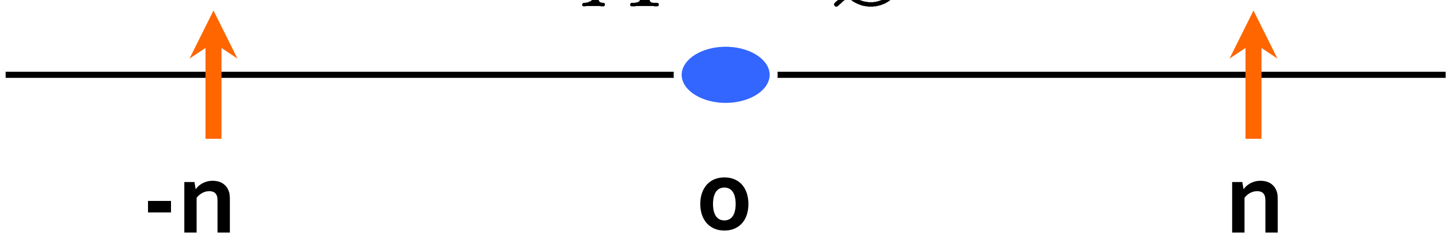
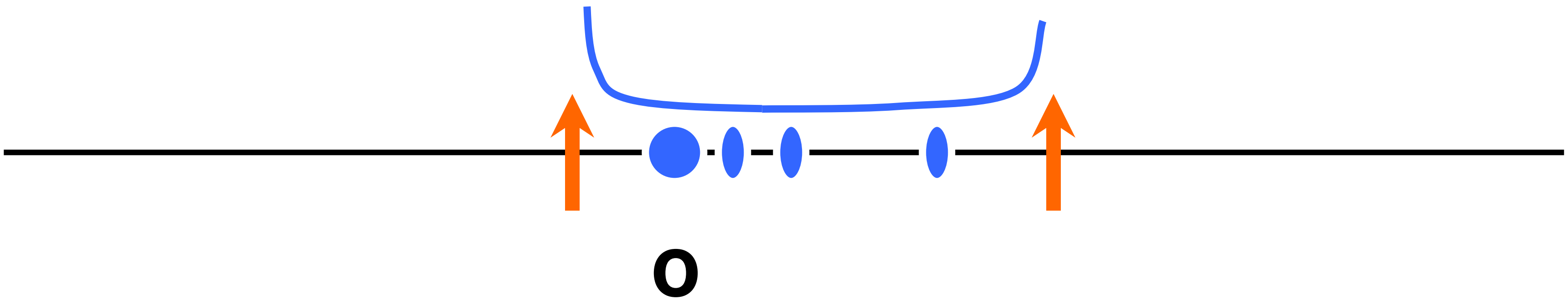


$$\Phi^{n}(\varnothing) = \mathrm{Tr}(nI)^{-1} = 1$$

$$\Phi_{-n}(\varnothing) = \mathrm{Tr}(nI)^{-1} = 1.$$

Step i+1

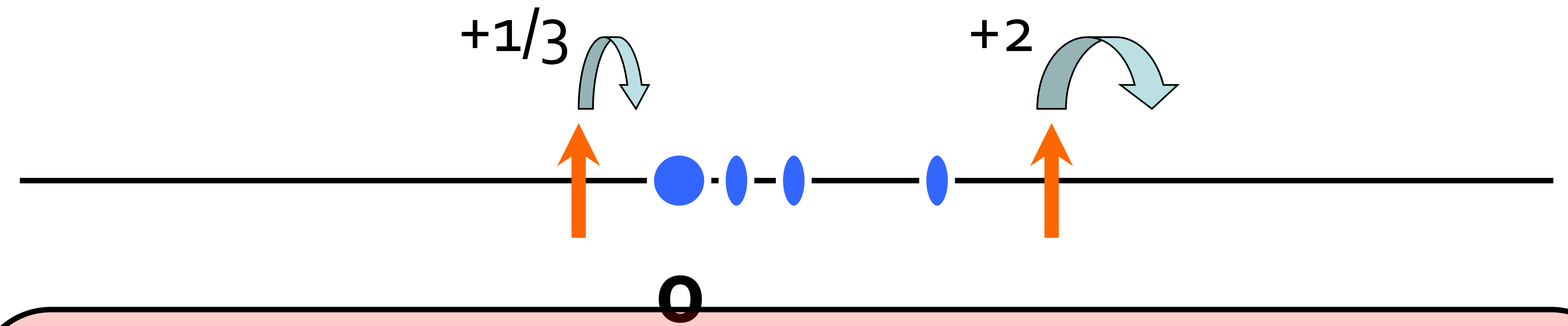
$A^{(i)}, A^{(i+1)}, A^{(i+2)}$



$$\Phi^u(A) \leq 1$$
$$\Phi_\ell(A) \leq 1.$$

Step i+1

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}$$



Lemma.

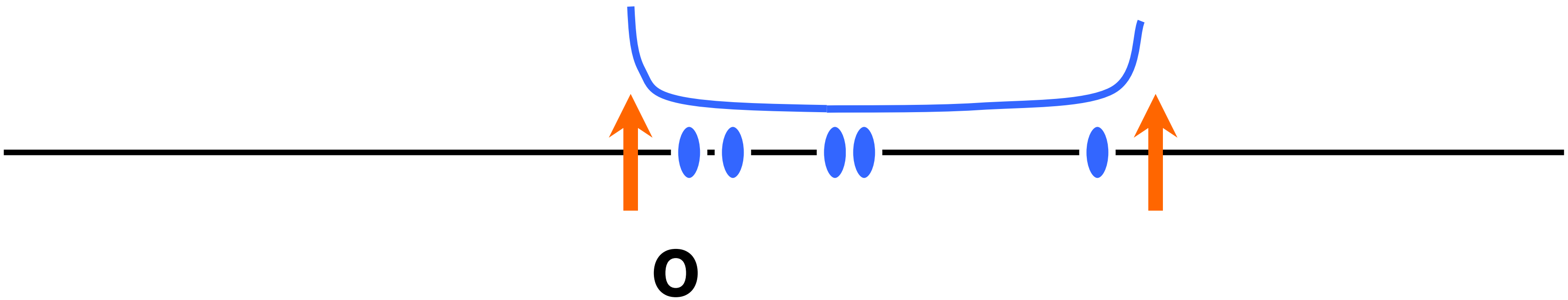
can always choose $+s\mathbf{v}\mathbf{v}^T$ so that potentials do not increase

