Towards a Theory of Learning Inductive Invariants

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CERTORA

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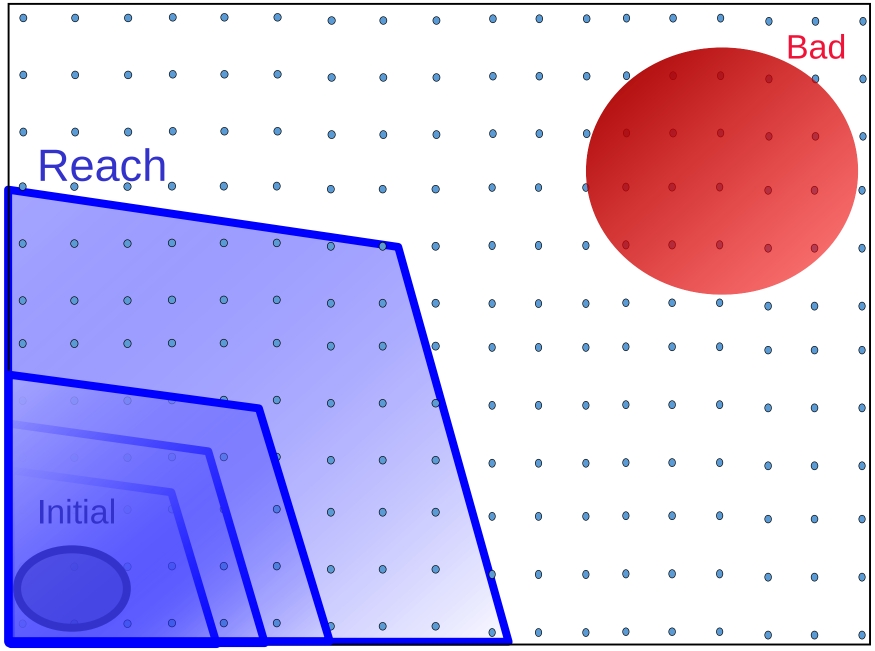
Safety of Transition Systems

<u>Init</u>:
$(x_1, \ldots, x_n) \coloneqq 0 \ldots 0$

<u>Bad</u>:
$(x_1, \ldots, x_n) \;=\; 1 \ldots 1$

<u>δ</u>:
$y_1, \ldots, y_n \coloneqq *$
$x_1, \ldots, x_n \coloneqq (x_1, \ldots, x_n) + 2 \cdot (y_1, \ldots, y_n) \pmod{2^n}$



Inductive Invariants

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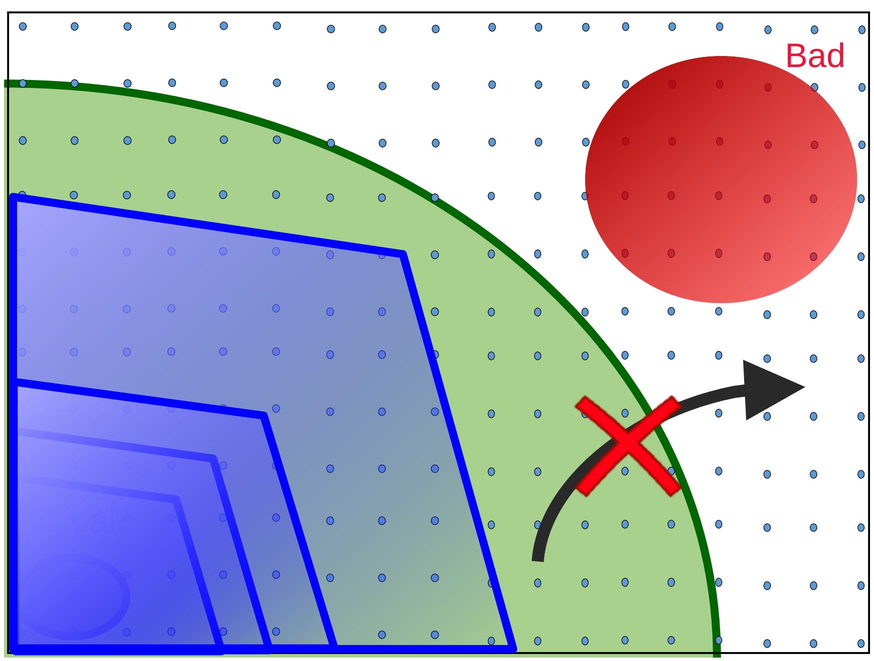
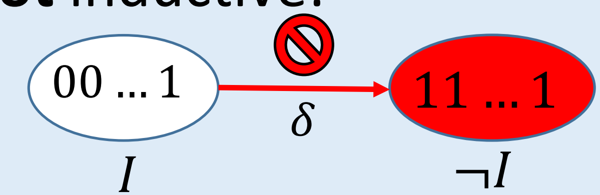


I: $(x_1, \dots, x_n) \neq 1 \dots 1$

Not inductive:



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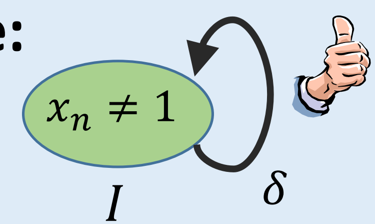
Inductive:



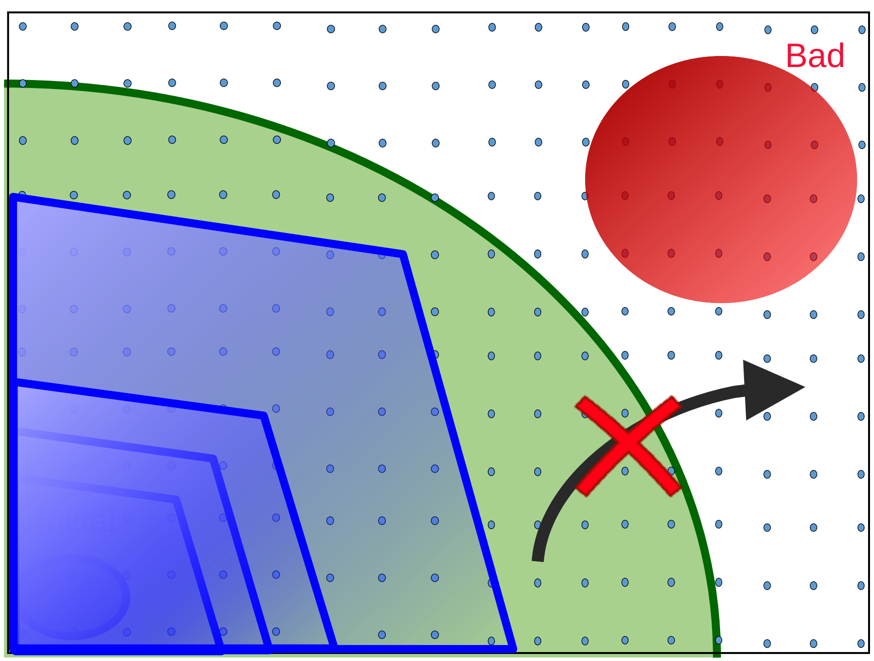
Inductive Invariants

Init: δ:

Goal:

Find inductive invariants automatically

$(x_1, \dots, x_n) = 1 \dots 1$



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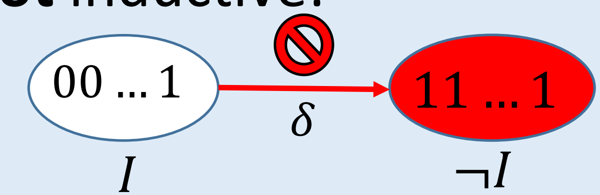

