# The Revolution in Graph Theoretic Optimization Problem

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Simons Open Lecture

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# JOINT WORK

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Alexsander Madry,
Jakub Pachocki,
Richard Peng,
Kanat Tangwongsan,
Shen Chen Xu

# OUTLINE

- Linear system solvers
- Regression and Image Denoising.
- Simple formulation and connection with solving linear systems.
- Overview of SDD solvers.
- A better $L_1$ formulation of denoising.
- Maximum flow using solvers
- New for 2013-14

# SPECTRAL GRAPH THEORY
# LAPLACIAN PARADIGM

Use graph algorithms to solve linear algebra problems.

Use linear algebra to solve graph problems.

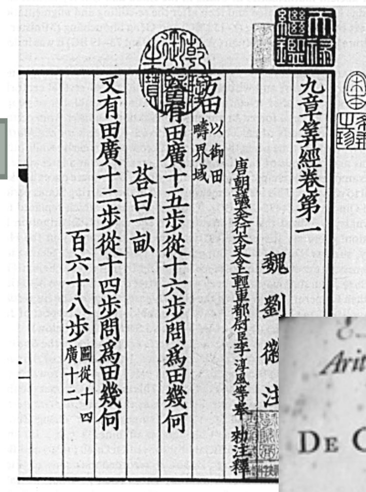
Use both to solve optimization problems

# OLDEST COMPUTATIONAL PROBLEM

$$\begin{pmatrix} 3 & 2 & -1 \\ 2 & -5 & 4 \\ -1 & 1/2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix}$$

# DIRECT LINEAR SYSTEM SOLVES

- [1st century CE] Gaussian Elimination: $O(n^3)$
- [Strassen `69] $O(n^{2.8})$
- [Coppersmith-Winograd `90] $O(n^{2.3755})$
- [Stothers `10] $O(n^{2.3737})$
- [Vassilevska Williams`11] $O(n^{2.3727})$
- [George `73], [Lipton-Rose-Tarjan `80], [Alon-Yuster `10] Faster direct methods for special non-zero structures.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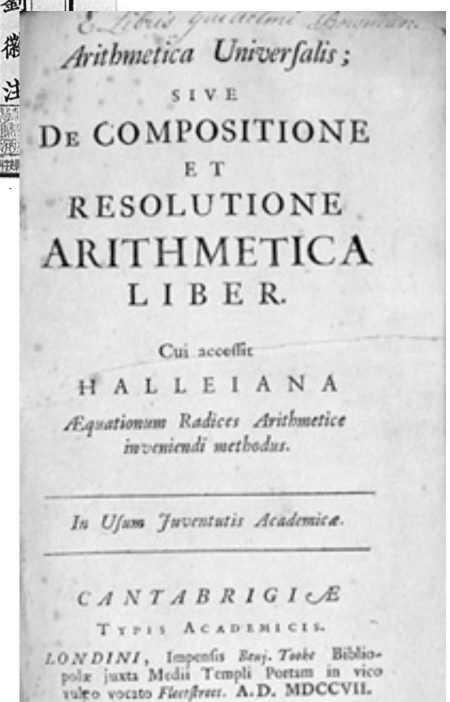
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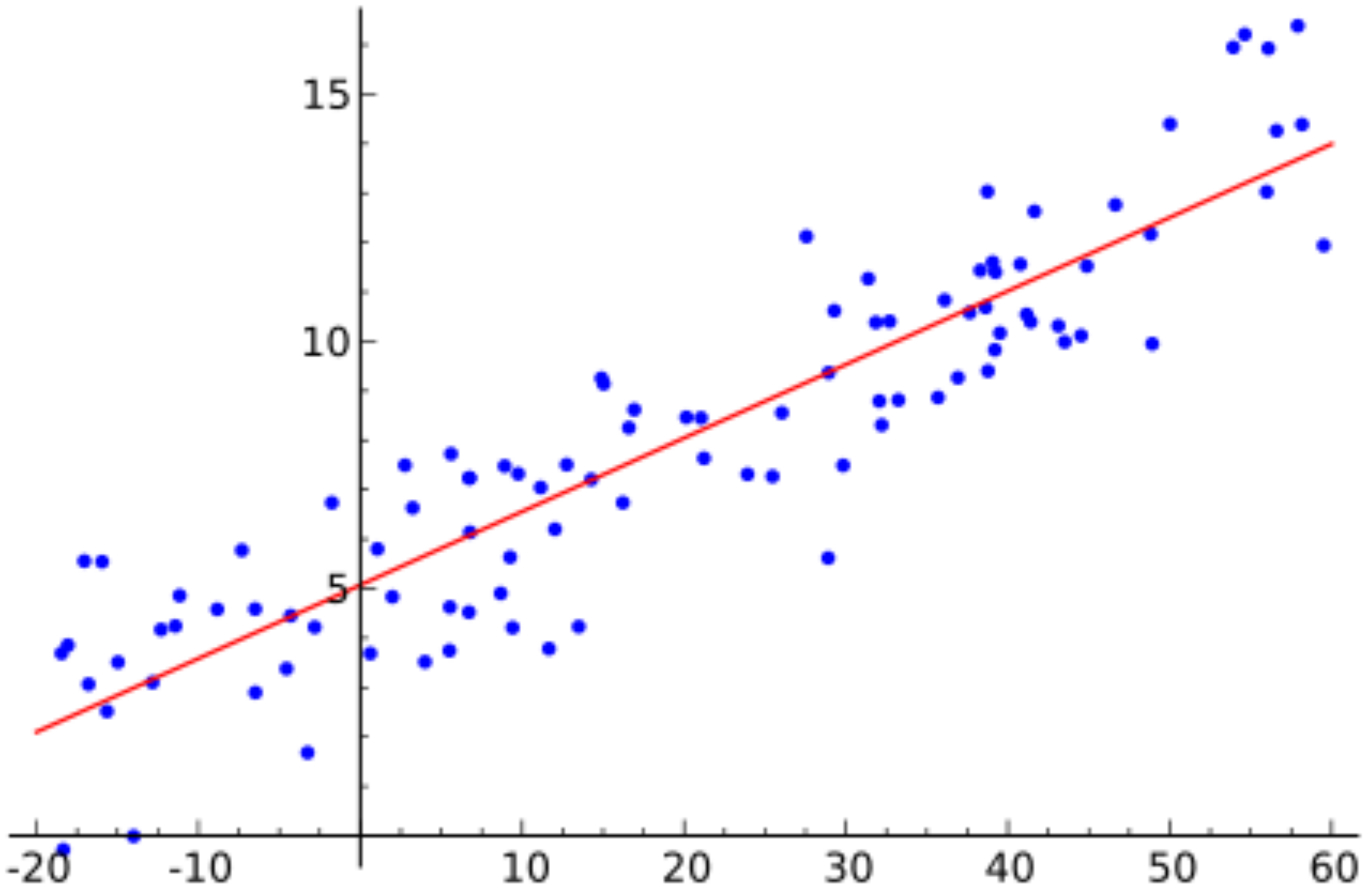
# REGRESSION



# OVER CONSTRAINED SYSTEMS

Over Constrained System: $A\ x = b$.

Solve system $A^T A\ x = A^T b$

- Matrix $A^T A$ is Symmetric Positive Semi-Definite (SPSD).
- Open Question:
  Find sub-quadratic time solvers for SPD systems?

We will need problems with an underlying graph.

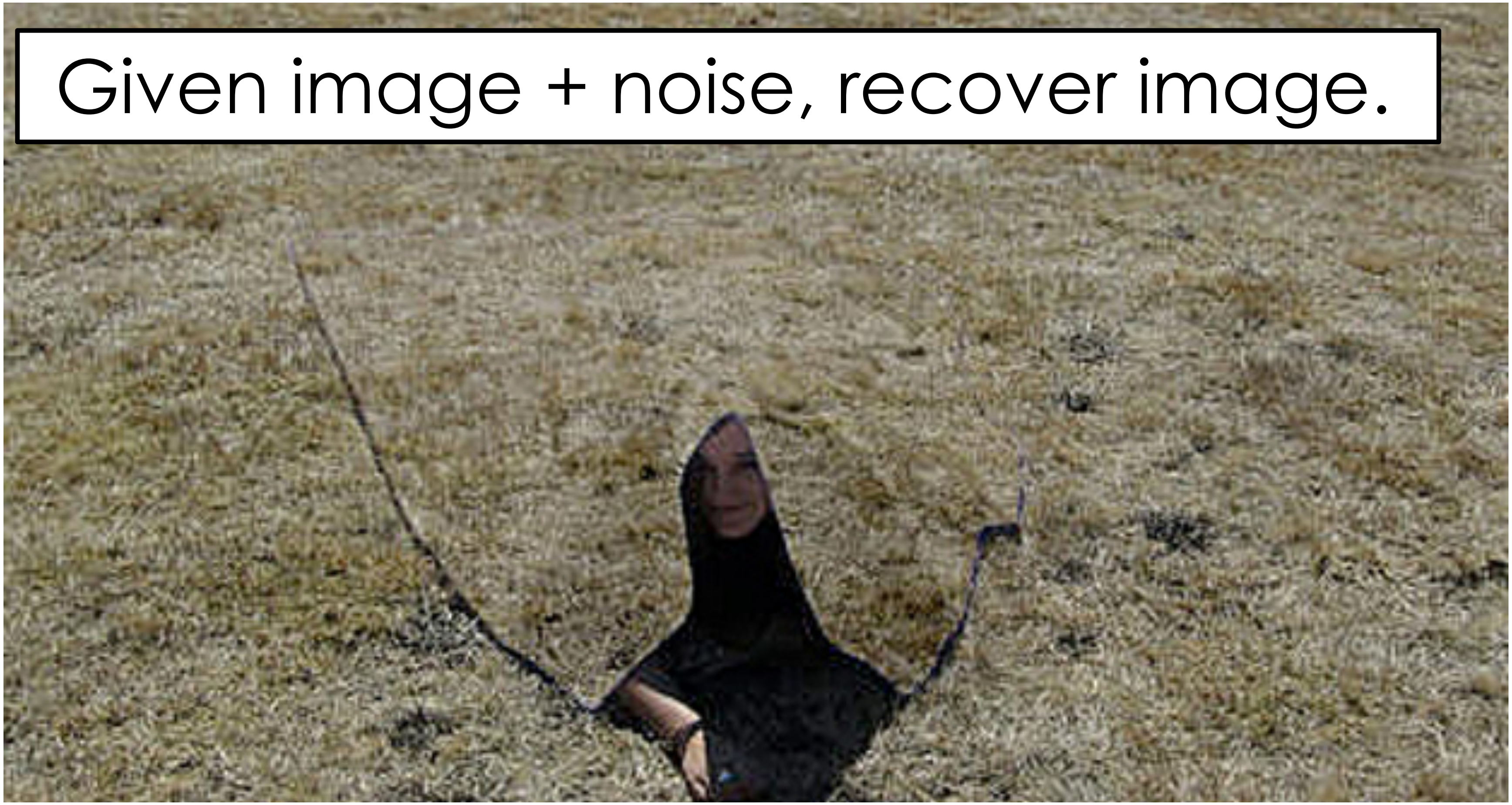
# APPROXIMATION ALGORITHMS

- Whole conferences NP-Approximation
- Same ideas and goals can be  applied problems in Polytime.
- Our goal is find  good approximation but much faster than known exact solutions.
- Maybe even faster exact solutions!

# CLASSIC REGRESSION PROBLEM

- Image Denoising
- Critical step in image segmentation and detection
- Good denoising makes the segmentation almost obvious.