$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step i+1

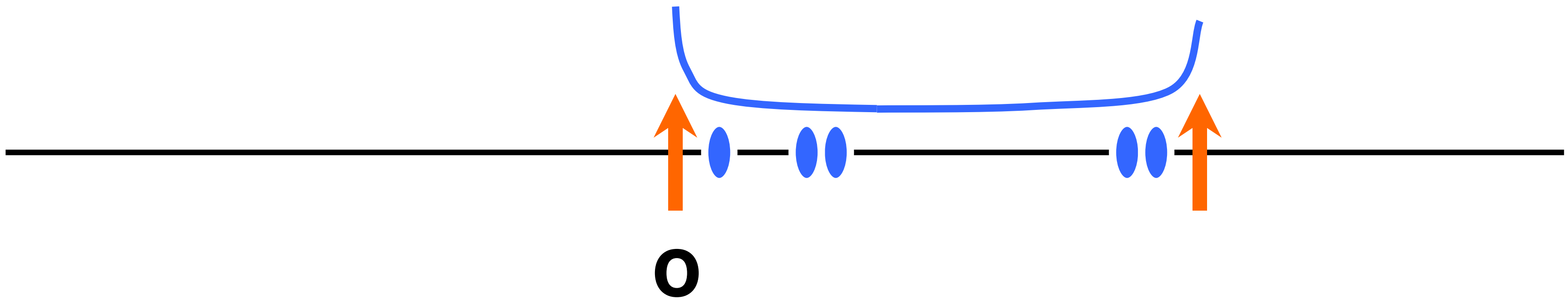
$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}$$



$$\Phi^u(A) \leq 1$$
$$\Phi_\ell(A) \leq 1.$$

Step i+1

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots$$



$$\Phi^u(A) \leq 1$$
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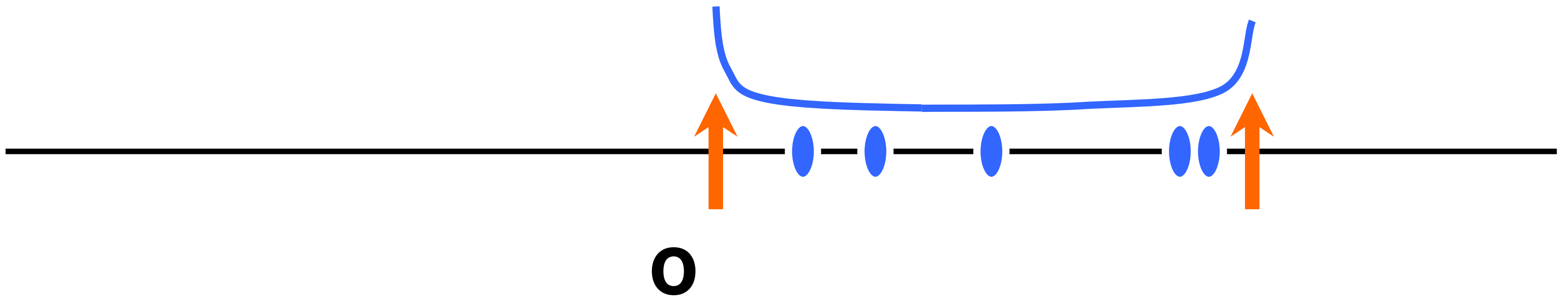
Step i+1

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots$$

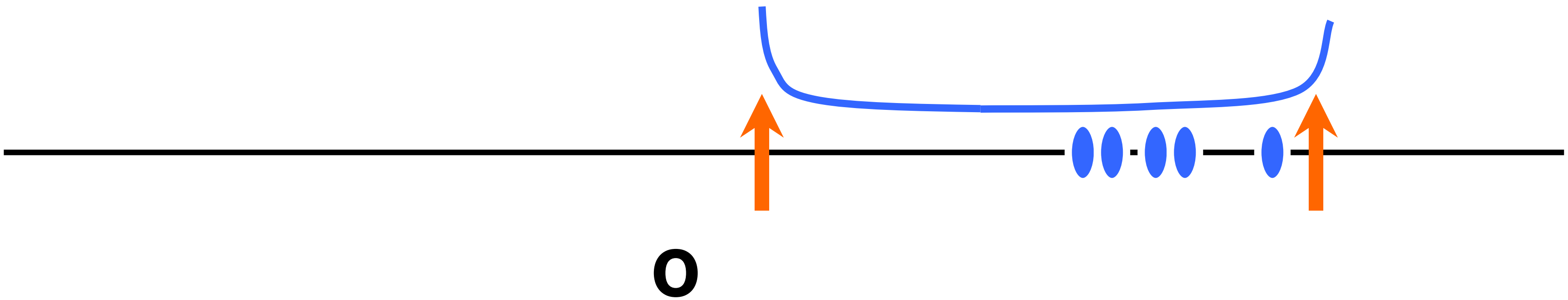


$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step i+1

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots$$



$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step 6n

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots, A^{(6n)}$$

$$\Phi^u(A) \leq 1$$
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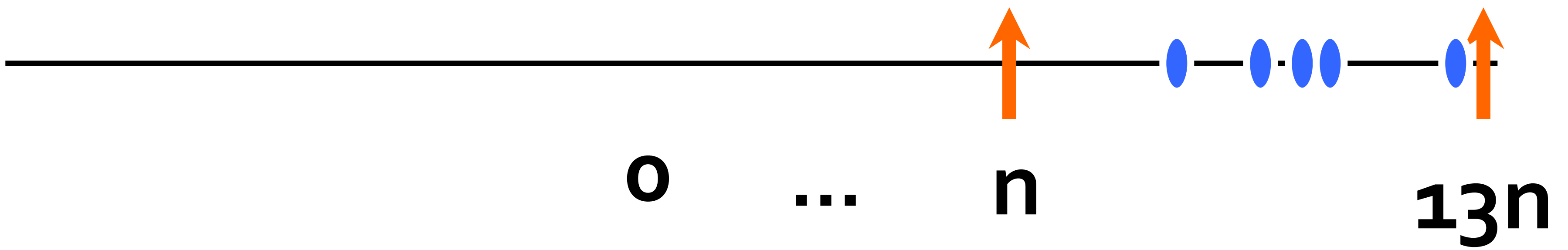
Step 6n

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots, A^{(6n)}$$



13-approximation with 6n vectors.