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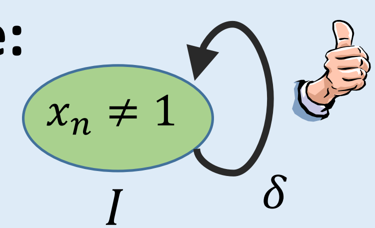
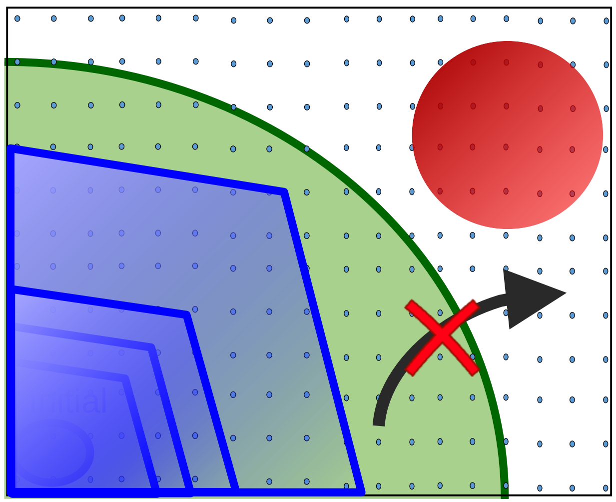


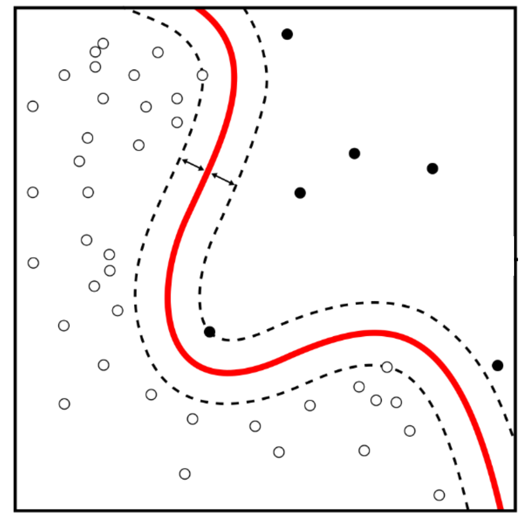
This Work

Invariant Inference



vs.

Exact Concept Learning



- Query-based learning models for invariant inference
- Invariant inference is harder than concept learning
- Complexity results for invariant inference algorithms from classification algorithms

Problem Setting: Polynomial-Length Inference

Boolean transition systems, $\Sigma = \{p_1, \ldots, p_n\}$

Given a transition system from a class $\mathcal{P}$ (over Σ),

Find an inductive invariant

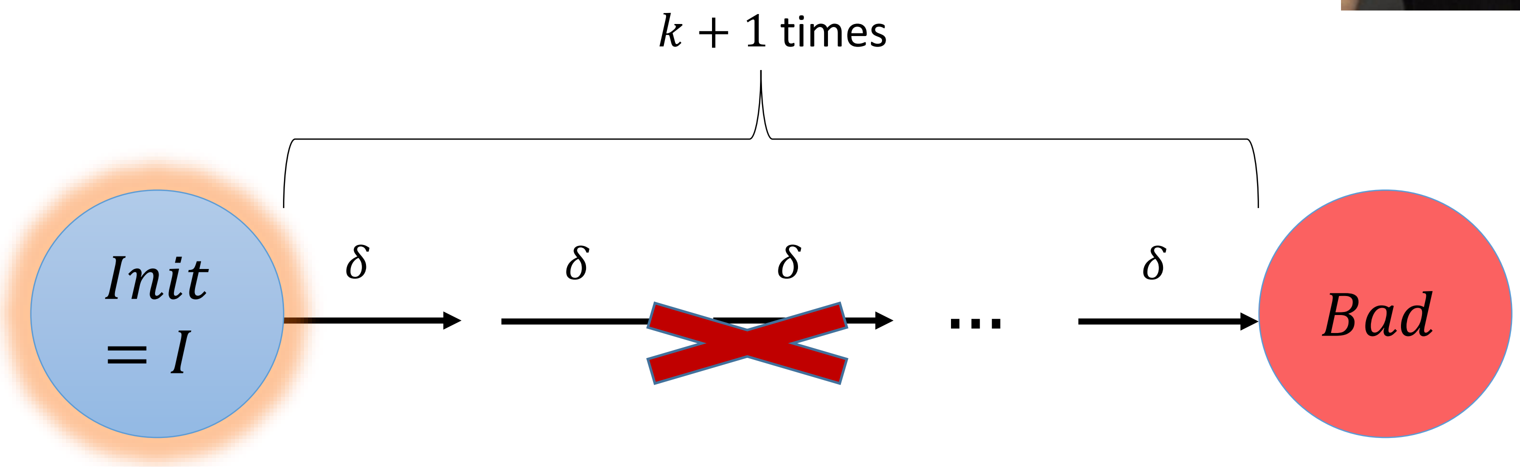
$I \in \mathrm{DNF}$ $|I| \leq \mathrm{poly}(n)$

(Decision problem is Σ_2^P-complete.)

[CADE'09] Complexity and Algorithms for Monomial and Clausal Predicate Abstraction. Lahiri, Qadeer
[POPL'20] Complexity and Information in Invariant Inference. Feldman, Immerman, Shoham, Sagiv

Interpolation-Based Inference

$I = Init$



[CAV'03] Interpolation and SAT-Based Model Checking, McMillan

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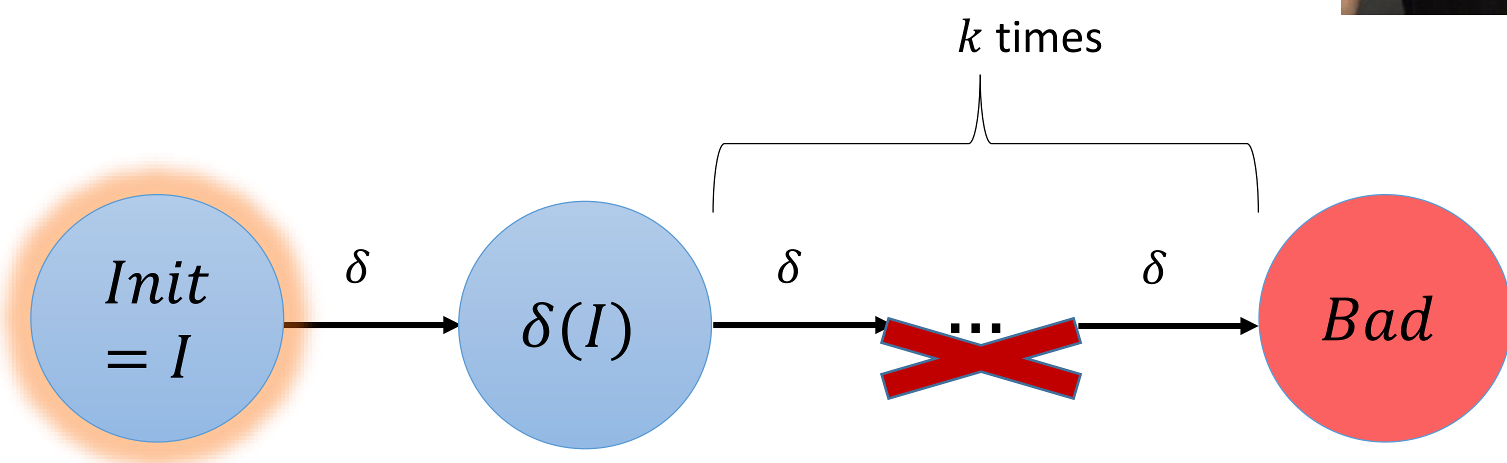