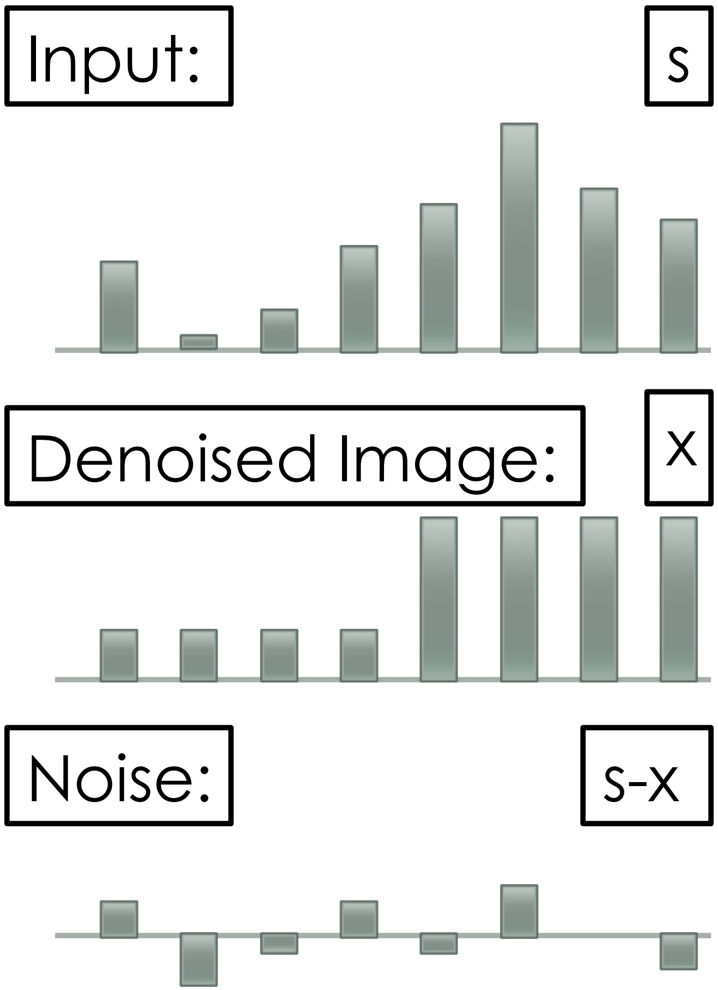
# CAMOUFLAGE DETECTION



# CAMOUFLAGE DETECTION

Hui-Han Chin

# IMAGE DENOISING: THE MODEL




- Assume there exist a 'original' noiseless image.
- Noise generated from some distribution.
- Input: original + noise.
- Goal: approx the original image.

# CONDITIONS ON X

- Noise is small: denoised image should still be close to image + noise.
- Real images are 'smooth' except at boundaries.

Function of (x-s)

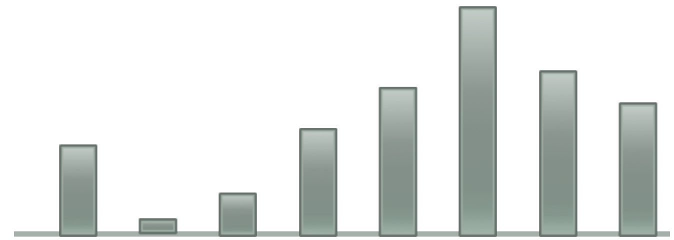
Fidelity(x, s)

Function computed on differences of neighboring pixels of x

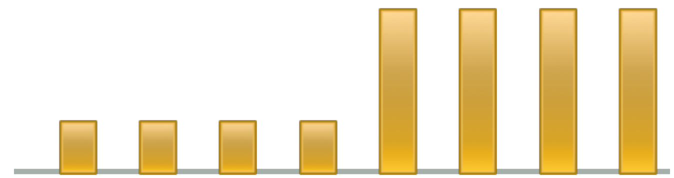
Smoothness(x)

# ENERGY FUNCTION

Noisy image: s



Candidate solution: x



Fidelity(x, s) + Smoothness(x)

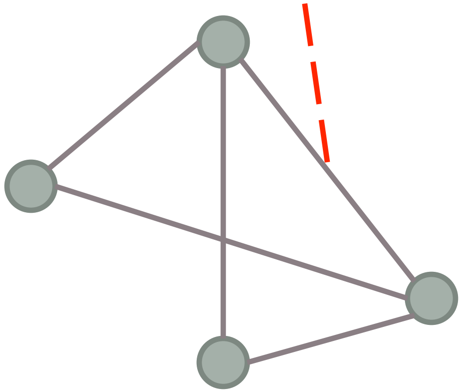
$(x_i - s_i)^2$ summed over all pixels i.

$(x_i - x_j)^2$ summed over all neighbor pixels i, j.

This is a toy example

# MATRICES ARISING FROM IMAGE PROBLEM HAVE NICE STRUCTURES

$$\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$



A is Symmetric Diagonally Dominant (SDD)

- Symmetric.
- Diagonal entry ≥ sum of absolute values of all off diagonals.

# OPTIMIZATION PROBLEMS IN CS

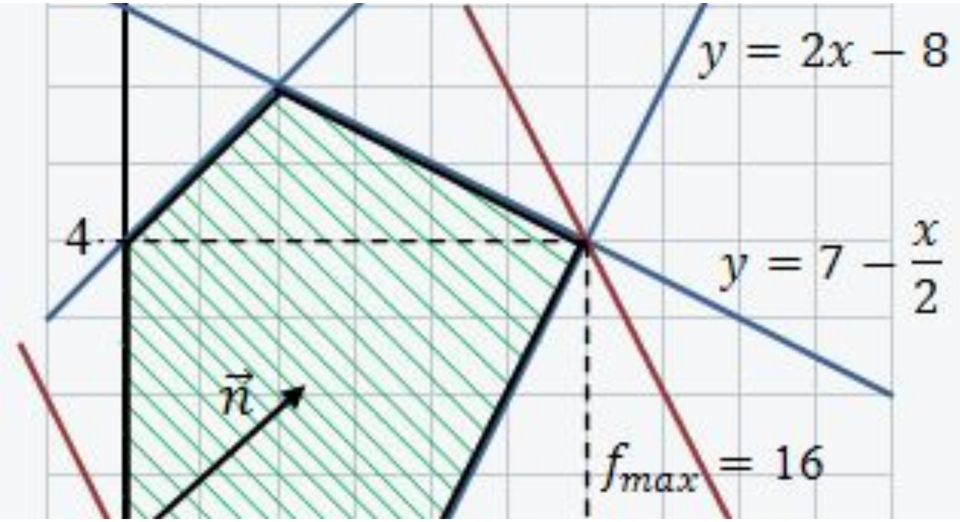
Many algorithm problems in CS are optimization problems with underlying graph.

- Maximum flow in a graph.
- Shortest path in a graph.
- Maximum Matching.
- Scheduling
- Minimum cut.

Some of these do not seem to have an underlying graph.

Longest common subsequence.

# LINEAR PROGRAMMING



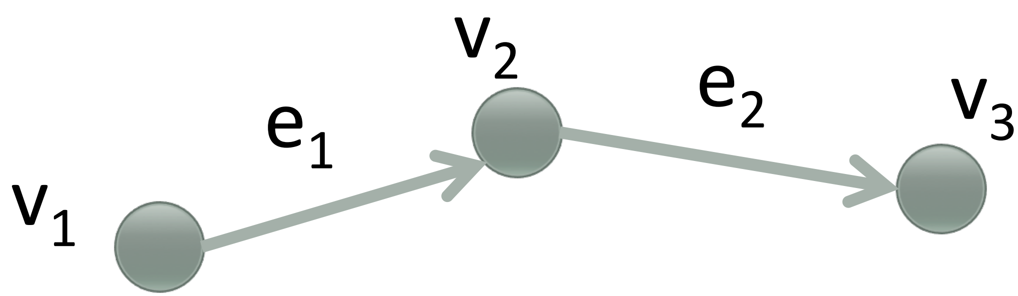

Many optimization problems can be written as an LP

EG: Single source shortest path.

Is this useful?

# LAPLACIAN PRIMER

- Matrix view of flows
- The Boundary Map B: Flow ➔ Residual Flow




$$
\begin{array}{c}
\\
v_1 \\
v_2 \\
v_3
\end{array}
\begin{array}{c}
\begin{array}{cc} e_1 & e_2 \end{array} \\
\begin{pmatrix} -1 & 0 \\ +1 & -1 \\ 0 & +1 \end{pmatrix}
\end{array}
\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ -3 \end{pmatrix}
$$

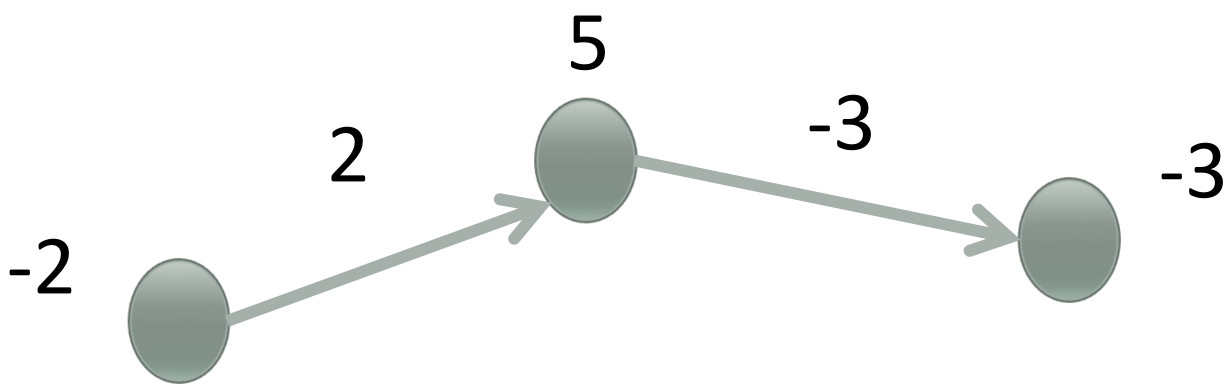
# THE BOUNDARY MAP B

Let $G = (V,E)$ be n vertex m oriented edges graph.

- Def: B is a Vertex by Edge matrix
  where $B_{ij} = +1$ if $v_i$ is head of $e_j$
  $-1$ if $v_i$ is tail of $e_j$
  $0$ otherwise

- Note: If f is a flow then Bf is residual vertex flow.

# BOUNDARY MATRIX




$$\begin{array}{cc} & \begin{array}{cc} \color{blue}{e_1} & \color{blue}{e_2} \end{array} \\ \begin{array}{c} \color{blue}{v_1} \\ \color{blue}{v_2} \\ \color{blue}{v_3} \end{array} & \begin{pmatrix} -1 & 0 \\ +1 & -1 \\ 0 & +1 \end{pmatrix} \end{array} \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ -3 \end{pmatrix}$$

# $B^T$ AND POTENTIAL DROPS

- Let v be a n-vector of potentials
- $B^Tv$ = vector of potential drops.
- $R^{-1}BTv$ = vector of potential drops.
  - R a diagonal matrix of resistive values
  - Ohms law:  Rule to go from potentials to flows.
- Today we set resistors all to one.
- Thus $B^Tv$ = vector of flows.

# GRAPH LAPLACIAN

- Def: $L := BB^T$ , Laplacian of G.
- Goal: Solve $Lv=b$
- Suppose v satisfies $BB^Tv=b$.

  thus $f := B^Tv$ is a flow s.t. $Bf=b$
- What can we say about f?

# CIRCULATIONS AND POTENTIAL FLOWS

- Def: $f_c$ in $R^m$ is a circulation if $Bf_c = 0$
- Def: $f_p$ in $R^m$ is a potential flow if $f_p = B^T v$
- Claim : $f_c$ is orthogonal to $f_p$
  then $f_c^T f_p = 0$

Pf: $f_c^T f_p = f_c^T (B^T v) = (Bf_c)^T v = 0^T v = 0$

# CIRCULATIONS AND POTENTIAL FLOWS

- Note: Dim(potential flows) = n-1

  (G connected)
- Claim : Dim(circulations) = m –n+1

  (The number of nontree edges.)

Pf: Given a spanning tree each nontree edge induces cyclic flow that is independent.