□

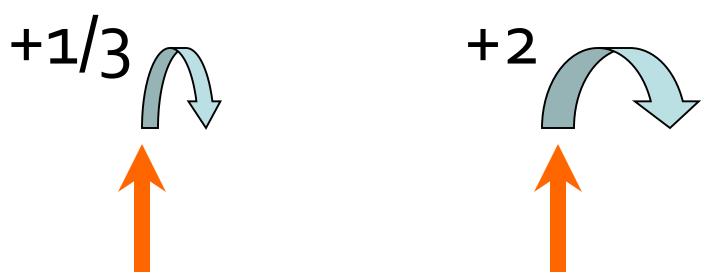
Goal

Lemma.

can always choose $+s\mathbf{v}\mathbf{v}^T$ so that *both* potentials do not increase.

$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$



$$+s\mathbf{v}\mathbf{v}^T$$

The Right Question

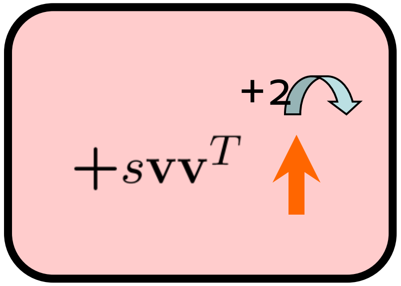
"Which vector should we add?"

The Right Question

~~“Which vector should we add?”~~

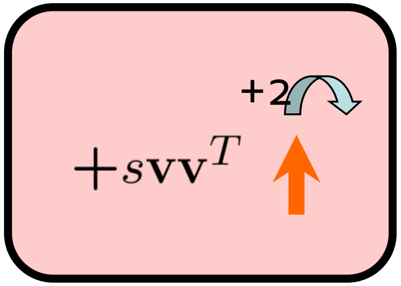
“Given a vector, how much of it can we add?”

Upper Barrier Update



Add $s\mathbf{v}\mathbf{v}^T$ & ***set*** $u' \leftarrow u + 2.$

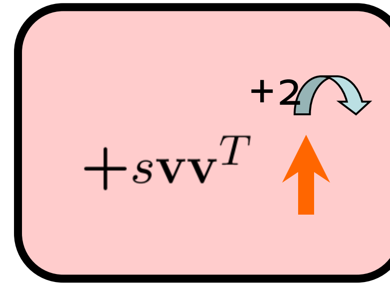
Upper Barrier Update



Add $s\mathbf{v}\mathbf{v}^T$ & ***set*** $u' \leftarrow u + 2.$

$$\Phi^{u'}(A + s\mathbf{v}\mathbf{v}^T)$$
$$= \mathrm{Tr}(u'I - A - s\mathbf{v}\mathbf{v}^T)^{-1}$$

Upper Barrier Update



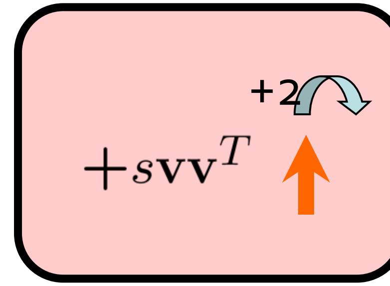
Add $s\mathbf{v}\mathbf{v}^T$ & ***set*** $u' \leftarrow u + 2.$

$$\Phi^{u'}(A + s\mathbf{v}\mathbf{v}^T)$$
$$= \mathrm{Tr}(u'I - A - s\mathbf{v}\mathbf{v}^T)^{-1}$$

$$\mathrm{Tr}(A + vv^T)^{-1} = \mathrm{Tr}A^{-1} - \frac{v^T A^{-2} v}{1 + v^T A^{-1} v}$$

Sherman-Morrisson

Upper Barrier Update



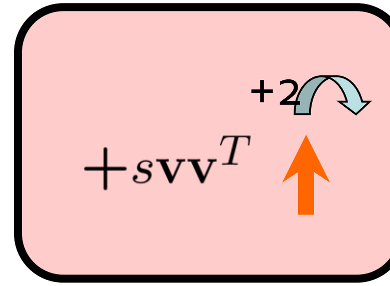
Add $s\mathbf{v}\mathbf{v}^T$ & ***set*** $u' \leftarrow u + 2.$

$$\Phi^{u'}(A + s\mathbf{v}\mathbf{v}^T)$$

$$= \mathrm{Tr}(u'I - A - s\mathbf{v}\mathbf{v}^T)^{-1}$$

$$= \Phi^{u'}(A) + \frac{\mathbf{v}^T(u'I - A)^{-2}\mathbf{v}}{1/s - \mathbf{v}^T(u'I - A)^{-1}\mathbf{v}}$$

Upper Barrier Update

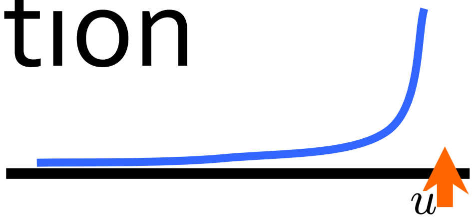


Add $s\mathbf{v}\mathbf{v}^T$ & ***set*** $u' \leftarrow u + 2.$

$$
\begin{aligned}
&\Phi^{u'}(A + s\mathbf{v}\mathbf{v}^T) \\
&= \mathrm{Tr}(u'I - A - s\mathbf{v}\mathbf{v}^T)^{-1} \\
&= \Phi^{u'}(A) + \frac{\mathbf{v}^T(u'I - A)^{-2}\mathbf{v}}{1/s - \mathbf{v}^T(u'I - A)^{-1}\mathbf{v}}
\end{aligned}
$$

want $\leq \Phi^{u}(A).$

Upper feasibility condition



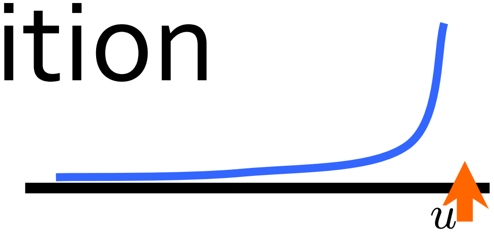
Rearranging:

$$\Phi^{u'}(A + s\mathbf{v}\mathbf{v}^T) \leq \Phi^u(A)$$

$$\iff$$

$$\frac{1}{s} \geq \mathbf{v}^T \left(\frac{(u'I - A)^{-2}}{\Phi^u(A) - \Phi^{u'}(A)} + (u'I - A)^{-1} \right) \mathbf{v}$$