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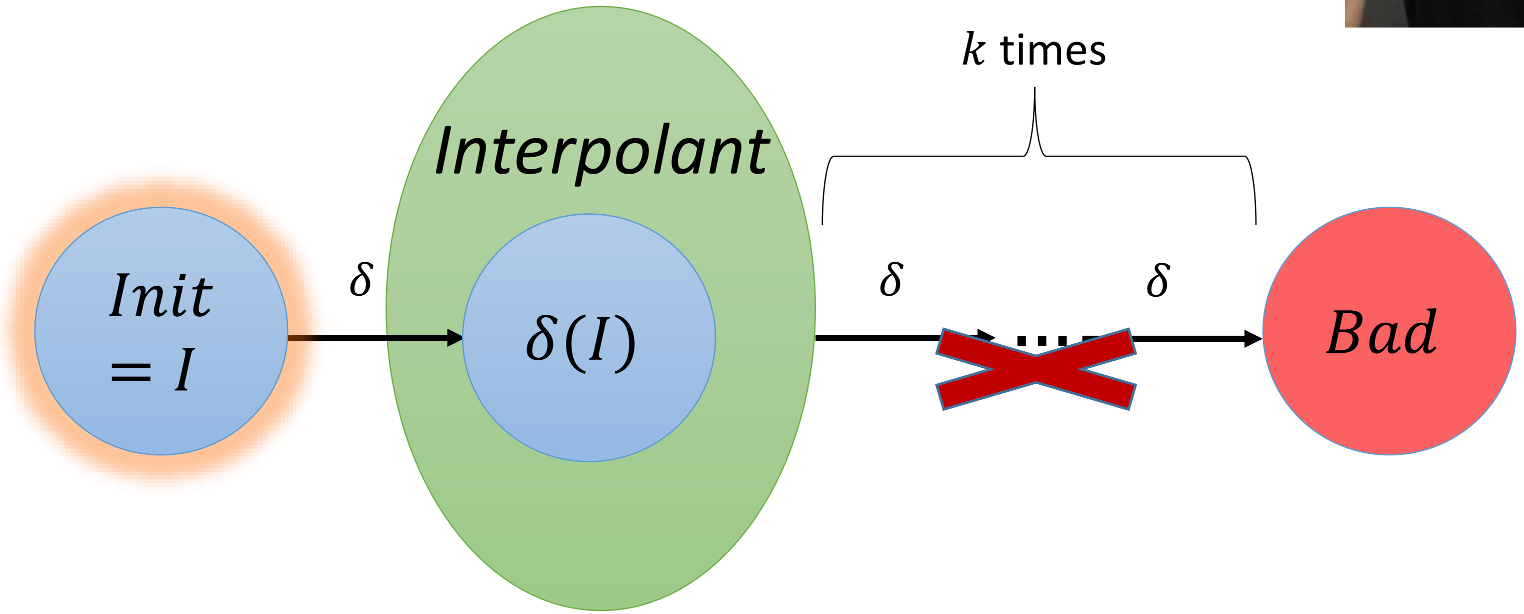

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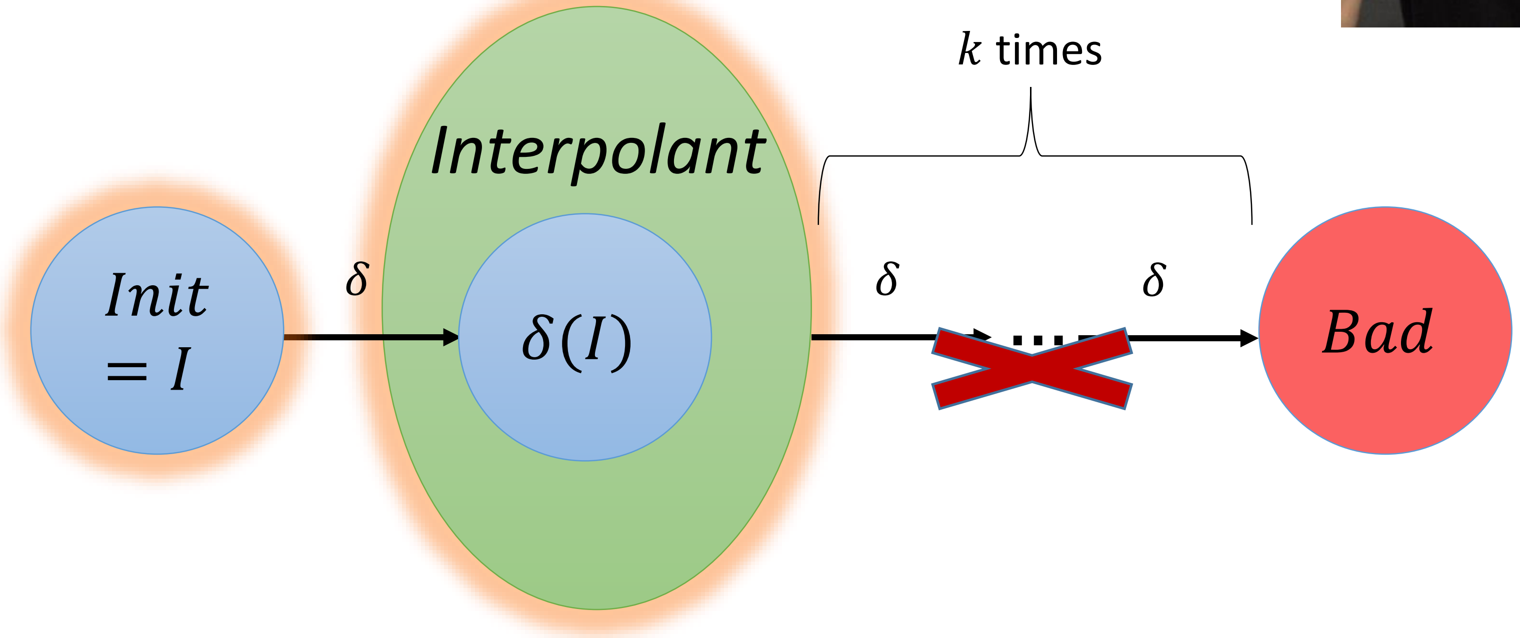
$$I = Init \lor Interpolant$$



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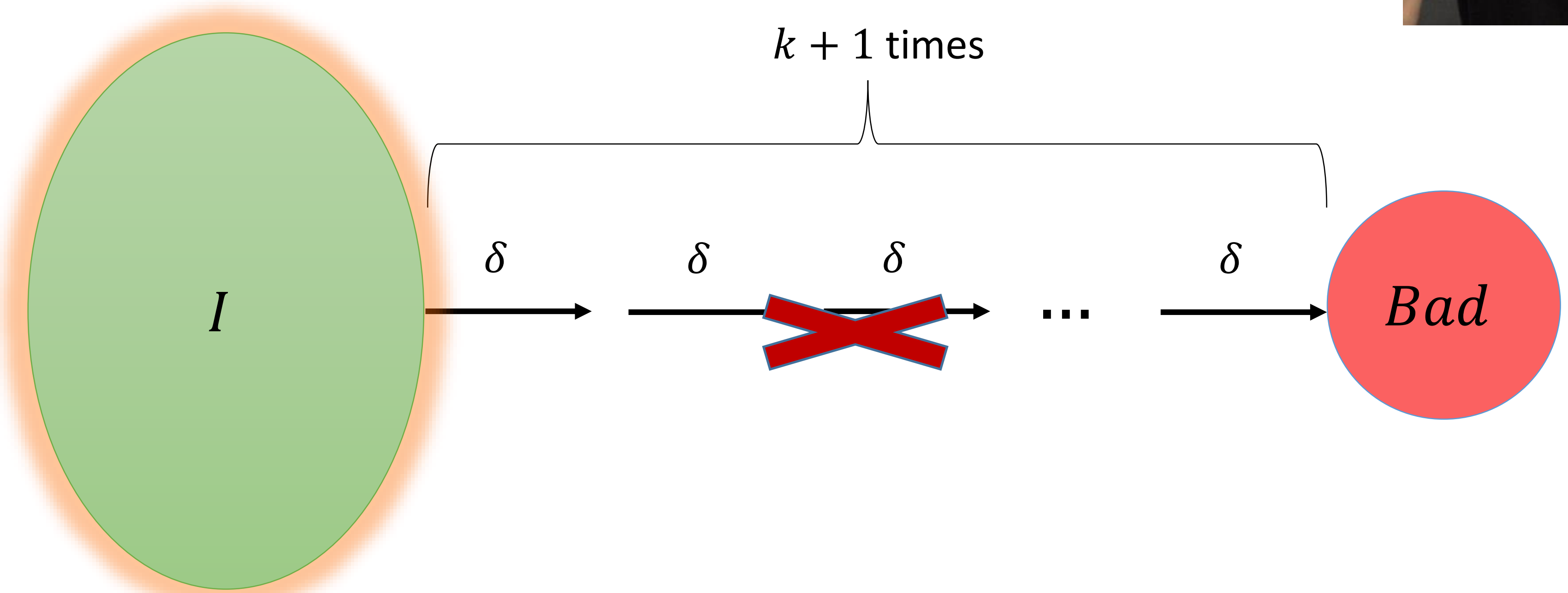