# POTENTIALS AND FLOWS

- Suppose $BB^Tv=b$ then $f=B^Tv$ is a flow s.t $Bf=b$
- Claim: min $f^Tf$ s.t. $Bf=b$ is a potential flow.

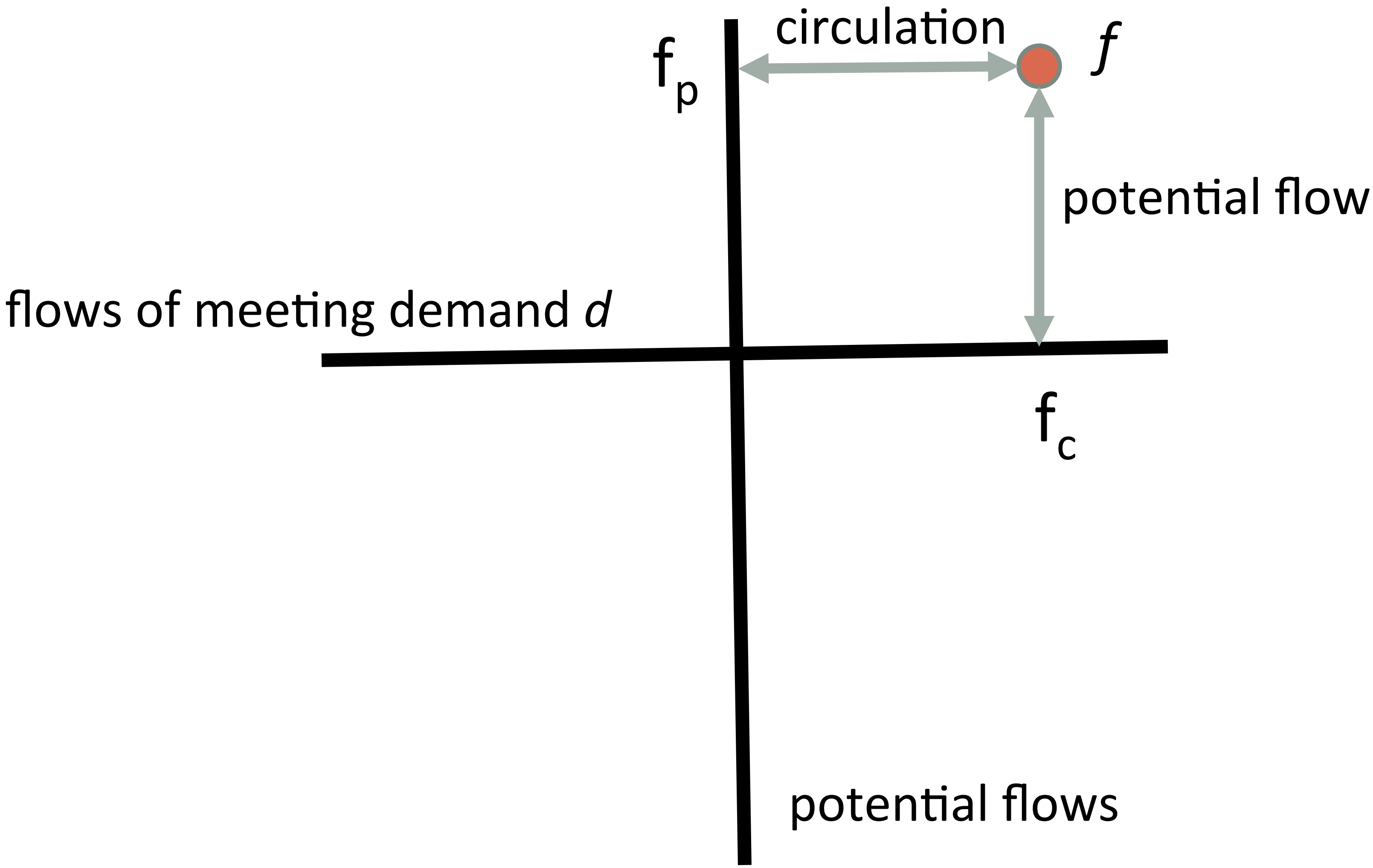
Pf: $f = f_p+f_c$

- $Bf_p = Bf_p + Bf_c = B(f_p+f_c) = b$
- $f^Tf = (f_p+f_c)^T(f_p+f_c)$
  $= f_p^2 + 2f_p^Tf_c + f_c^2$
  $= f_p^2 + f_c^2 \quad \geq f_p^2$

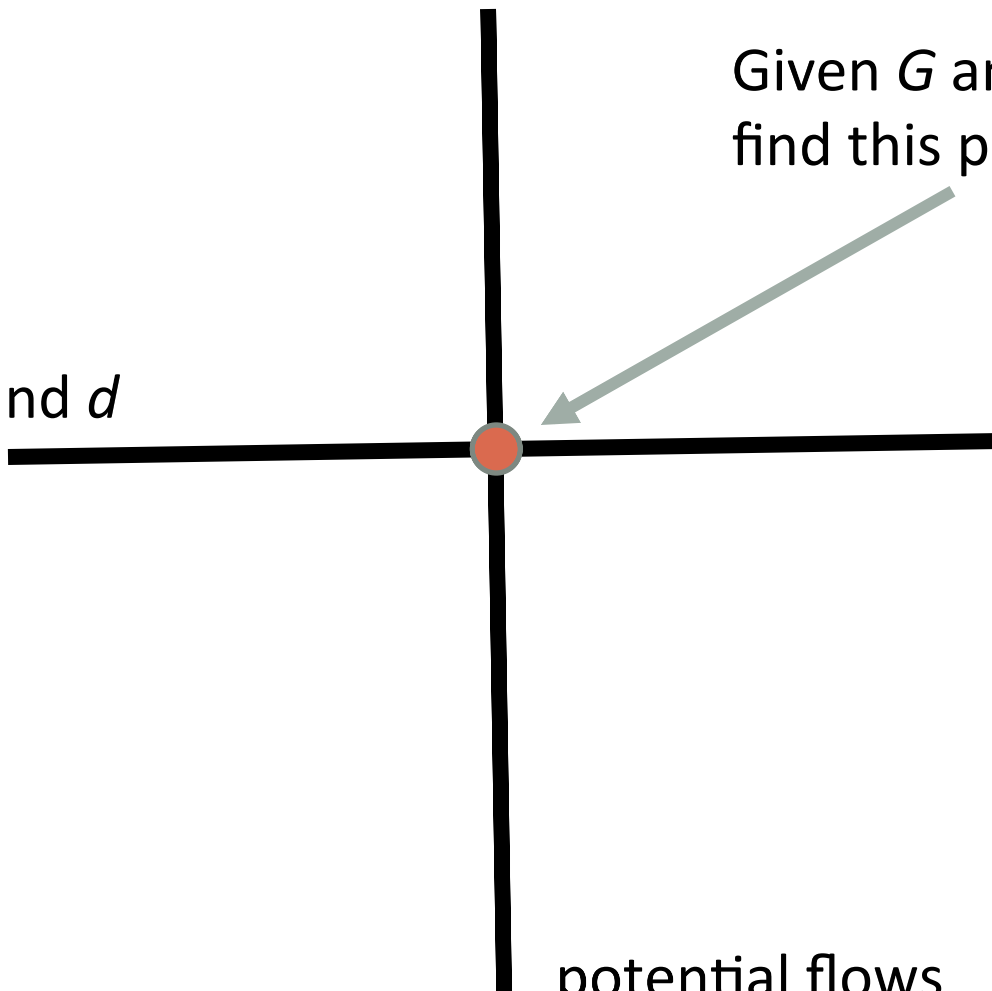
# GRAPH LAPLACIAN SOLVERS

- Def: $L := BB^T$, Laplacian of G.
- Two dual approaches to approximately solving $Lv = b$
- 1) Find a potential that minimizes $Lv-b$
- 2) find a minimum energy flow f s.t. $Bf=b$