Upper feasibility condition



Rearranging:

$$\Phi^{u'}(A + s\mathbf{v}\mathbf{v}^T) \leq \Phi^{u}(A)$$

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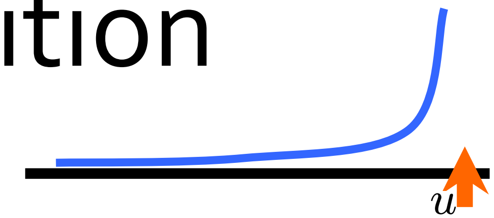
$$\frac{1}{s} \geq \mathbf{v}^T \left(\frac{(u'I - A)^{-2}}{\Phi^{u}(A) - \Phi^{u'}(A)} + (u'I - A)^{-1} \right) \mathbf{v}$$

$$\boxed{\frac{1}{s} \geq U_A \bullet \mathbf{v}\mathbf{v}^T}$$

Upper feasibility condition



Rearranging:

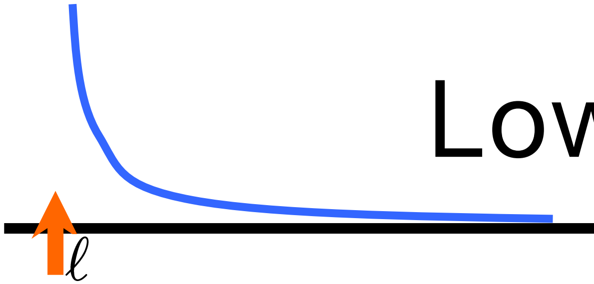
$$\Phi^{u'}(A + s\mathbf{v}\mathbf{v}^T) \leq \Phi^u(A)$$

s=0 always feasible

$$\Longleftrightarrow$$

$$\frac{1}{s} \geq \mathbf{v}^T \left(\frac{(u'I - A)^{-2}}{\Phi^u(A) - \Phi^{u'}(A)} + (u'I - A)^{-1} \right) \mathbf{v}$$

$$\frac{1}{s} \geq U_A \bullet \mathbf{v}\mathbf{v}^T$$



Lower Feasibility

Similarly:

$$\Phi_{\ell'}(A + s\mathbf{v}\mathbf{v}^T) \leq \Phi_{\ell}(A)$$

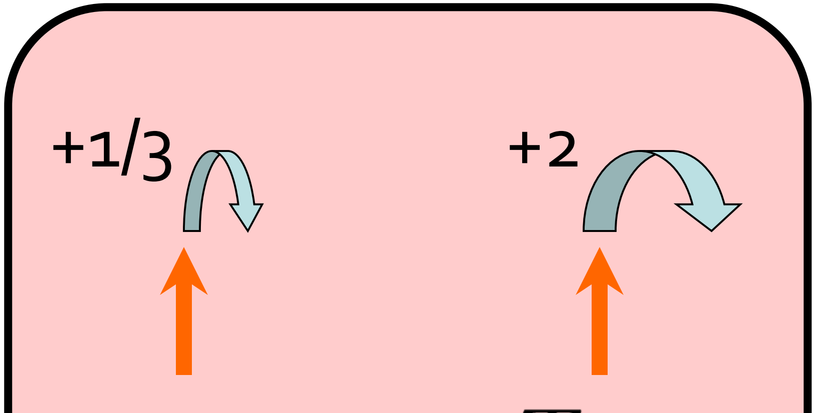
$$\Longleftrightarrow$$

$$\frac{1}{s} \leq \mathbf{v}^T \left(\frac{(A - \ell' I)^{-2}}{\Phi_{\ell'}(A) - \Phi_{\ell}(A)} - (A - \ell' I)^{-1} \right) \mathbf{v}$$

$$\boxed{\frac{1}{s} \leq L_A \bullet \mathbf{v}\mathbf{v}^T}$$

Goal

Show that we can always add some vector while respecting *both* barriers.



$$+s\mathbf{v}\mathbf{v}^T$$

Both Barriers

Goal

There is always a vector with

$$U_A \bullet \mathbf{v}\mathbf{v}^T \leq L_A \bullet \mathbf{v}\mathbf{v}^T$$

Both Barriers

can add

must add

There is always a vector with

$$U_A \bullet \mathbf{v}\mathbf{v}^T \leq L_A \bullet \mathbf{v}\mathbf{v}^T$$

can add

Both Barriers

must add

There is always a vector with

$$U_A \bullet \mathbf{v}\mathbf{v}^T \leq L_A \bullet \mathbf{v}\mathbf{v}^T$$

Then, can squeeze scaling factor in between:

$$U_A \bullet \mathbf{v}\mathbf{v}^T \leq \frac{1}{s} \leq L_A \bullet \mathbf{v}\mathbf{v}^T$$

Average over all $\boldsymbol{v_e}$

Goal

$$\exists \mathbf{v}, U_A \bullet \mathbf{v}\mathbf{v}^T \leq L_A \bullet \mathbf{v}\mathbf{v}^T$$

Average over all $\boldsymbol{v_e}$

Goal

$$\sum_{\mathbf{v}} U_A \bullet \mathbf{v}\mathbf{v}^T \leq \sum_{\mathbf{v}} L_A \bullet \mathbf{v}\mathbf{v}^T$$

Average over all $\boldsymbol{v_e}$

Goal

$$\sum_{\mathbf{v}} U_A \bullet \mathbf{v}\mathbf{v}^T \leq \sum_{\mathbf{v}} L_A \bullet \mathbf{v}\mathbf{v}^T$$

$$\sum_{\mathbf{v} \in \{v_e\}} U_A \bullet \mathbf{v}\mathbf{v}^{\mathbf{T}} = U_A \bullet \left(\sum_e v_e v_e^T \right)$$

$$= U_A \bullet I$$

$$= \mathrm{Tr}(U_A).$$

Average over all $\boldsymbol{v_e}$

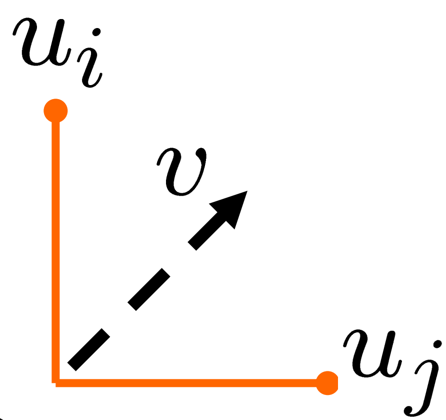
Goal

$$\sum_{\mathbf{v}} U_A \bullet \mathbf{v}\mathbf{v}^T \leq \sum_{\mathbf{v}} L_A \bullet \mathbf{v}\mathbf{v}^T$$