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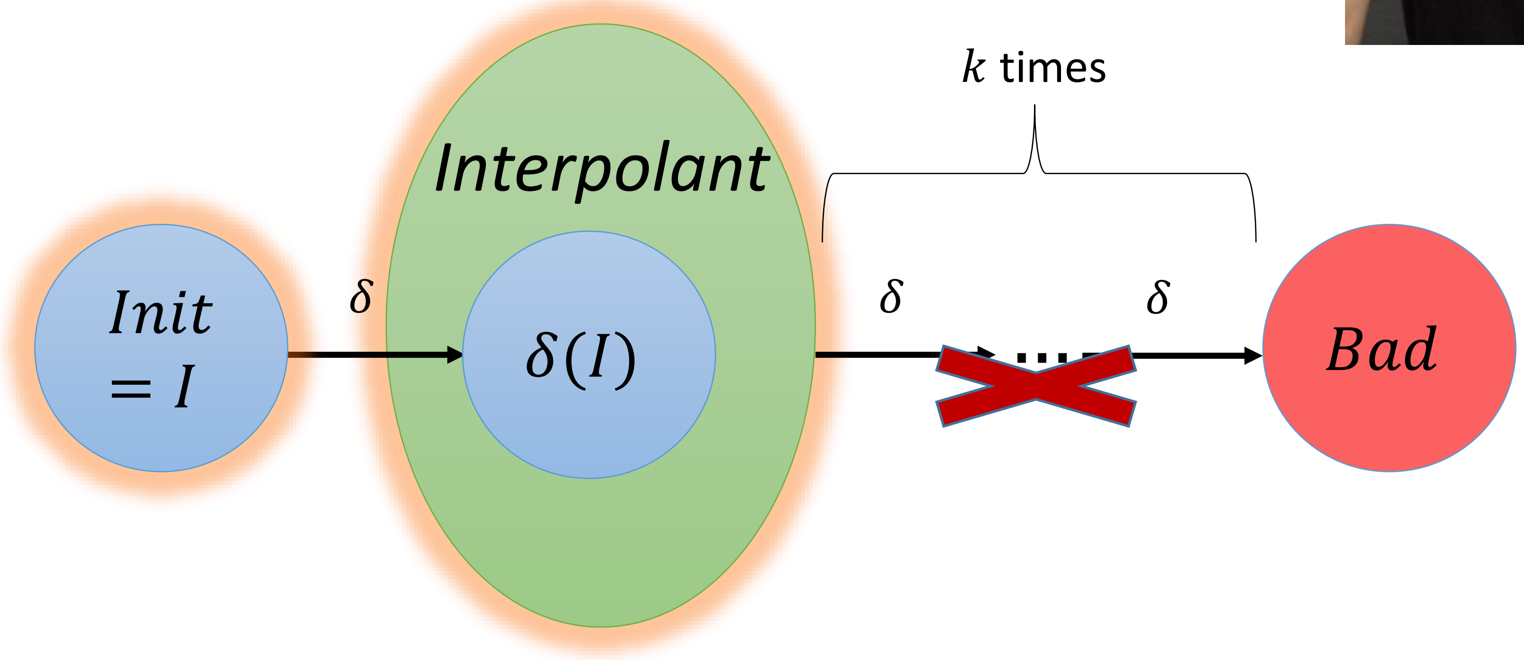

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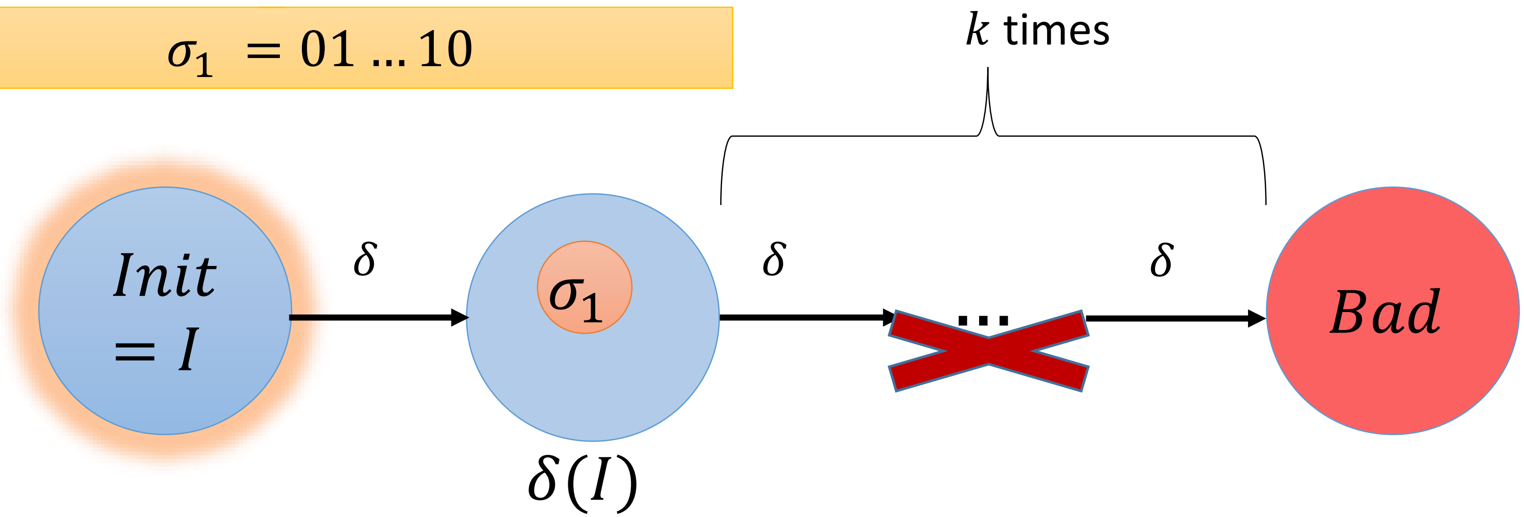
$Interpolant_1 = (x_1 = 0 \wedge x_2 = 1 \wedge \cdots \wedge x_n = 1 \wedge x_n = 0)$

$\sigma_1 = 01 \ldots 10$



[HVC'12] Computing Interpolants without Proofs. Chockler, Ivrii, Matsliah
[LPAR'13] Instantiations, Zippers and EPR Interpolation. Bjørner, Gurfinkel, Korovin, Lahav

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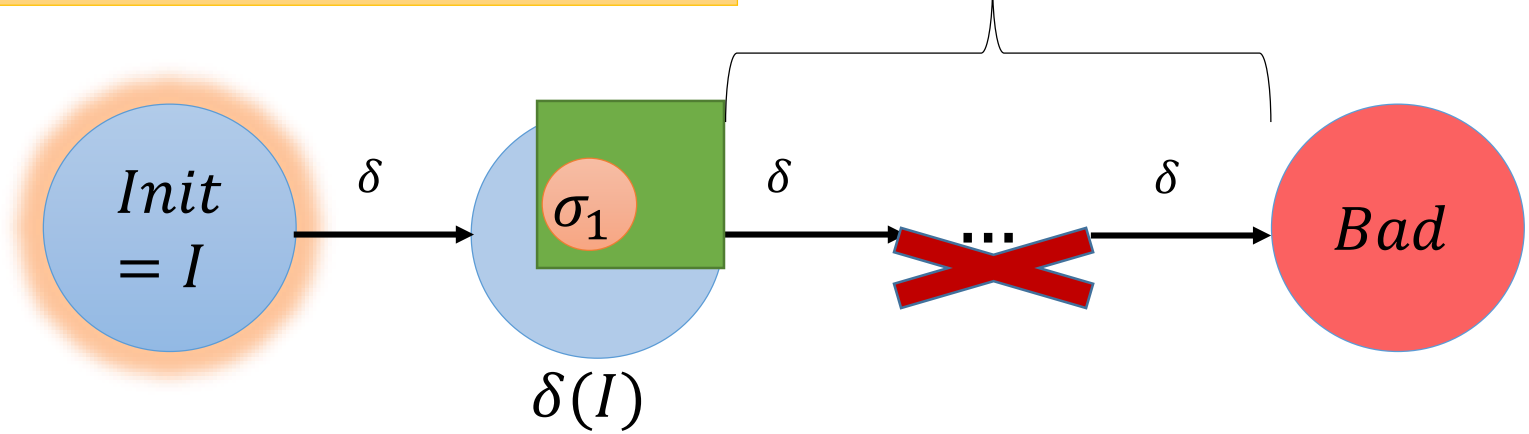