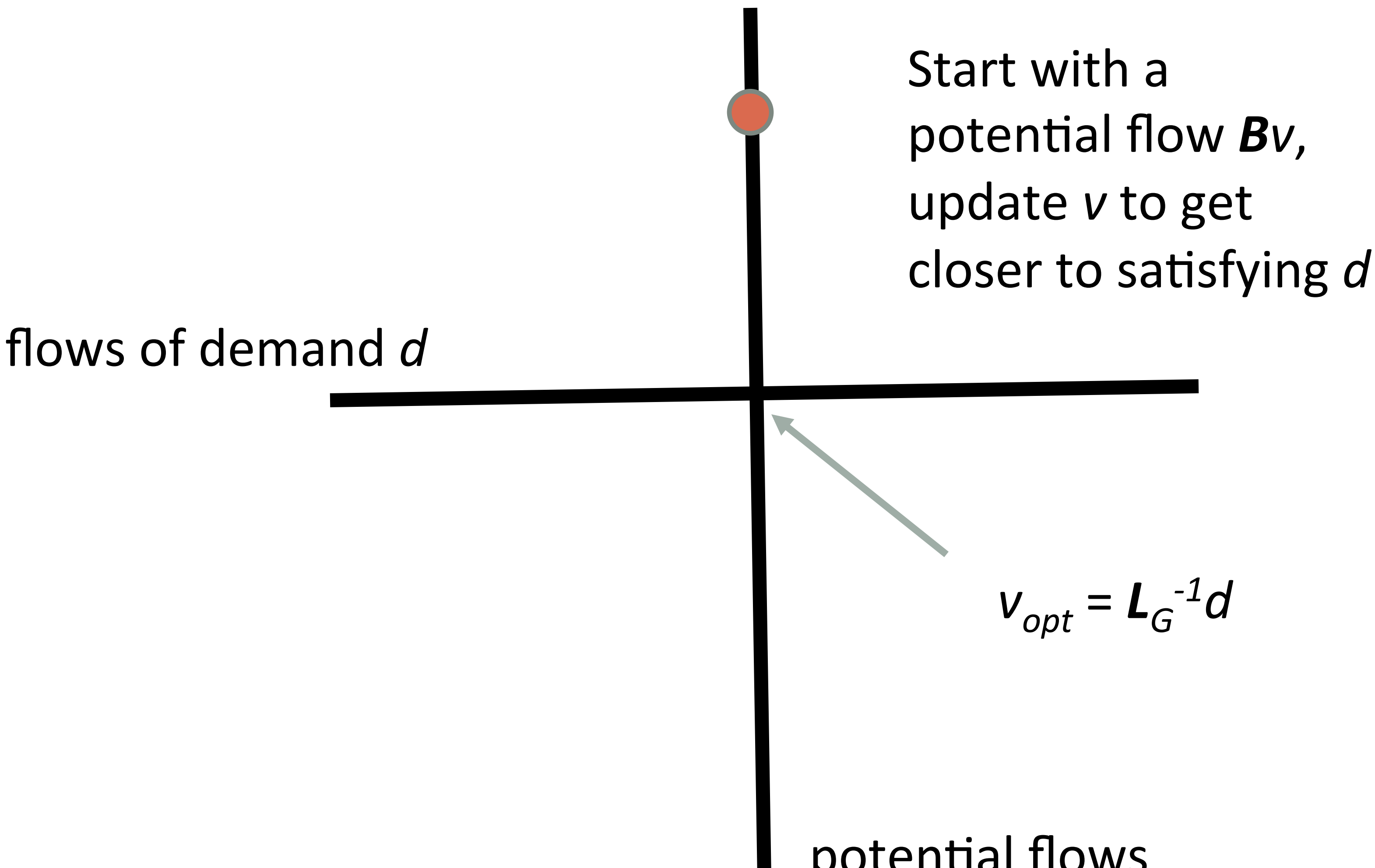
# THE SPACE OF FLOWS



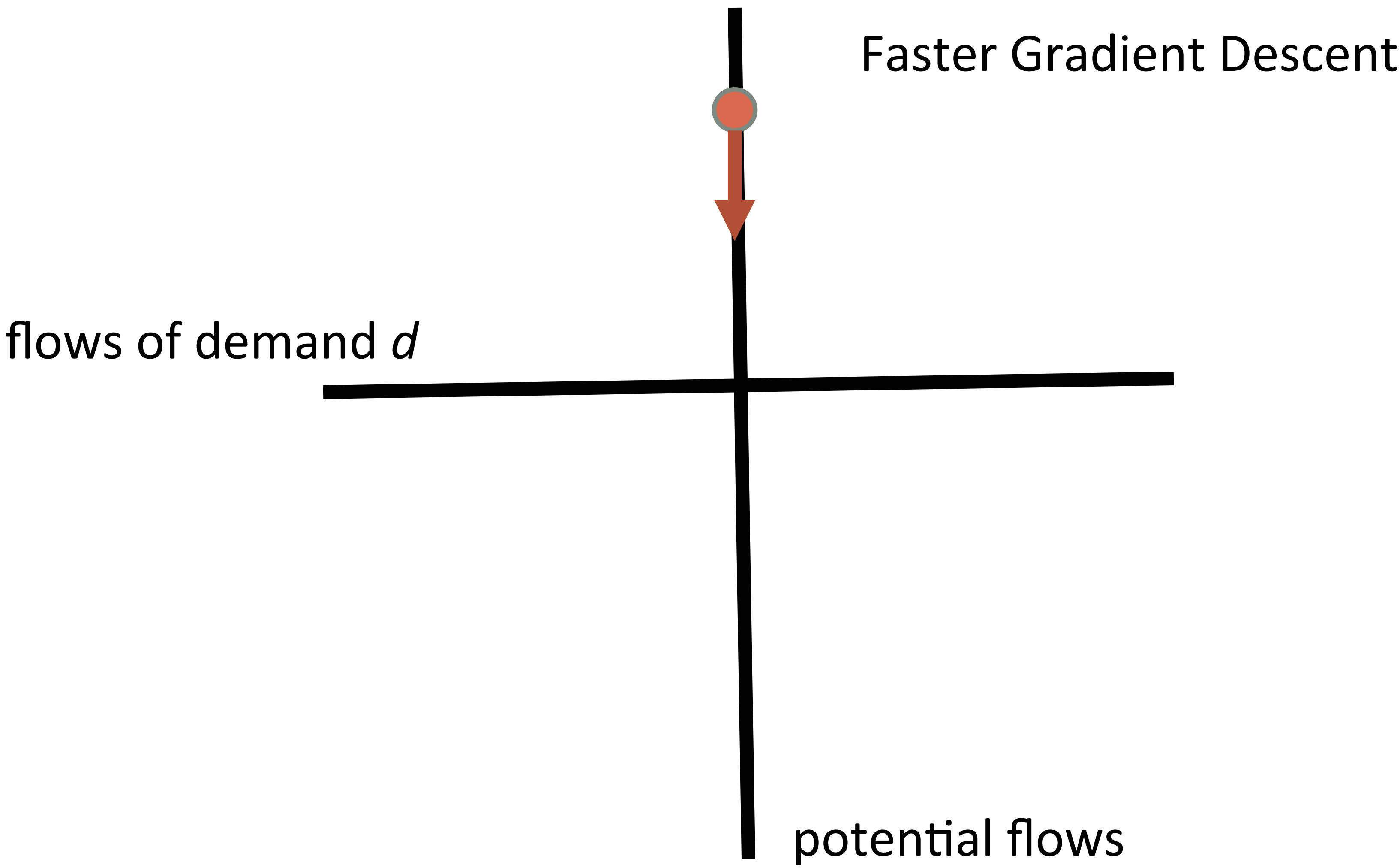
# SOLVING LAPLACIANS



# DUAL APPROACH: SOLVING A LINEAR SYSTEM



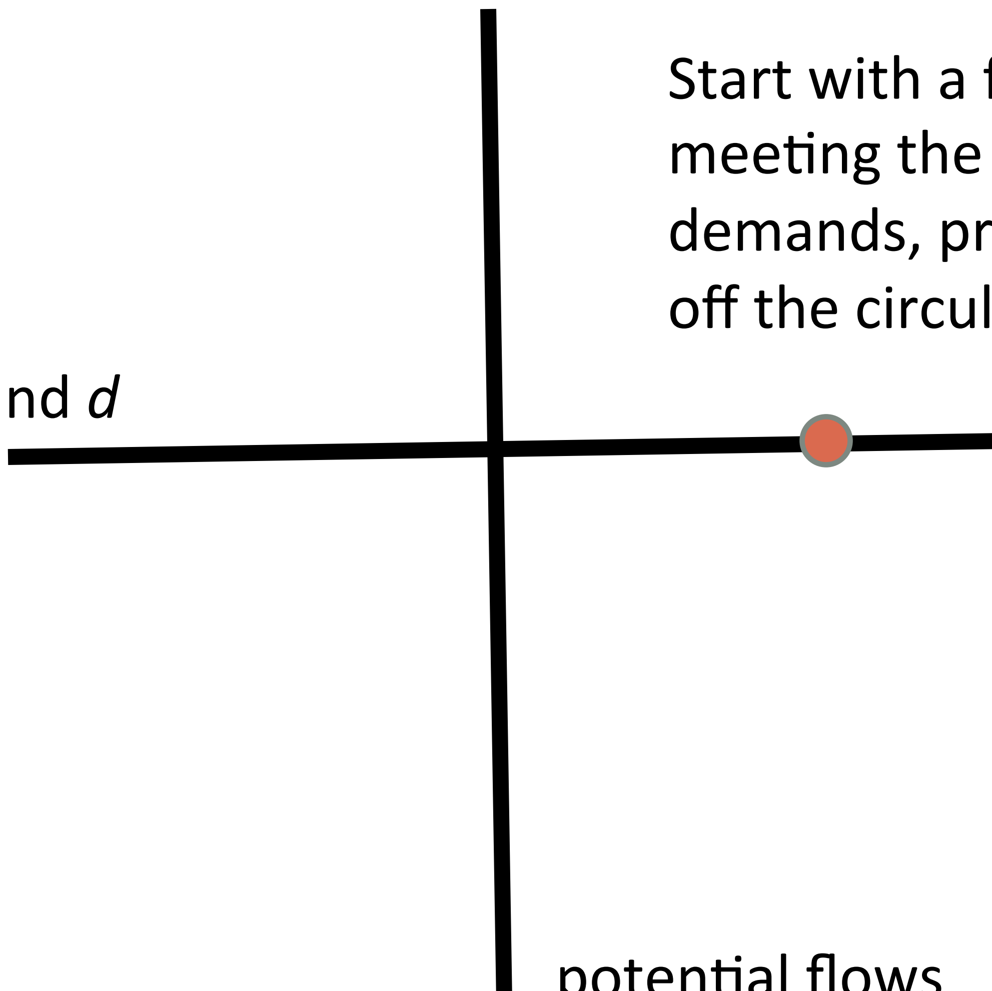
# DUAL APPROACH: SINGLE STEP (ST'04,KMP '10, '11)