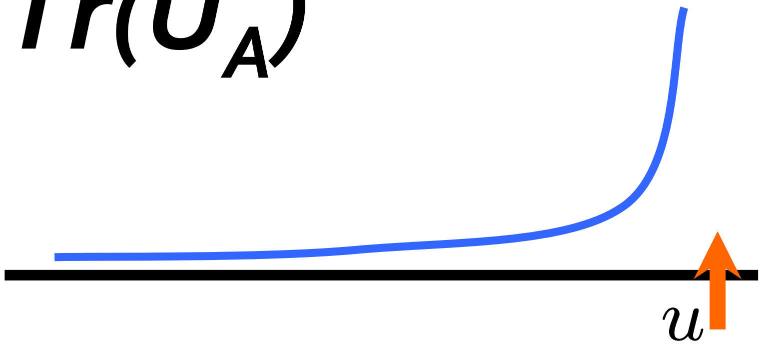
$$\sum_{\mathbf{v}\in\{v_e\}} U_A \bullet \mathbf{v}\mathbf{v}^{\mathbf{T}} = U_A \bullet \left(\sum_e v_e v_e^T\right)$$

$$U_A \bullet I$$

$$= \mathrm{Tr}(U_A).$$

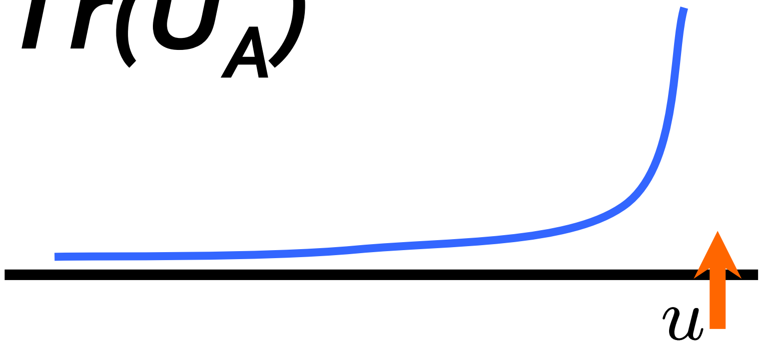


Bounding $\mathbf{\mathit{Tr(U_A)}}$



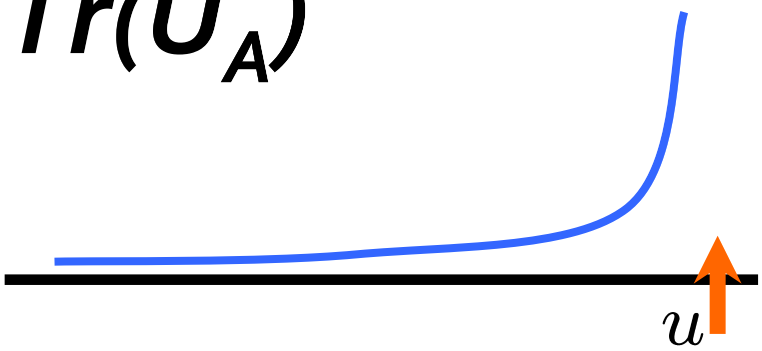
$$\frac{\mathrm{Tr}(u'I - A)^{-2}}{\Phi^{u}(A) - \Phi^{u'}(A)} + \mathrm{Tr}(u'I - A)^{-1}$$

Bounding *Tr(U_A)*



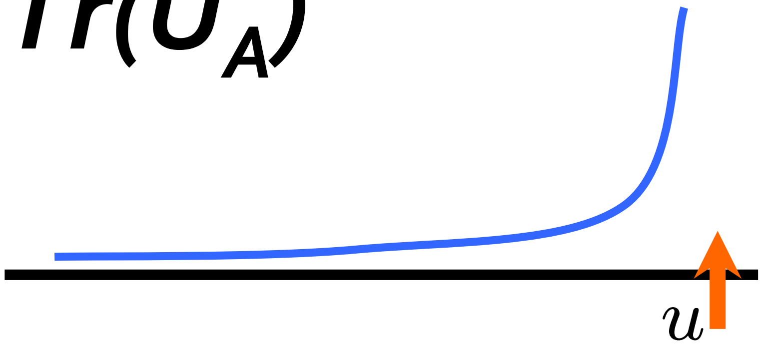
$$\frac{\mathrm{Tr}(u'I - A)^{-2}}{\Phi^{u}(A) - \Phi^{u'}(A)} + \boxed{\Phi^{u'}(A)}$$

Bounding *Tr(U_A)*



$$\frac{\mathrm{Tr}(u'I - A)^{-2}}{\Phi^{u}(A) - \Phi^{u'}(A)} + \boxed{\leq \Phi^{u}(A)}$$

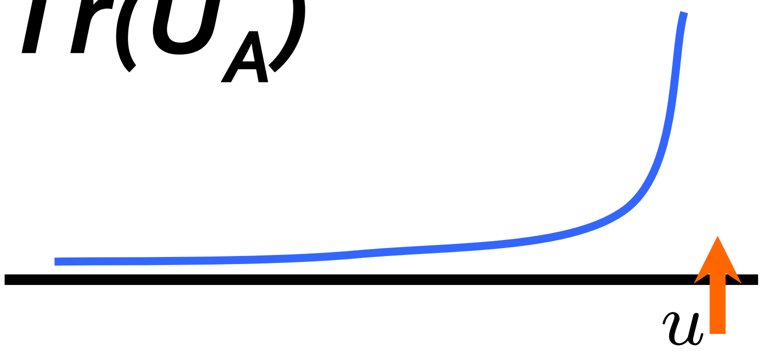
Bounding $Tr(U_A)$



$$\frac{\mathrm{Tr}(u'I - A)^{-2}}{\Phi^{u}(A) - \Phi^{u'}(A)} + \boxed{\leq 1}$$

induction

Bounding *Tr(U_A)*

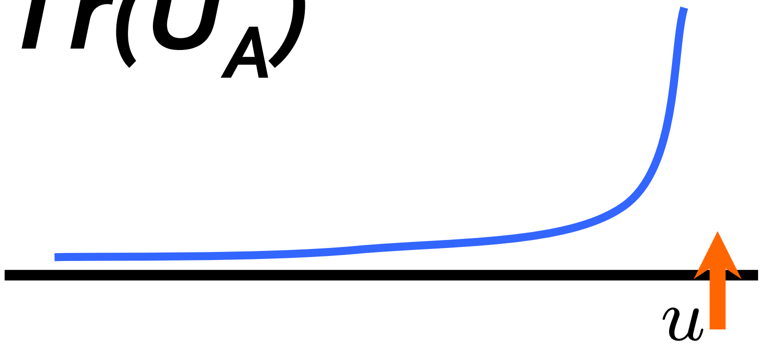


$$\frac{\boxed{-\frac{\partial}{\partial u'}\Phi^{u'}(A)}}{\Phi^{u}(A) - \Phi^{u'}(A)} + \boxed{\leq 1}$$

induction

(Recall $\Phi^u(A) = \mathrm{Tr}(uI - A)^{-1}$.)