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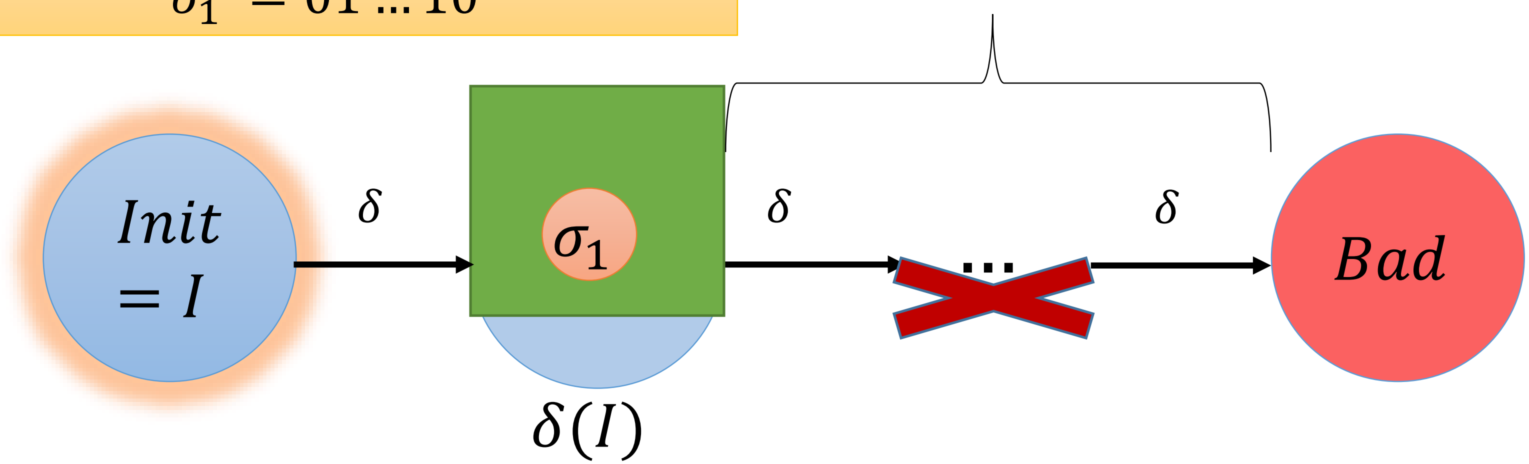

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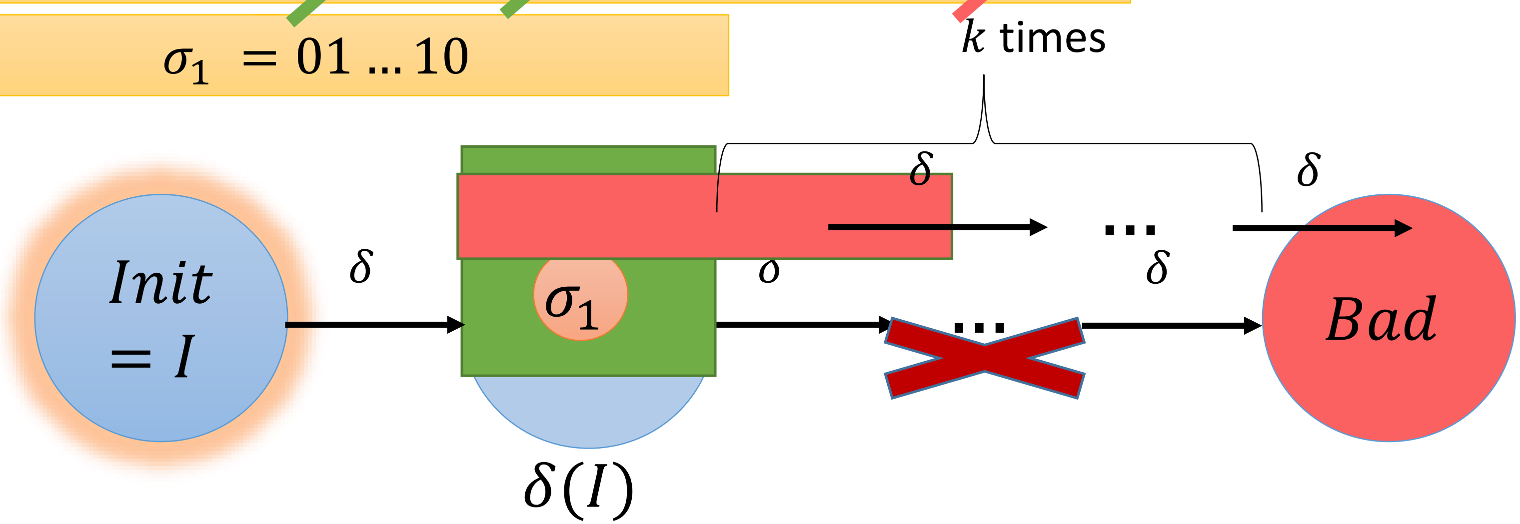

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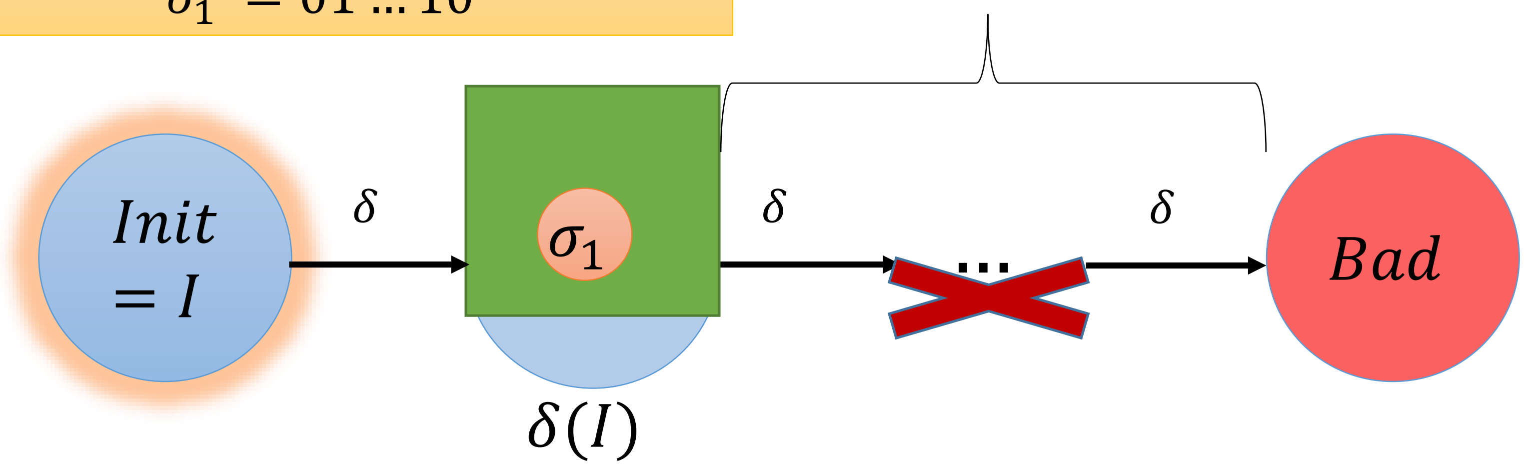

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Model-Based Interpolation

<u>Init</u>:

$(x_1, \dots, x_n) \coloneqq \color{blue}{0 \dots 0}$

<u>Bad</u>:

$(x_1, \dots, x_n) = \color{blue}{1 \dots 1}$

<u>δ</u>:

$y_1, \dots, y_n \coloneqq *$

$x_1, \dots, x_n \coloneqq (x_1, \dots, x_n) + \color{blue}{2} \cdot (y_1, \dots, y_n) \pmod{2^n}$

$Interpolant_1 = (x_1 \neq 0 \wedge x_2 \neq 1 \wedge \cdots \wedge x_n \neq 1 \wedge x_n = 0)$

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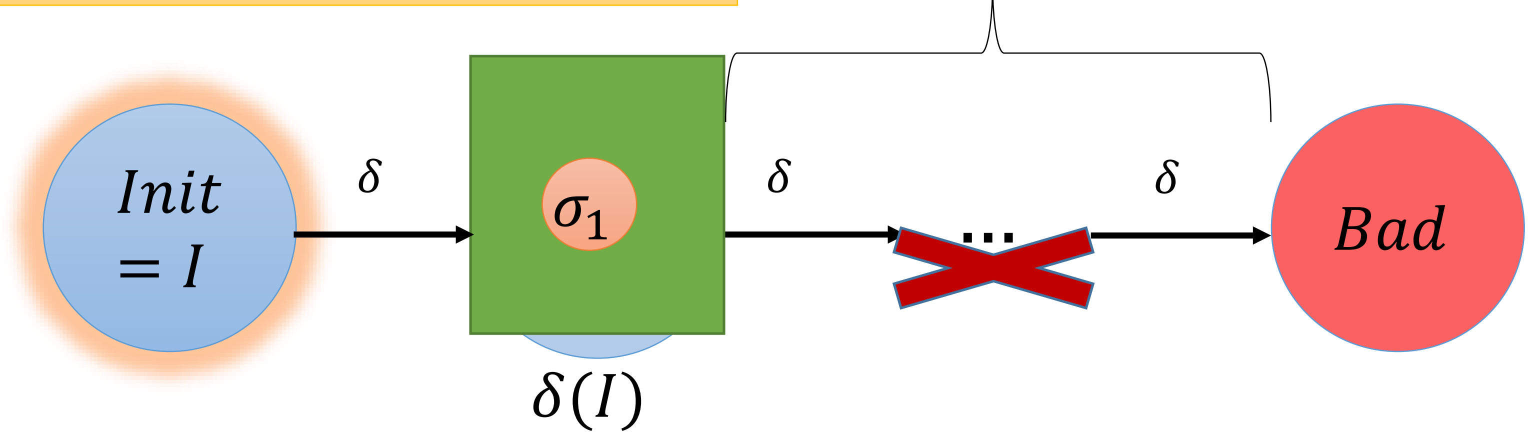

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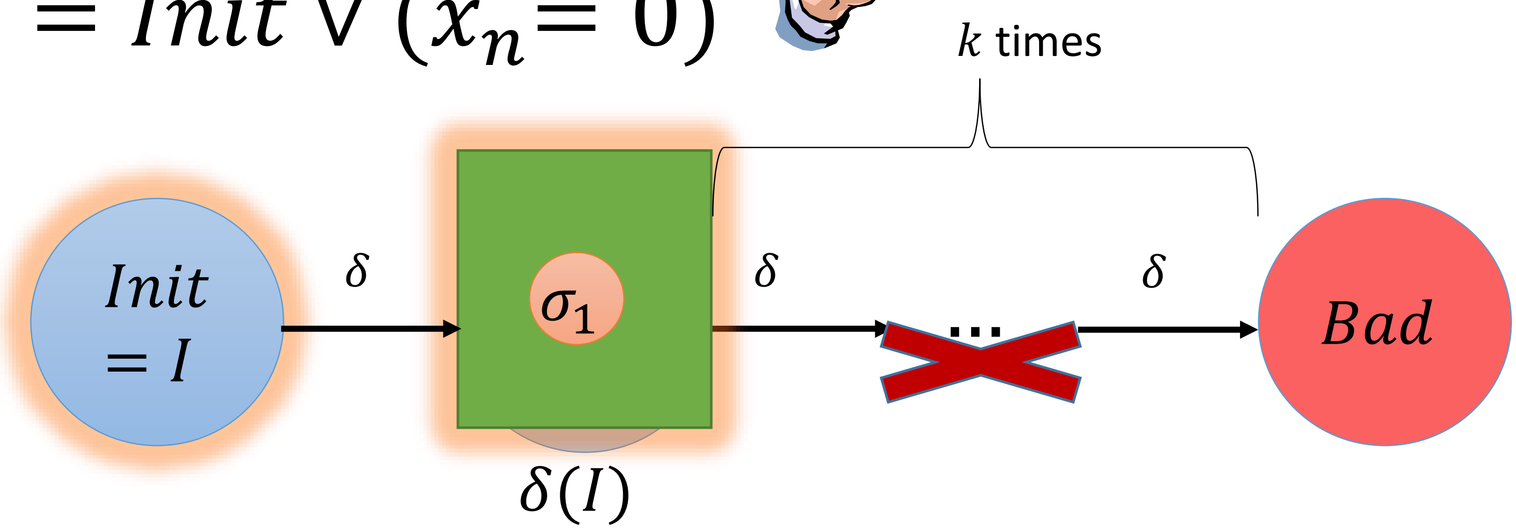
$$I = Init \vee (x_n = 0)$$



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Model-Based Interpolation

Inferring invariant in DNF:

$$\underbrace{\left(\ell_1^1 \wedge \cdots \wedge \ell_{k_1}^1\right)}_{\text{gen}(\sigma_1)} \vee \; \ldots \; \vee \underbrace{\left(\ell_1^m \wedge \cdots \wedge \ell_{k_m}^m\right)}_{\text{gen}(\sigma_m)}$$

I := false

while $(\_, \sigma')$ counterexample
      to Inductive(δ, I):

  I := $I \vee$ **generalize**(σ')

generalize(σ'):

  drop literals from σ'

  while BMC$(\delta, \sigma', k) \cap \text{Bad} = \emptyset$

Model-Based Interpolation

```
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while (_,σ′) counterexample
      to Inductive(δ,I):
   I  := I ∨ generalize(σ′)
```

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Understanding Invariant Inference

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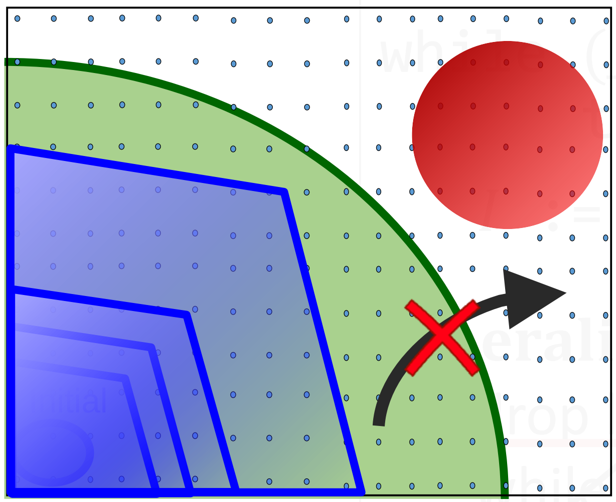
Complexity bounds from exact classification algorithms

Rich SAT queries allow exponentially faster inference

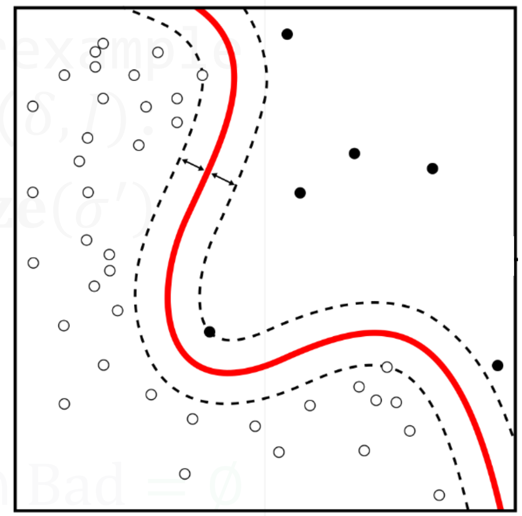
Understanding Invariant Inference

Invariant Inference



vs.