# PRIMAL APPROACH: SOLVING A FLOW PROBLEM

Start with a flow meeting the demands, project off the circulation

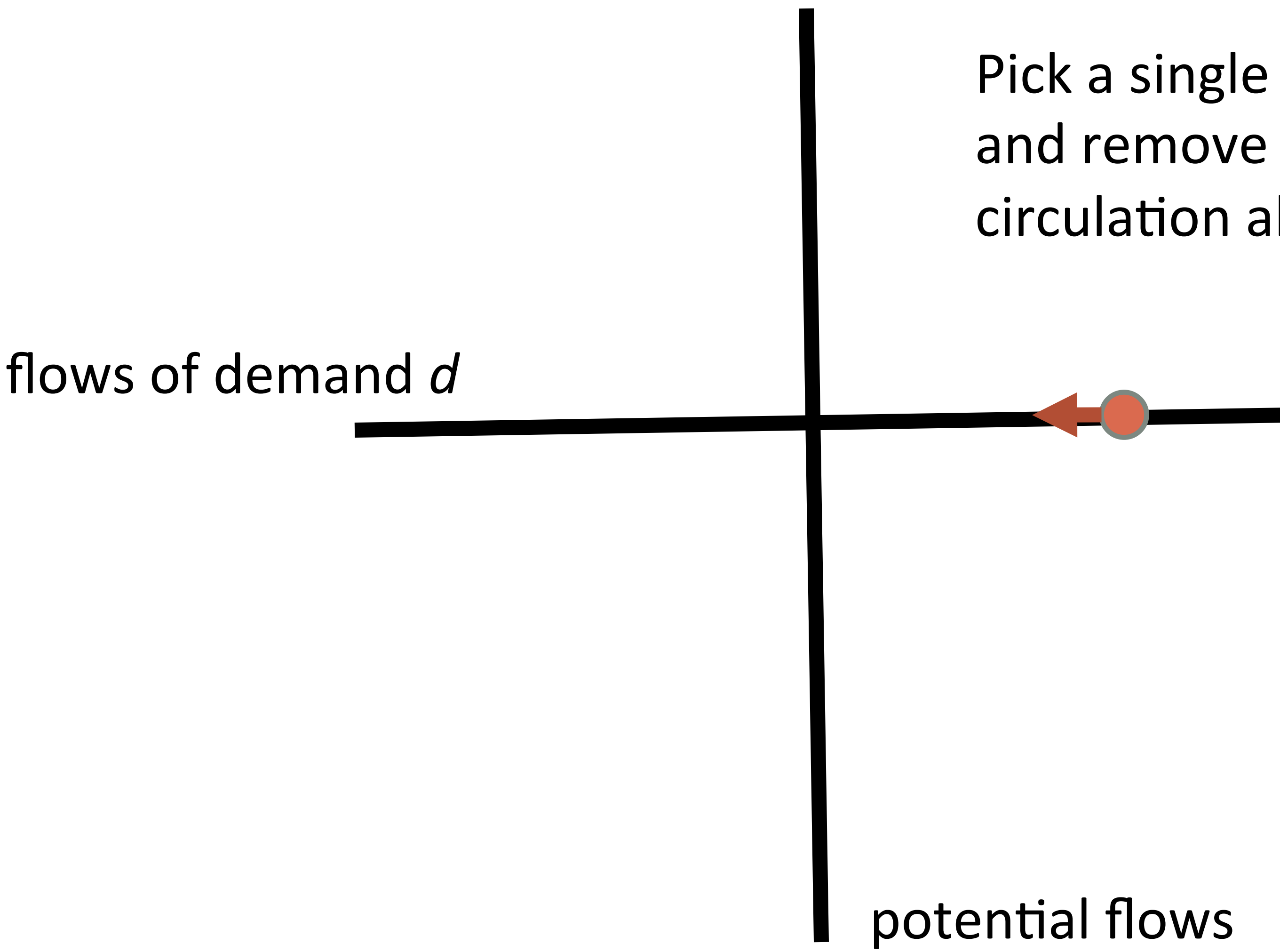
flows of demand *d*



potential flows

# PRIMAL APPROACH: SINGLE STEP (KOSZ '13)

Pick a single cycle and remove the circulation along it



# POTENTIAL BASED SOLVERS [SPIELMAN-TENG`04] [KOUTIS-M-PENG`10, `11]

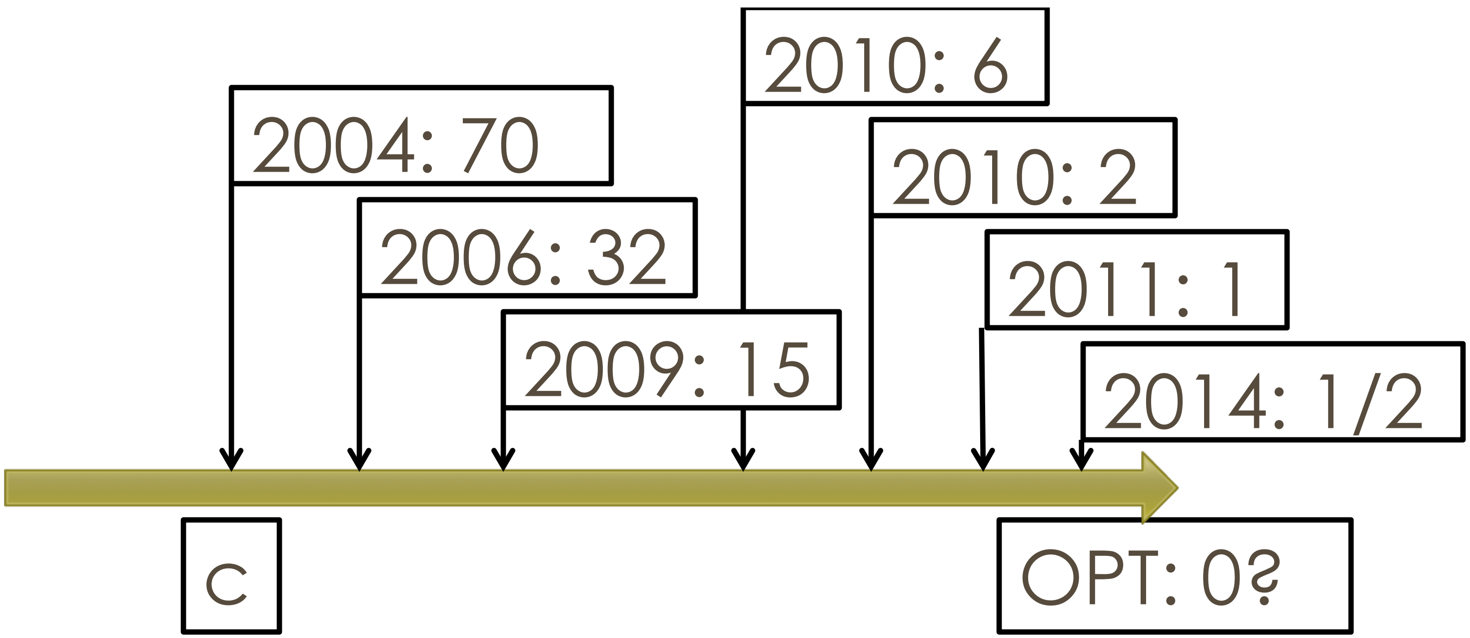
**Input**: n by n SDD matrix A with m non-zeros
vector b
**Output**: Approximate solution Ax = b
**Runtime**: O(m log n )

[Blelloch-Gupta-Koutis-M-Peng-Tangwongsan. `11]:
Parallel solver, $O(m^{1/3})$ depth and nearly-linear work

# ZENO'S DICHOTOMY PARADOX

$O(m\log^{\boxed{c}} n)$

Fundamental theorem of Laplacian solvers: improvements decrease c by factor between [2,3]



# FLOW BASED SOLVERS [KELNER-ORECCHIA-SIDFORD-ZHU `13] [LEE-SIDFORD `13]

**Input**: n by n SDD matrix A with m non-zeros, demand b
**Output**: Approximate minimum energy electrical flow
**Runtime**: $O(m \log^{1.5} n)$

# POTENTIAL BASED SOLVER AND ENERGY MINIMIZATION

- Suppose that A is SPD:

  Claim: minimizing $\frac{1}{2} x^T A x - x^T b$

  gives solution to $Ax = b$.

  Note: Gradient = $Ax-b$

Thus solving these systems are quadratic minimization problems!

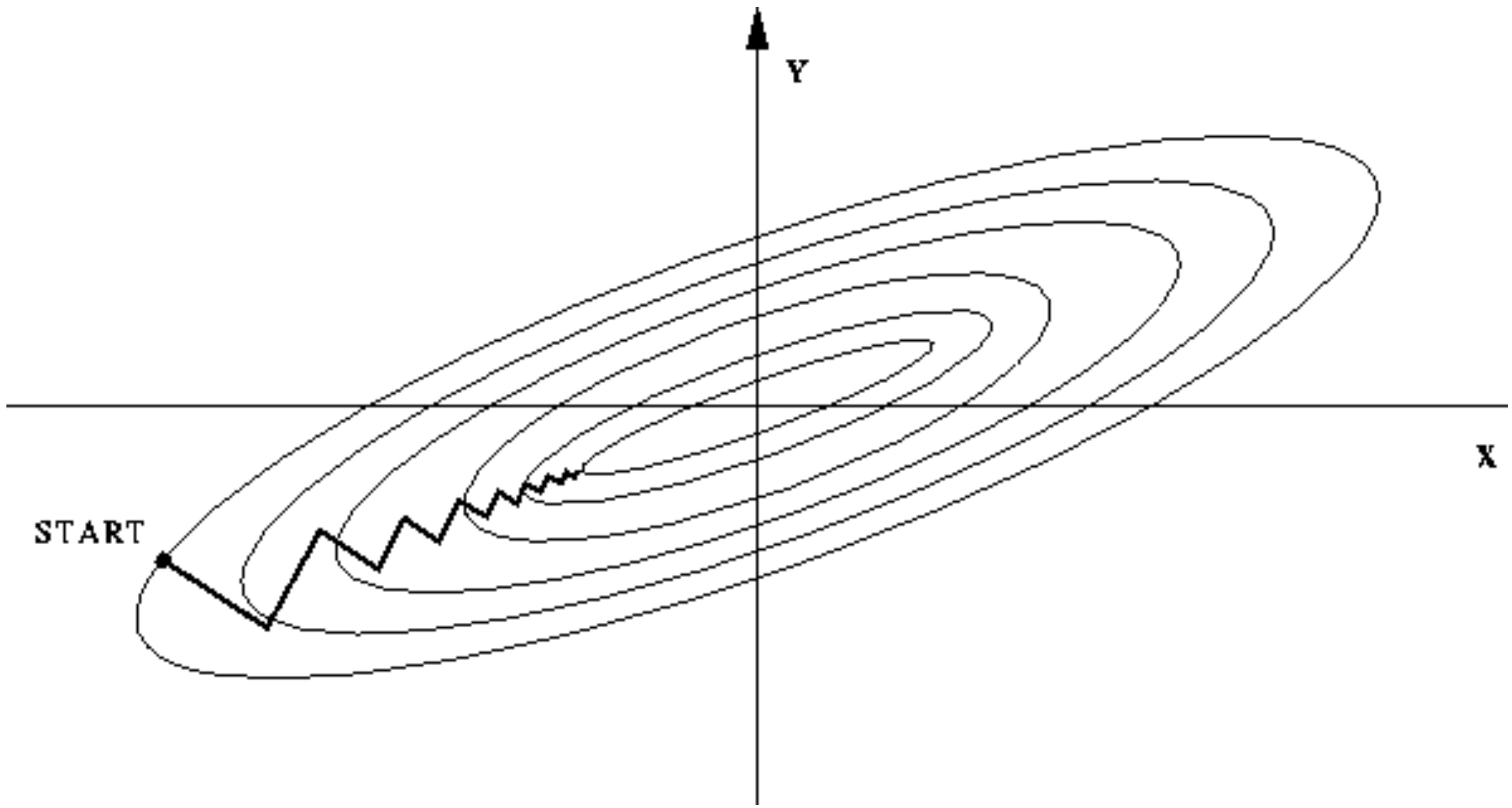
# ITERATIVE METHOD
# GRADIENT DESCENT

- Goal: approx solution to Ax = b
- Start with initial guess $u^0 = 0$
- Compute new guess

$$u^{(i+1)} = u^{(i)} + (b - Au^{(i)})$$

This maybe slow to converge or not converge at all!

# STEEPEST DESCENT



# PRECONDITIONED ITERATIVE METHOD

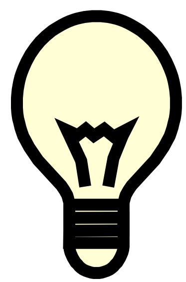
- Goal: approx solution to $B^{-1}Ax = B^{-1}b$
- Start with initial guess $u^0 = 0$
- Compute new guess

$$u^{(i+1)} = u^{(i)} - B^{-1}(b - Au^{(i)})$$

Recursive solve Bz=y where y=($b$-A$u^{(i)}$).

# PRECONDITIONING WITH A GRAPH

[Vaidya `91]: Since A is a graph, B should be as well.
Apply graph theoretic techniques!



And use Chebyshev acceleration

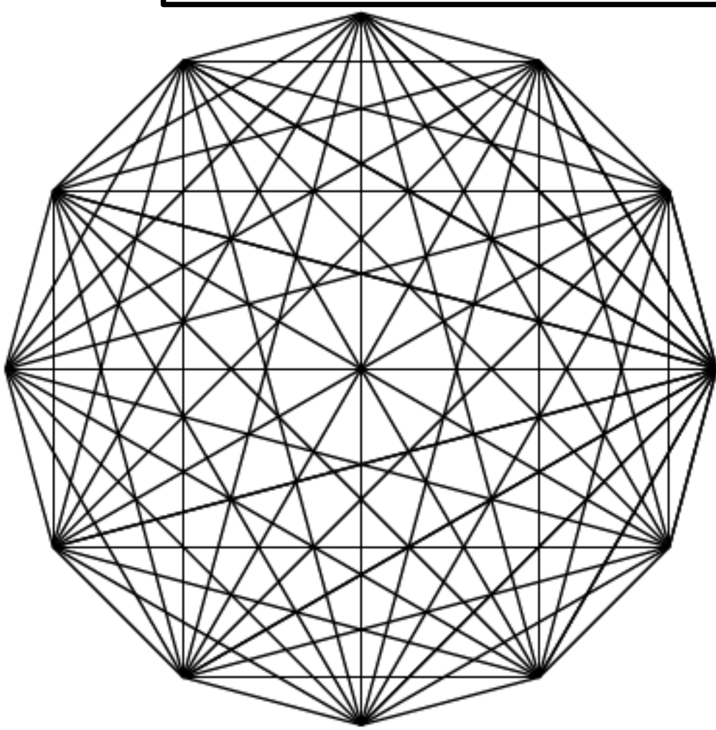
# GRAPH SPARSIFIERS

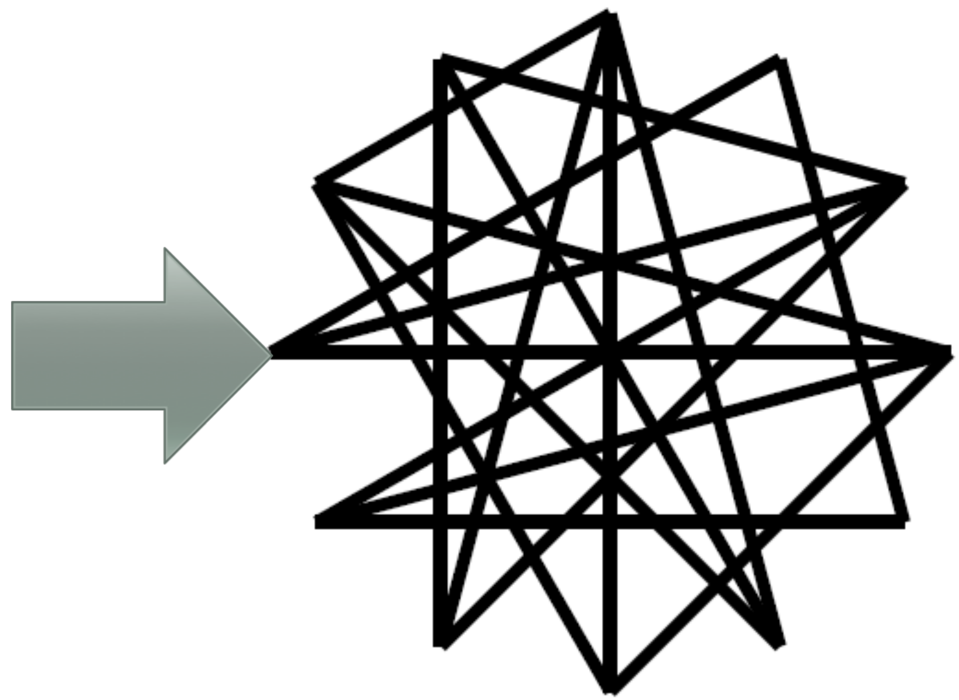
Sparse Equivalents of Dense Graphs that preserve some property

- Spanners: distance, diameter.
- [Benczur-Karger '96] Cut sparsifier: weight of all cuts.
- Spectral sparsifiers: eigenstructure

# EXAMPLE: COMPLETE GRAPH

O(n logn) sampling edges uniform suffice!