Bounding *Tr(U_A)*



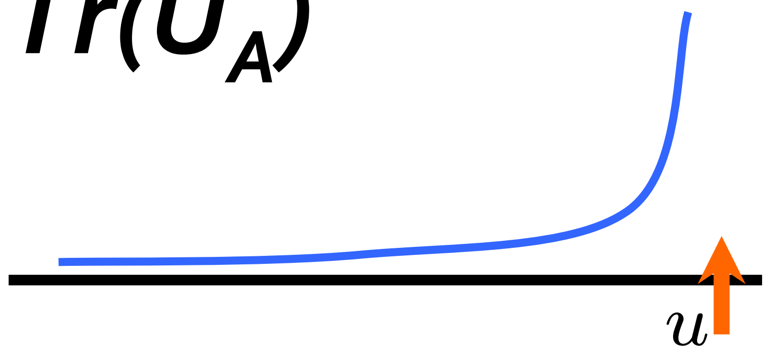
$$\frac{-\frac{\partial}{\partial u'}\Phi^{u'}(A)}{\geq \delta_u\left(-\frac{\partial}{\partial u'}\Phi^{u'}(A)\right)} + \leq 1$$

convexity

induction

(Recall $\Phi^u(A) = \mathrm{Tr}(uI - A)^{-1}$.)

Bounding *Tr(U_A)*



$$\frac{-\frac{\partial}{\partial u'}\Phi^{u'}(A)}{\geq \delta_u\left(-\frac{\partial}{\partial u'}\Phi^{u'}(A)\right)} + \leq 1$$

convexity

induction

$$\mathrm{Tr}(U_A) \leq \frac{1}{\delta_u} + 1$$

Taking Averages

Goal

$$\exists \mathbf{v}, U_A \bullet \mathbf{v}\mathbf{v}^T \leq L_A \bullet \mathbf{v}\mathbf{v}^T$$

$$\sum_{\mathbf{v} \in \{v_e\}} U_A \bullet \mathbf{v}\mathbf{v}^{\mathbf{T}} \leq \frac{1}{\delta_u} + 1.$$

Taking Averages

$$\exists \mathbf{v}, U_A \bullet \mathbf{v}\mathbf{v}^T \le L_A \bullet \mathbf{v}\mathbf{v}^T$$

$$\sum_{\mathbf{v} \in \{v_e\}} U_A \bullet \mathbf{v}\mathbf{v}^{\mathbf{T}} \le \frac{1}{\delta_u} + 1.$$

$$\sum_{\mathbf{v} \in \{v_e\}} L_A \bullet \mathbf{v}\mathbf{v}^{\mathbf{T}} \ge \frac{1}{\delta_\ell} - 1.$$

Taking Averages

$$\exists \mathbf{v}, U_A \bullet \mathbf{v}\mathbf{v}^T \leq L_A \bullet \mathbf{v}\mathbf{v}^T$$

$$\sum_{\mathbf{v} \in \{v_e\}} U_A \bullet \mathbf{v}\mathbf{v}^{\mathbf{T}} \leq \frac{1}{2} + 1.$$

=3/2

$$\sum_{\mathbf{v} \in \{v_e\}} L_A \bullet \mathbf{v}\mathbf{v}^{\mathbf{T}} \geq \frac{1}{\delta_\ell} - 1.$$

Taking Averages

$$\exists \mathbf{v}, U_A \bullet \mathbf{v}\mathbf{v}^T \leq L_A \bullet \mathbf{v}\mathbf{v}^T$$

$$\sum_{\mathbf{v} \in \{v_e\}} U_A \bullet \mathbf{v}\mathbf{v}^{\mathrm{T}} \leq \frac{1}{2} + 1. \quad =3/2$$

$$\sum_{\mathbf{v} \in \{v_e\}} L_A \bullet \mathbf{v}\mathbf{v}^{\mathrm{T}} \geq \frac{1}{1/3} - 1. \quad 2$$

Taking Averages

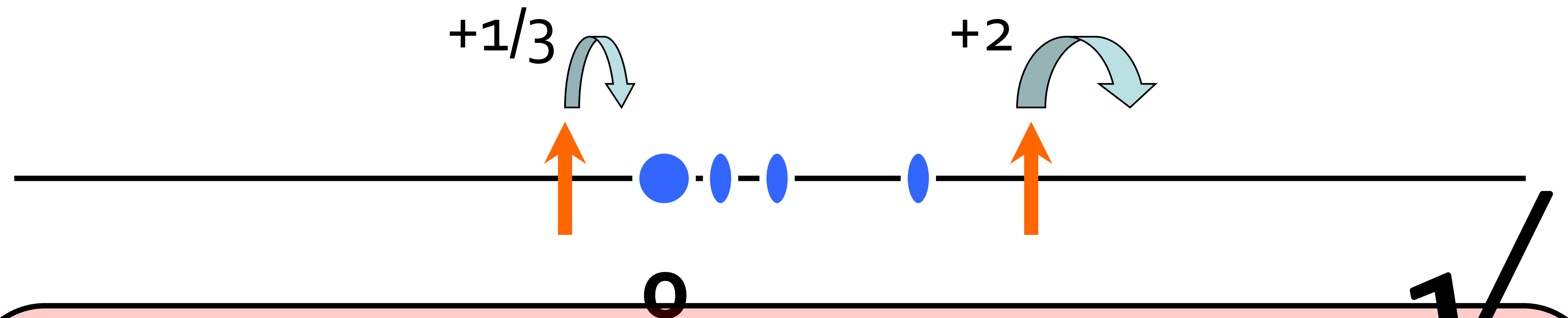
$$\exists \mathbf{v}, U_A \bullet \mathbf{v}\mathbf{v}^T \le L_A \bullet \mathbf{v}\mathbf{v}^T$$ √

$$\sum_{\mathbf{v}\in\{v_e\}} U_A \bullet \mathbf{v}\mathbf{v}^{\mathrm{T}} \le \frac{1}{2} + 1.$$ =3/2

$$\sum_{\mathbf{v}\in\{v_e\}} L_A \bullet \mathbf{v}\mathbf{v}^{\mathrm{T}} \ge \frac{1}{1/3} - 1.$$ 2