Exact Concept Learning



Complexity bounds from
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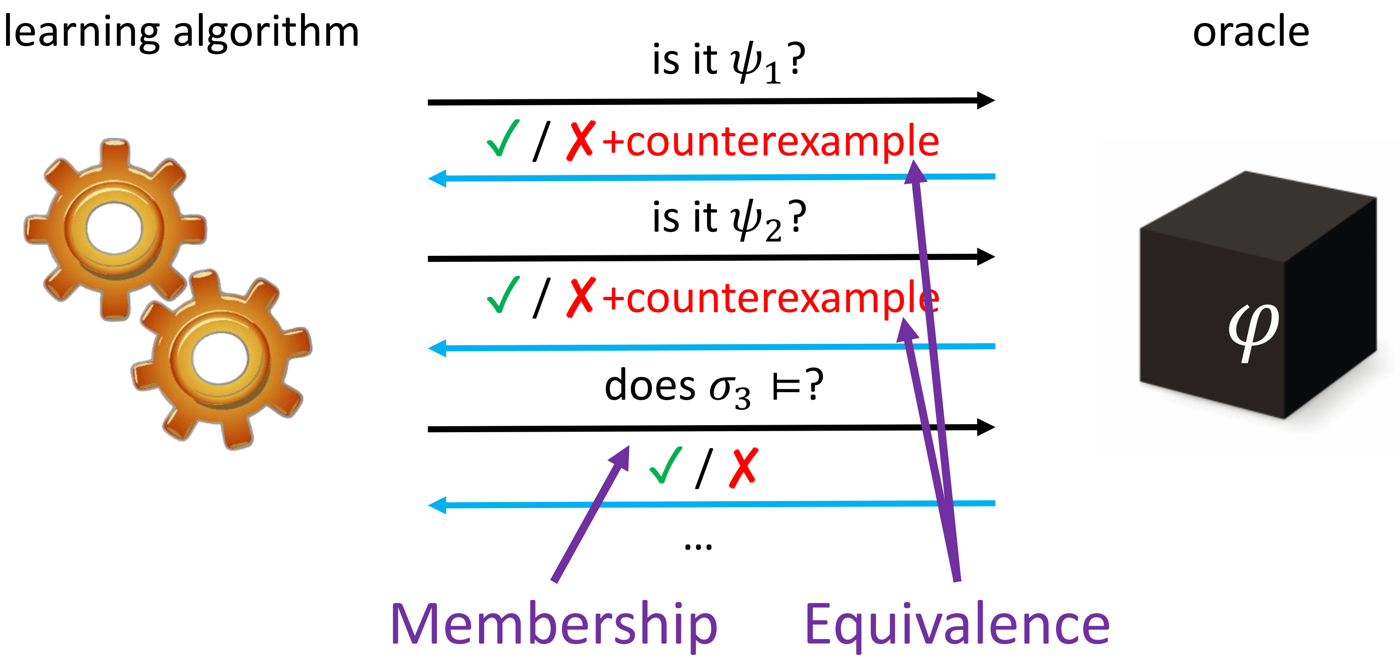
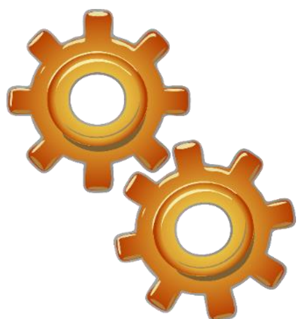
Exact Concept Learning with Equivalence & Membership Queries



[ML'87] Queries and Concept Learning. Angluin

Invariant Inference with Equivalence & Membership Queries

learning algorithm



Need to implement this

is it ψ_1?

✓ / ✗+counterexample

is it ψ_2?

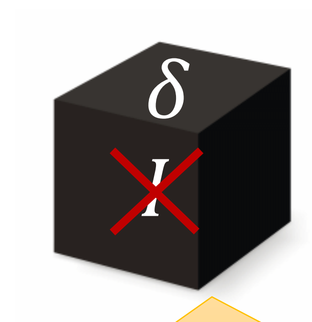
✓ / ✗+counterexample

does $\sigma_3 \vDash$?

✓ / ✗

…

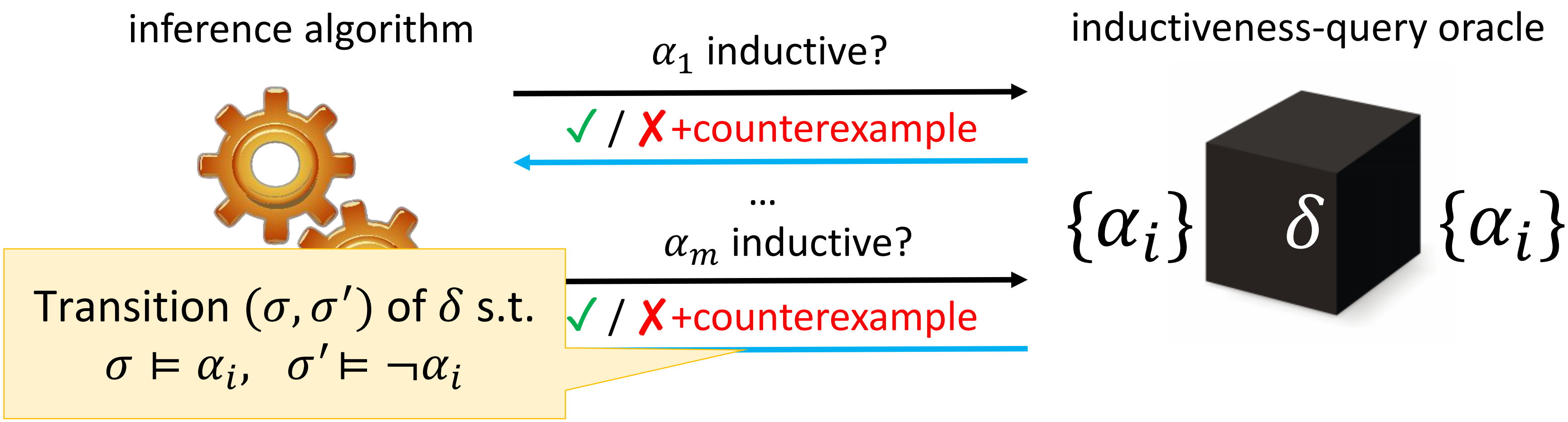
oracle



Need to implement this

[CAV'14] ICE: A Robust Framework for Learning Invariants. Garg, Löding, Madhusudan, Neider

Inductiveness-Query Model



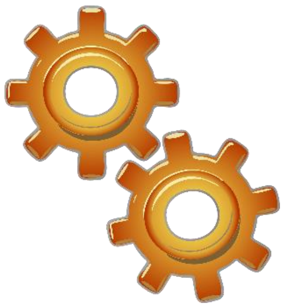
Algorithms cannot access the transition relation directly, only perform inductiveness queries

Complexity: # inductiveness queries worst case amongst possible counterexamples

[CAV'14] ICE: A Robust Framework for Learning Invariants. Garg, Löding, Madhusudan, Neider

Inductiveness-Query Model

inference algorithm



α_1 inductive?

✓ / ✗+counterexample

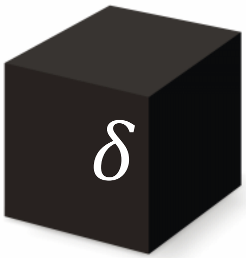
…

α_m inductive?