# SPECTRAL SPARSIFICATION BY EFFECTIVE RESISTANCE

What probability P(e) should we sample edge?

Answer: [Spielman-Srivastava `08]:

- Set P(e) = R(u,v) effective resistance from u to v.
- For each sample set edge set weight to 1/P(e)
- Sample O(n log n) times.

spectral sparsifier with O(nlogn) edges for any graph

Fact: $\sum_e R(e) = n-1$

# THE CHICKEN AND EGG PROBLEM

How To Calculate Sample Probablity?

[Spielman-Srivastava `08]: Use Solver

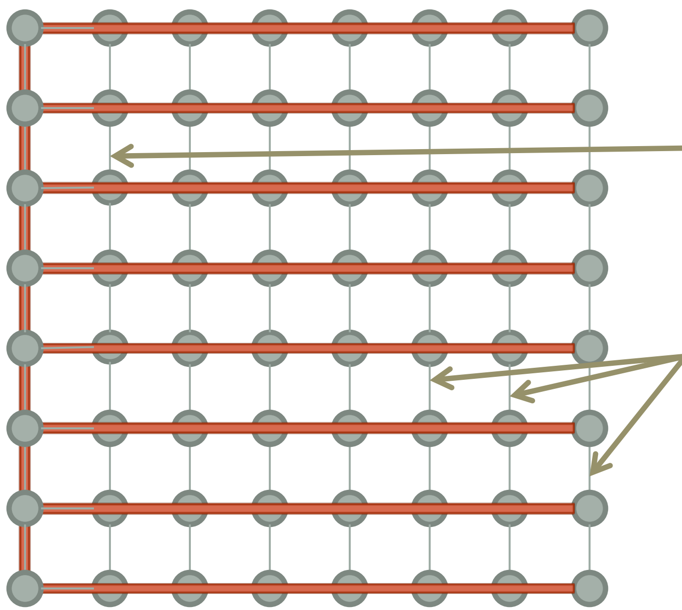
[Spielman-Teng `04]: Need Sparsifier

Workaround: upper bound using Low Stretch Spanning Trees

# CHOICE OF TREES MATTER

$n^{1/2}$-by-$n^{1/2}$ unit weighted mesh

'haircomb' tree is both shortest path tree and max weight spanning tree



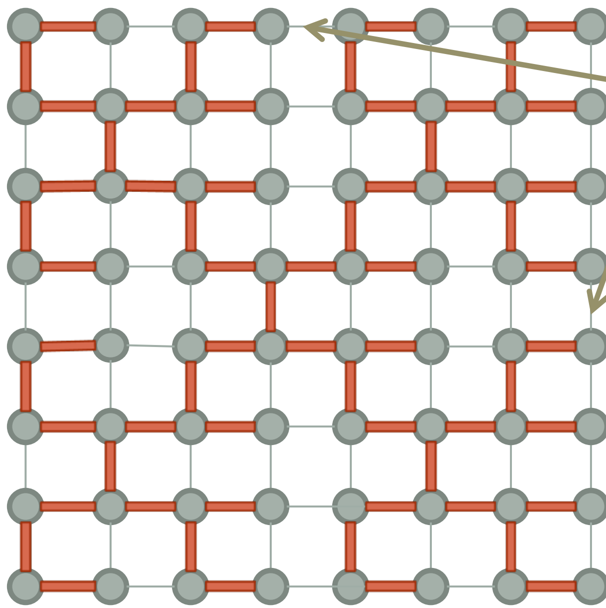
stretch(e)= O(1) 🙂

stretch(e)=O( $n^{1/2}$) ☹

total stretch = O($n^{3/2}$) ☹

# AN O(N LOG N) STRETCH TREE

Recursive 'C' Construction



stretch(e) still $O(n^{1/2})$ ☹

But only $O(n^{1/2})$ such edges ☺

logn levels, total = O(nlogn) ☺

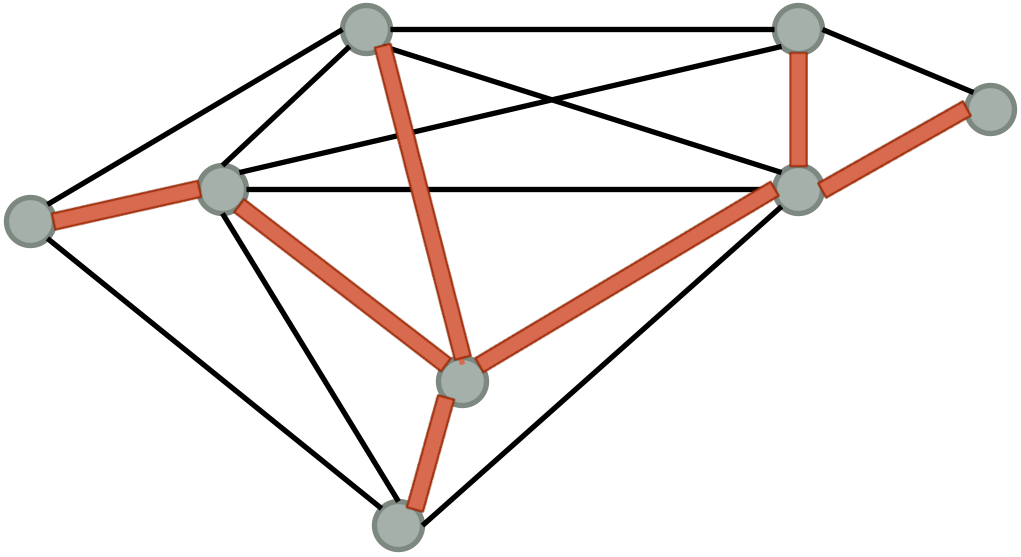
Able to obtain good trees for any graph by leveraging this type of tradeoffs

# LOW STRETCH SPANNING TREES

| |
|---|
| [Alon-Karp-Peleg-West '91]: |
| [Elkin-Emek-Spielman-Teng '05]:<br>A low stretch spanning tree with |
| [Abraham-Bartal-Neiman '08,<br>Koutis-M-Peng `11,<br>Abraham-Neiman `12]:<br>A spanning tree with<br>total stretch O(m log n) in<br>O(m log n) time. |

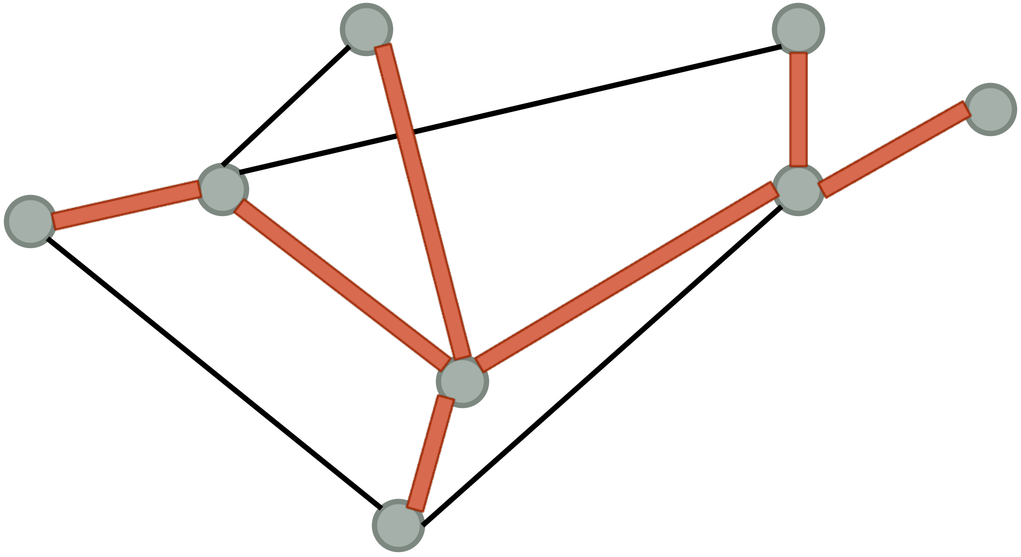
# SOLVER IN ACTION

Sample off tree edges where
P(e) = stretch of e.



# SOLVER IN ACTION

Eliminate degree 1 or 2 nodes



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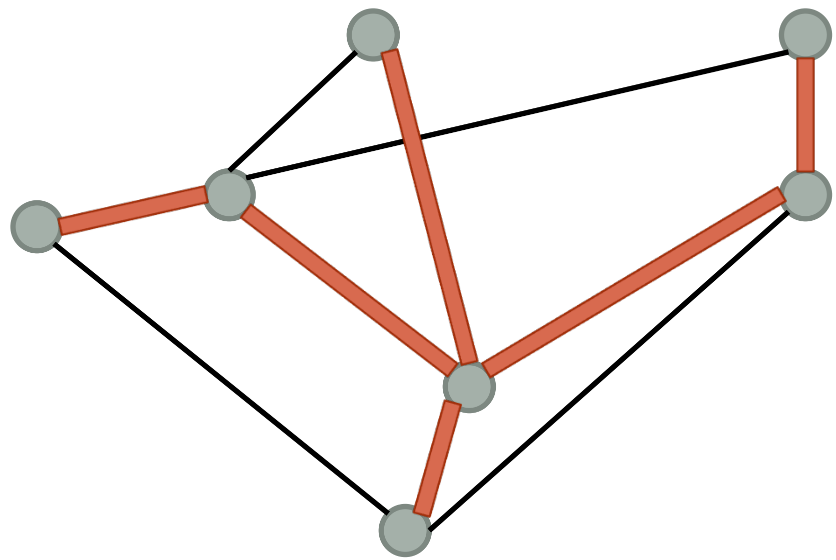

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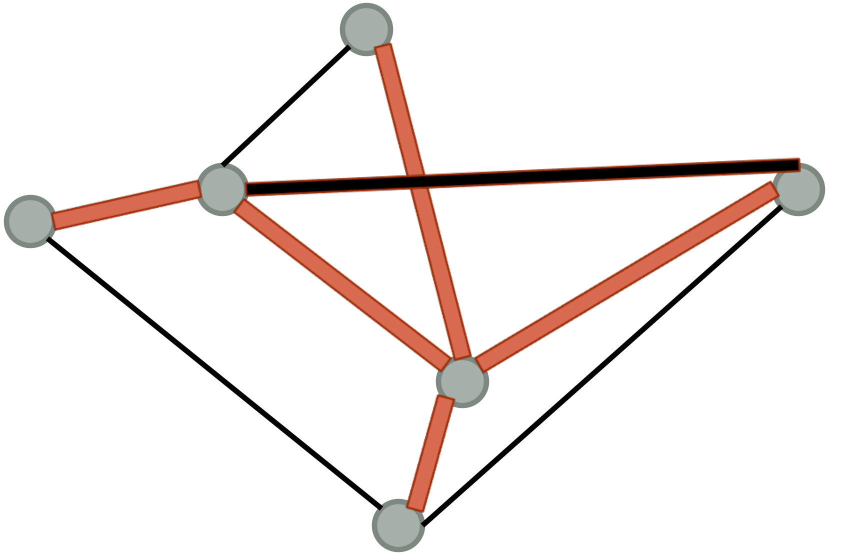