Step i+1

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}$$



Lemma.

can always choose $+s\mathbf{v}\mathbf{v}^T$

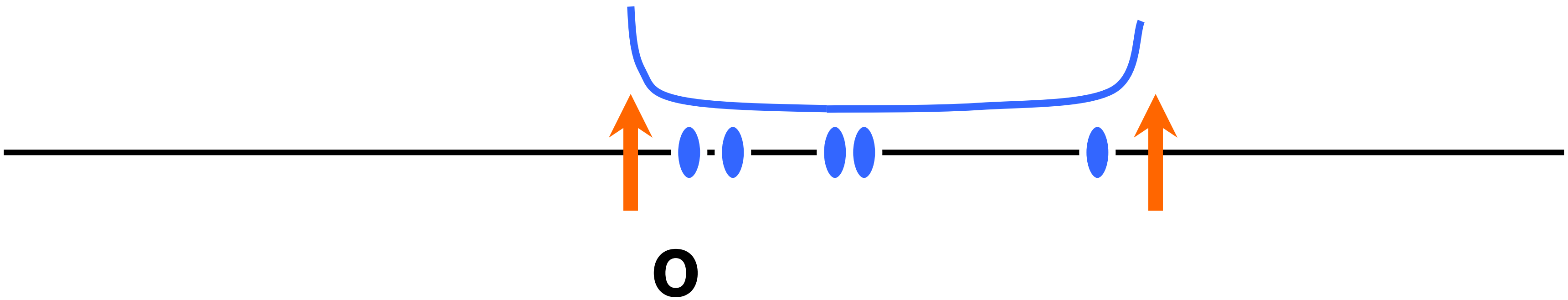
so that potentials do not increase.

$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step i+1

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}$$

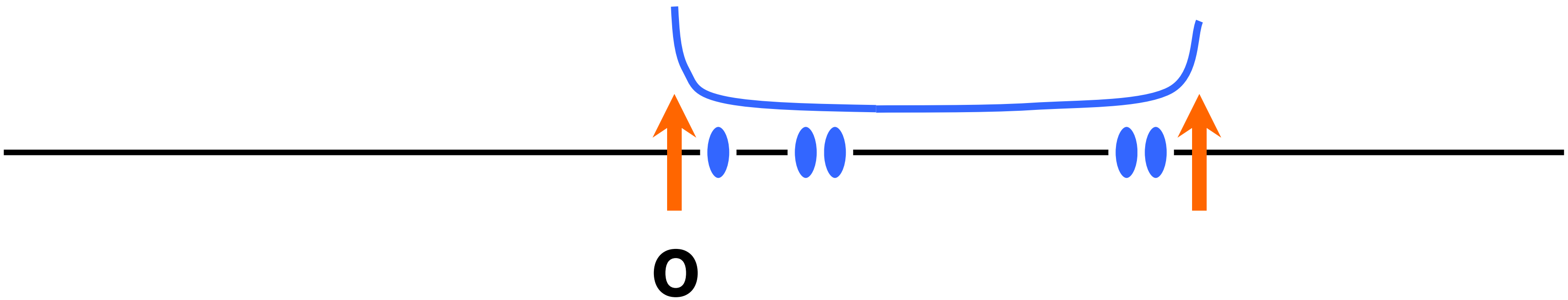


$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step i+1

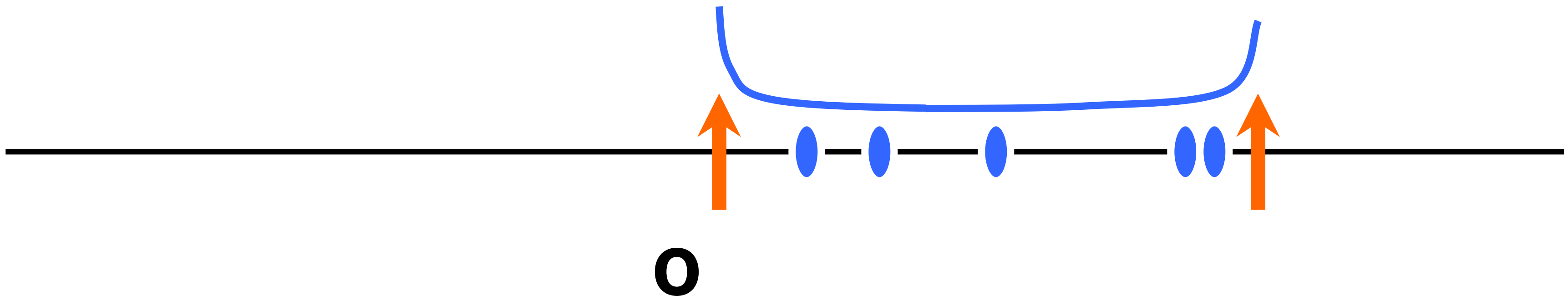
$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots$$



$$\Phi^u(A) \le 1$$
$$\Phi_\ell(A) \le 1.$$

Step i+1

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots$$

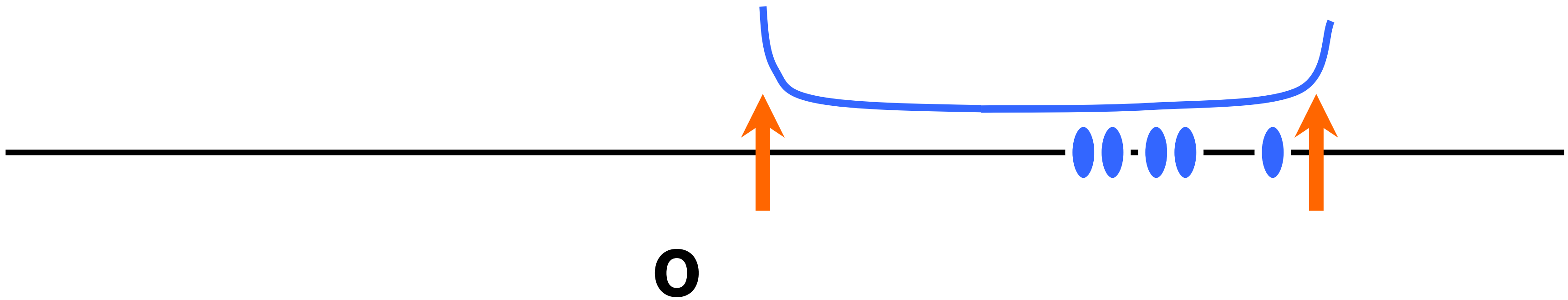


$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step i+1

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots$$



$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step 6n

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots, A^{(6n)}$$

$$\Phi^u(A) \leq 1$$
$$\Phi_\ell(A) \leq 1.$$

Step 6n

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots, A^{(6n)}$$

13-approximation with 6n vectors.

□

Done!

Spectral Sparsification Theorem: $\surd$

Given $\sum_{i\leq m} v_i v_i^T = I_n$ there are $s_i \geq 0$ with:

- $I \preceq \sum_i s_i v_i v_i^T \preceq 13 \cdot I$
- $\mathrm{supp}(s) \leq 6n.$

Nearly Optimal bound

Fixing dn steps and tightening parameters gives

$$\kappa = \frac{(\sqrt{d}+1)^2}{(\sqrt{d}-1)^2}$$

(zeros of Laguerre polynomials).

This is within a factor of 2 of the optimal Ramanujan Bound [LPS, Alon-Boppana].

□

Why does this work?

$$\Phi^u(A) = \mathrm{Tr}(uI - A)^{-1} \leq 1$$

Efficiently bounds

$\lambda(A)$

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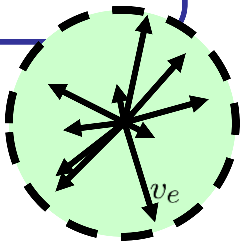
$$\Phi^u(A) = \mathrm{Tr}(uI - A)^{-1} \leq 1$$

Efficiently bounds

$\lambda(A)$

$$= \mathbb{E}_{v_e} v_e^T (uI - A)^{-1} v_e$$



$I = \sum_e v_e v_e^T$

Why does this work?

$$\Phi^u(A) = \mathrm{Tr}(uI - A)^{-1} \leq 1$$

Efficiently bounds

$\lambda(A)$

$= \mathbb{E}_{v_e} v_e^T (uI - A)^{-1} v_e$

guarantees good vector

$\Phi^{u'}(A + s\mathbf{v}\mathbf{v}^T) \leq \Phi^u(A)$

Why does this work?

$$\Phi^u(A) = \mathrm{Tr}(uI - A)^{-1} \leq 1$$

Efficiently bounds $\lambda(A)$

$$= \mathbb{E}_{v_e} v_e^T (uI - A)^{-1} v_e$$

guarantees good vector

$$\Phi^{u'}(A + s\mathbf{v}\mathbf{v}^T) \leq \Phi^u(A)$$

Major Themes

- Electrical model of **interlacing** is useful
- Can use barrier potential to **iteratively** construct matrices with desired spectra
- Analysis of progress is **greedy / local**
- Requires **fractional weights** on vectors

Instead of directly reasoning about $\lambda_i(A)$, reason about $(zI - A)^{-1}$.

Open Questions

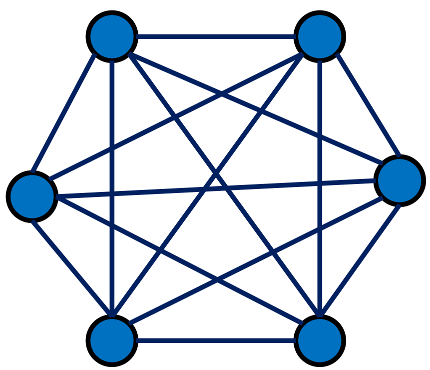
Fast algorithm currently $O(n^4)$

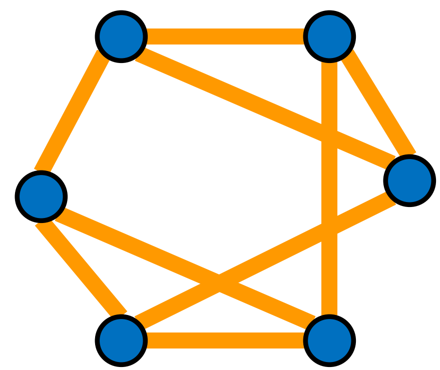
Optimization proof?

More applications

There are no weights here…

G=K$_n$ **H** = random d-regular x **(n/d)**





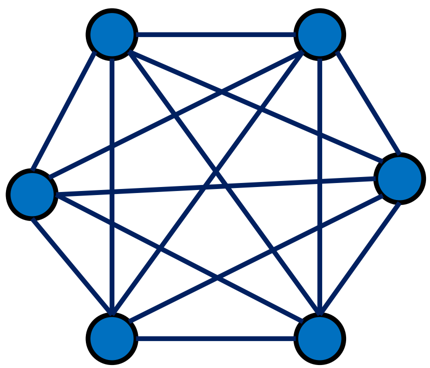
$|E_G| = O(n^2)$ $|E_H| = O(dn)$

$$d = 2/\epsilon^2$$

$$\forall x, \quad \frac{x^T L_G x}{x^T L_H x} \simeq 1 \pm \epsilon$$

And off by a factor of 2

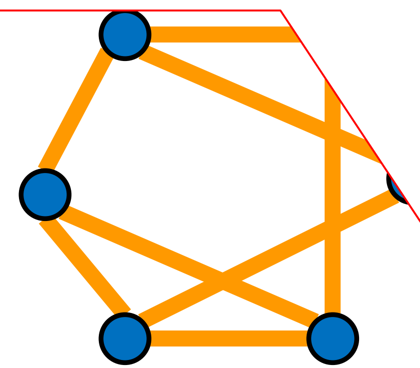
G=K$_n$



$|E_G| = O(n^2)$

H = ra

We get $4n/\epsilon^2$



$|E_H| = O(dn)$

$d = 2/\epsilon^2$

$$\forall x, \quad \frac{x^T L_G x}{x^T L_H x} \simeq 1 \pm \epsilon$$

Tomorrow

$2/\epsilon^2$ degree unweighted approximations for K_n

"Ramanujan Graphs"

Tomorrow

$2/\epsilon^2$ degree unweighted approximations for K_n

"Ramanujan Graphs"

$$\mathbb{E}p^{(k)} = \left(1 - \frac{1}{m}\frac{\partial}{\partial x}\right)^k x^n$$

$$4n/\epsilon^2$$

This is not a dream.