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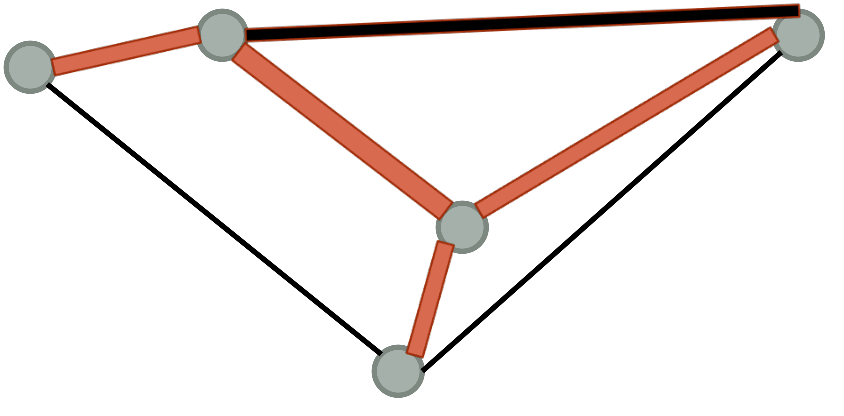

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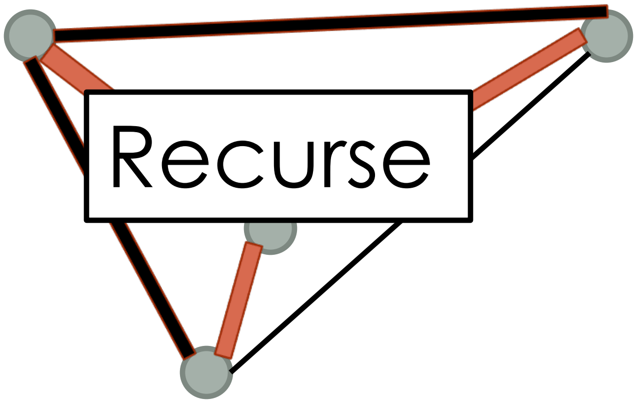
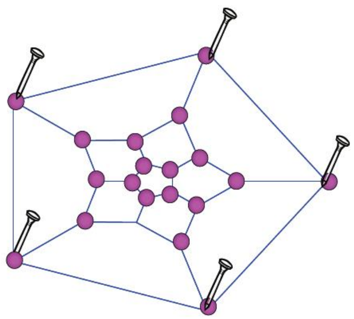
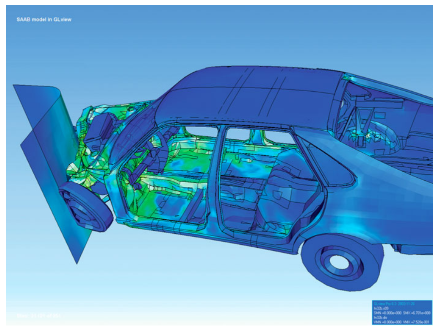
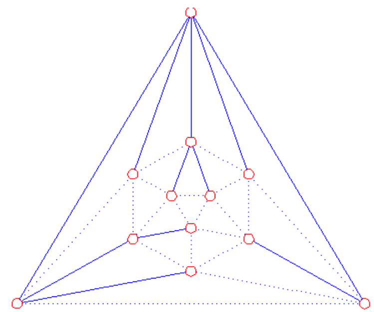
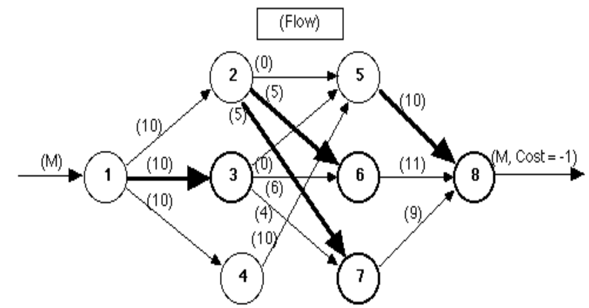


# THEORETICAL APPLICATIONS OF SDD SOLVERS: MULTIPLE ITERATIONS



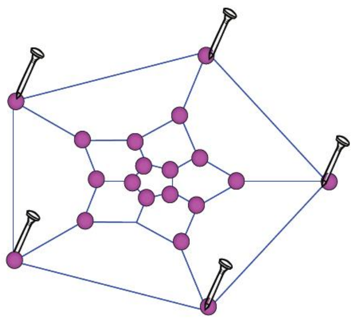


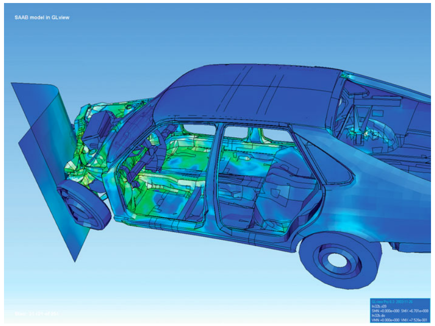


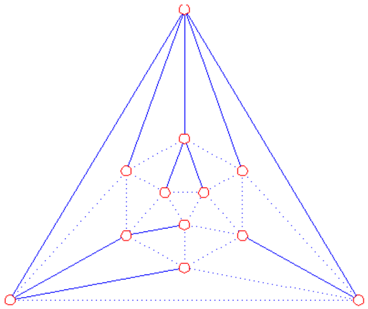


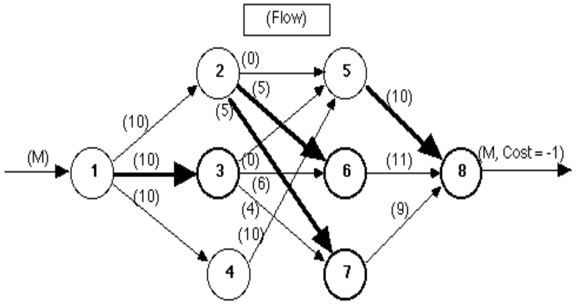

[Tutte `62] Planar graph embeddings.
[Boman-Hendrickson-Vavasis `04] Finite Element PDEs
[Zhu-Ghahramani-Lafferty, Zhou-Huang-Scholkopf `03,05] learning on graphical models.
[Kelner-Mądry `09] Generating random spanning trees in $O(mn^{1/2})$ time by speeding up random walks.

# THEORETICAL APPLICATIONS OF SDD SOLVERS: MULTIPLE ITERATIONS










[Daitsch-Spielman `08] Directed maximum flow, Min-cost-max-flow, lossy flow all can be solved via LP interior point where pivots are SDD systems in $O(m^{3/2})$ time.

# BACK TO IMAGE DENOISING

# PROBLEM WITH QUADRATIC OBJECTIVE

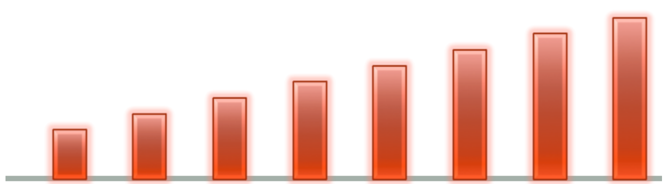
- Result too ‘smooth’, objects become blurred
- Quadratic functions favor the removal of boundaries

# FUNCTION ACCENTUATING BOUNDARIES

$L_1$ smoothness term

If $a<b<c$, $|a-b|+|b-c|$ doesn't depend on $b$

s: sharp boundary



Same smoothness term

Better fidelity

# TOTAL VARIATION OBJECTIVE

- [Rudin-Osher-Fatemi, 92] Total Variation objective: $L_2^2$ fidelity term, $L_1$ smoothness.

Fidelity(x, s) + Smoothness(x)

$(x_i - s_i)^2$ summed over all pixels i.

$|x_i - x_j|$ summed over all neighbor pixels i, j.

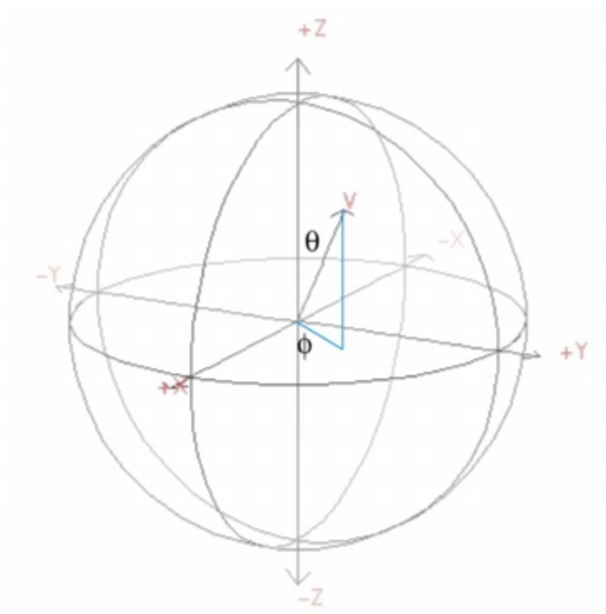
# TOTAL VARIATION MINIMIZATION

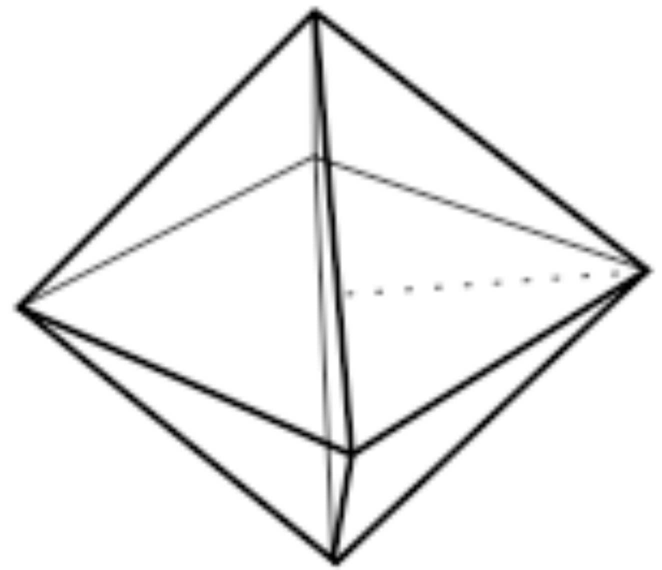
Higher weight on smoothness term

Effect: sharpen boundaries

Overdoing makes image cartoon like

# WHAT'S HARD ABOUT $L_1$?



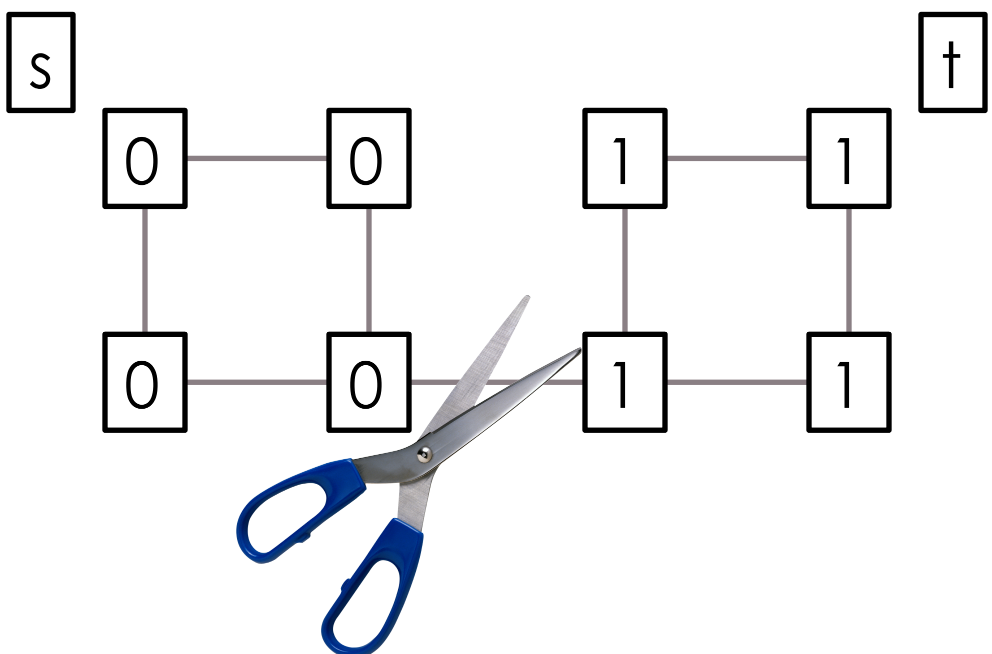
Absolute value function on n variables has $2^n$ points of discontinuity, $L_2^2$ has none.

# MIN CUT PROBLEM AS $L_1$ MINIMIZATION

Minimum s-t cut:

minimize $\Sigma \, |x_i - x_j|$

subject $x_s=0$, $x_t=1$



# MINCUT VIA. $L_2$ MINIMIZATION

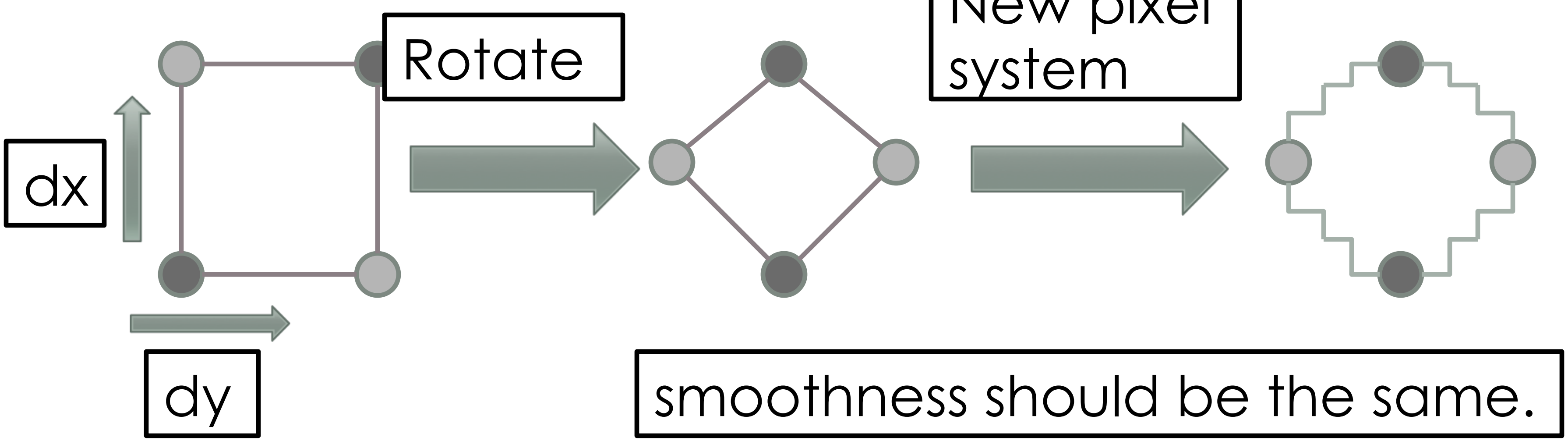
[Christiano-Kelner-Mądry-Spielman-Teng `11]: undirected max flow and mincut can be approximated using $\tilde{O}(m^{1/3})$ SDD solves.

- Multiplicative weights update method
- Gradually update the linear systems being solved

Total: $\tilde{O}(m^{4/3})$

# ISOTROPIC VERSION

[Osher]: 2D images can be rotated, instead of |dx| + |dy|, smoothness term should be $\sqrt{(dx^2 + dy^2)}$



# ALTERNATE VIEW

Alternate interpretation of absolute value: $|t| = \sqrt{t^2}$

$|dx| = \sqrt{(dx^2)}$

Each group can be viewed as an 'edge' in minimum cut problem

This took one year

# TV USING $L_2$ MINIMIZATION

- [Chin-Madry-M-Peng `12]: Can use methods based on electrical flows to obtain $(1+\varepsilon)$ approximation in $O(mk^{1/3}\varepsilon^{-8/3})$ time.
- Interpolates between various versions involving $L_1$ and $L_2$ objectives.

# WHAT IS NEW FOR 2013 AND 2014!

- Faster approximate Flow algorithms!
- Faster solvers!
- Faster exact flow algorithms!
- Faster LSST algorithms
- Parallel LSSTs and solvers

# GENERALIZED GRADIENT DECENT

- [Lee-Rao-Srivastava`13]: gradient descent view of maxflow using electrical flows
- Generalized  Decent for Classes of Lipschitz convex functions.
  - Nesterov optimization
  - Soft max of $L_\infty$
  - Smoothed version of $L_1$

# FASTER APPROXIMATE FLOW ALGORITHMS!

Sherman 13: Approximate maxflow and Mini-cut in $\tilde{O}(m^{1+\delta} poly(k, \varepsilon^{-1}))$ time

Uses:

Madry 10: Fast Approximate Mini-Cut Algorithm

# FASTER APPROXIMATE FLOW ALGORITHMS!

Kelner-Lee-Orecchi-Sidford’ 13:
Approximate maxflow and multi-commodity flow in $\tilde{O}(m^{1+\delta}\mathrm{poly}(k, \varepsilon^{-1}))$ time

Uses:

Racke’ 08: Approximate Oblivious routing ideas.

# EVEN FASTER SOLVERS

- Cohen-Kyng-Pachocki-Peng-Rao `13
  SDD linear systems Faster solver in
  - $O(m\log^{1/2} n)$ time given a LSST.

- The log appears in two places in KMP:

1. Matrix Chernoff Bounds
2. LSST tree construction

# LOW DIAMETER DECOMPOSITION

Awerbuch 85: O(log n) diameter clusters with o(m) inter-cluster edges.

M,Peng,Xu 13: O( c log n) diameter clusters with O(m/c) expected inter-cluster edges:
O(m) work and O(c $\log^2$ n) depth

Algorithm: Do BFS from each noted using an exponential delay start time.

# FASTER TREE GENERATION

- Koutis-M-Peng `11, Abraham-Neiman `12]:

LSST with stretch O(m log n) in O(m log n) time.

We do not know how to beat these bounds!

We find a tree that is good enough!

# LSST RELAXATION

Allow Steiner nodes.
But still embeddable!

Relax the definition of stretch.

Recall $STR_T(e=(a,b)) = dist_T(a,b)$
New Def: $STR_T^P(e=(a,b)) = |dist_T(a,b)|^P$
for $P < 1$

# FASTER TREE ALGORITHM FOR $L^P$- STRETCH

| | Runtime | Stretch |
|---|---|---|
| **AKPW** | $O(m \log \log n)$ | $O(\log^{O(1)} n)$ |
| **Bartal/AN** | $O(m \log n)$ | $O(\log n)$ |

Can we get the best of both worlds?

# NEARLY LINEAR TIME, POLYLOG DEPTH SOLVERS

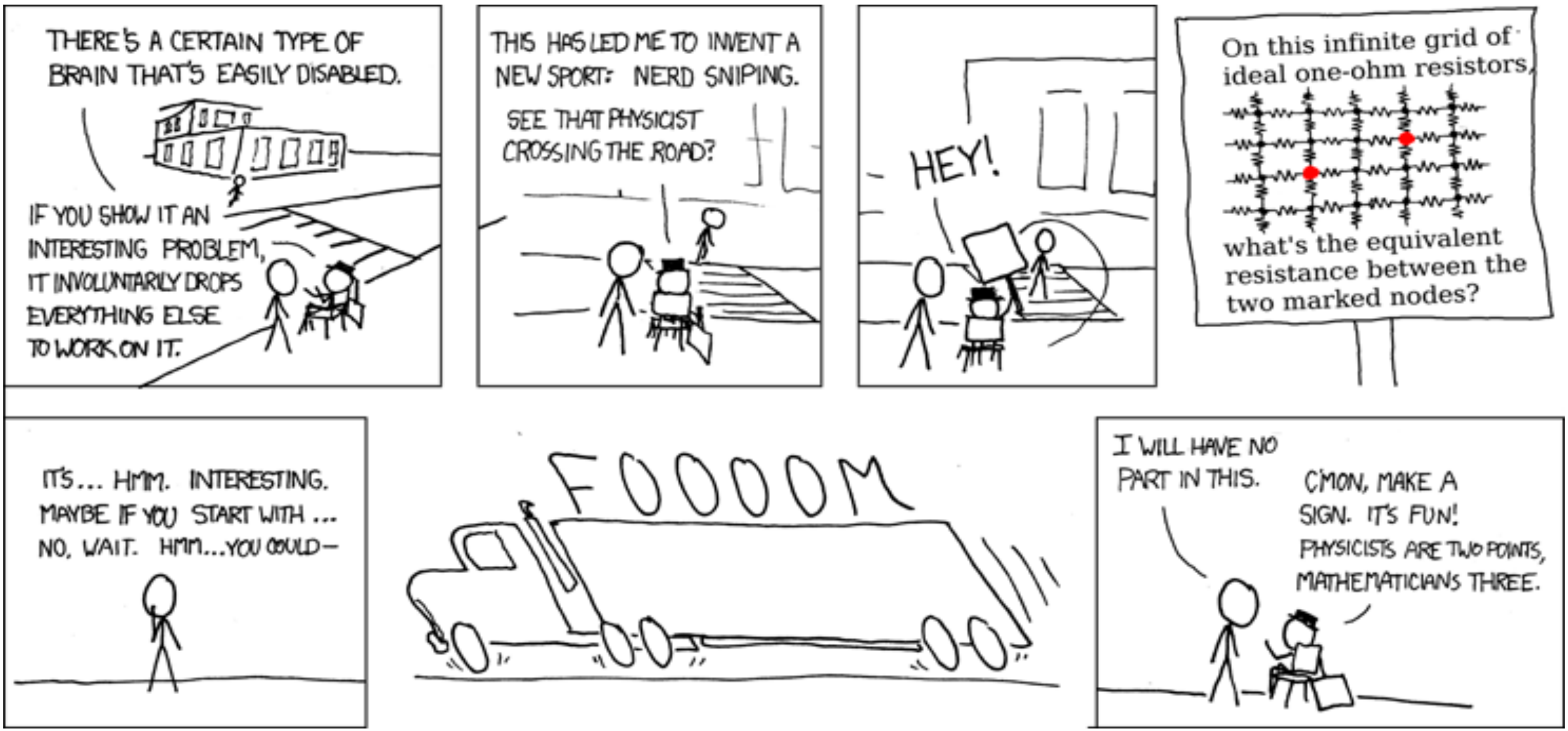
[Peng-Spielman `13]

**Input**: SDD matrix $\mathbf{M}$ with m non-zeros, condition number $\kappa$
**Output**: Sparse product $\mathbf{Z_1 \ldots Z_k} \approx_{\varepsilon} \mathbf{M}^{-1}$
**Cost**: $O(\log^c m \log^c \kappa \log(1/\varepsilon))$ time
$O(m \log^c m \log^c \kappa \log(1/\varepsilon))$ work

Approximation $\approx_{\varepsilon}$ in matrix sense

# FUTURE WORK

- Practical/parallel implementations?
  - The win over sequential is parallel!
- Near linear time exact max flow?
  - $\log(1/\varepsilon)$ dependency in runtime?
- Sub-quadratic SPD solver?



THERE'S A CERTAIN TYPE OF BRAIN THAT'S EASILY DISABLED.
IF YOU SHOW IT AN INTERESTING PROBLEM, IT INVOLUNTARILY DROPS EVERYTHING ELSE TO WORK ON IT.
THIS HAS LED ME TO INVENT A NEW SPORT: NERD SNIPING.
SEE THAT PHYSICIST CROSSING THE ROAD?
HEY!
On this infinite grid of ideal one-ohm resistors, what's the equivalent resistance between the two marked nodes?
IT'S... HMM. INTERESTING. MAYBE IF YOU START WITH ... NO, WAIT. HMM...YOU COULD—
FOOOOM
I WILL HAVE NO PART IN THIS.
C'MON, MAKE A SIGN. IT'S FUN! PHYSICISTS ARE TWO POINTS, MATHEMATICIANS THREE.