✓ / ✗+counterexample

inductiveness-query oracle

$\{\alpha_i\}$



δ

$\{\alpha_i\}$

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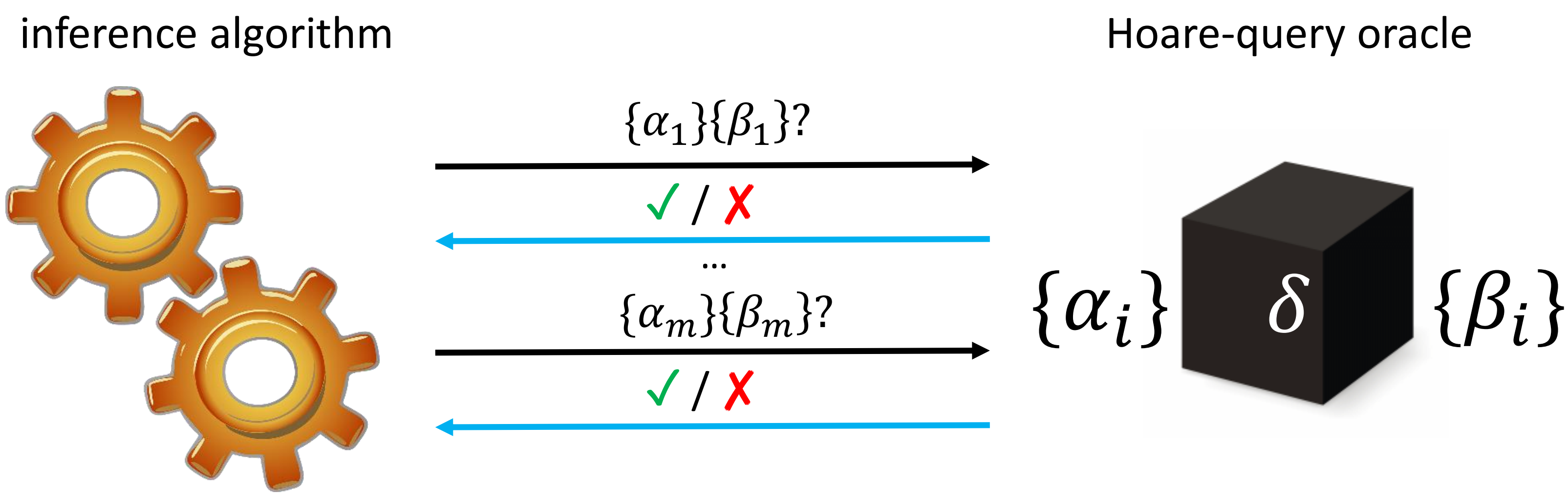
I inductive?

✓ / ✗

?

Hoare-Query Model

Capable of modeling several interesting algorithms



Algorithms cannot access the transition relation directly, only perform Hoare queries

Hoare-Query Model

Capable of modeling several interesting algorithms

model of $\neg\{I\}\delta\{I\}$, $O(n)$ queries

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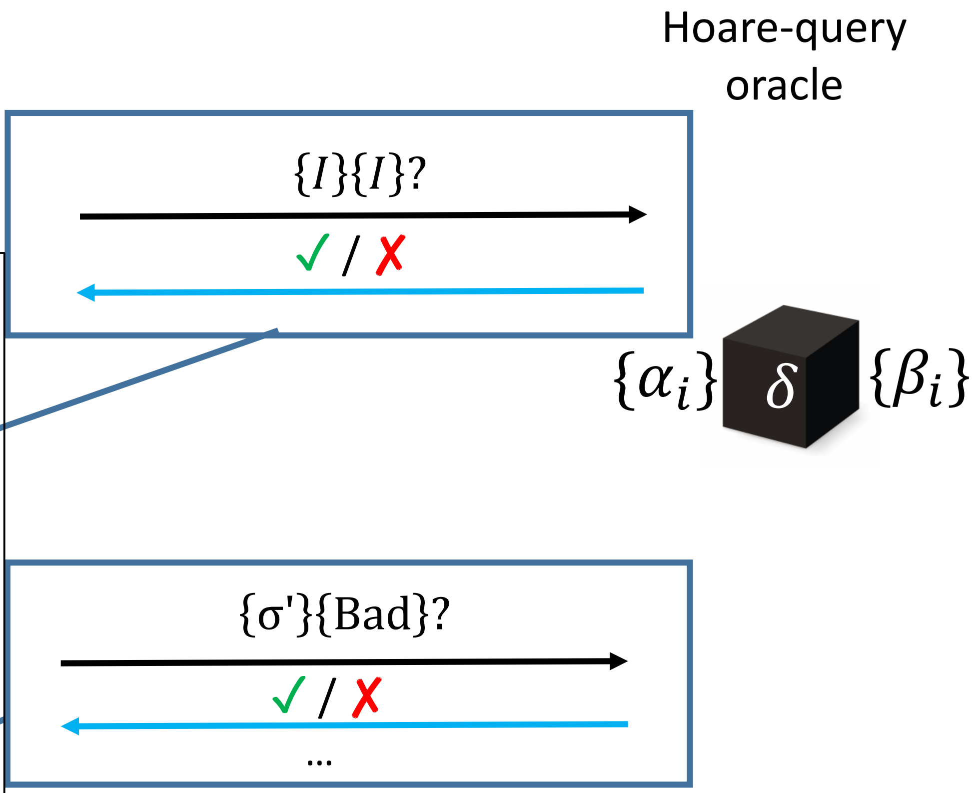

Hoare > Inductiveness

<u>Thm</u>: There exists a class of transition systems $\mathcal{P}$, so that for solving polynomial-length inference:

1. **∃** Hoare-query algorithm with $\text{poly}(n)$ queries
2. **∀** inductiveness-query algorithm requires $2^{\Omega(n)}$ queries

a simple case of IC3/PDR

$\Rightarrow$ ICE cannot model PDR,

and the extension of [VMCAI'17] is necessary

[POPL'20] Complexity and Information in Invariant Inference. Feldman, Immerman, Shoham, Sagiv
[VMCAI'17] IC3 - Flipping the E in ICE. Vizel, Gurfinkel, Shoham, Malik.

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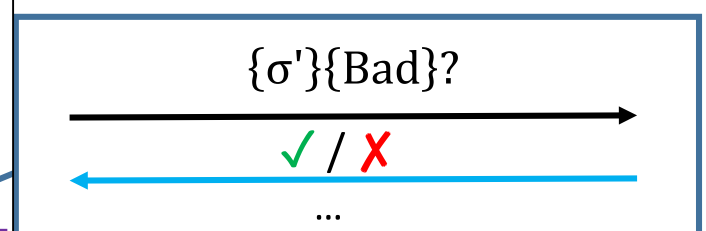

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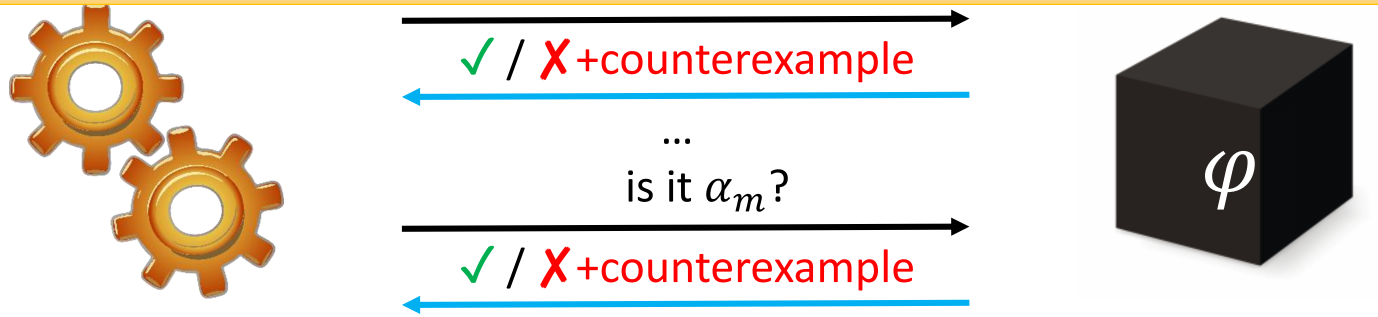
Learning from Counterexamples to Equivalence Queries

Thm: Learning from counterexamples to induction is **harder** than learning from labeled examples.



Positive/negative examples:

$\sigma^+ \vDash \varphi$, $\sigma^- \vDash \neg\varphi$

Counterexamples to induction:

$\sigma \vDash \neg\varphi$ **or** $\sigma' \vDash \varphi$

Learning monotone DNF:

subexponential

this work: $\mathbf{2^{\Omega(n)}}$

[ML'87] Queries and Concept Learning, Angluin
[COLT'12] Tight Bounds on Proper Equivalence Query Learning of DNF, Hellerstein et al.
[POPL'20] Complexity and Information in Invariant Inference. Feldman, Immerman, Shoham, Sagiv

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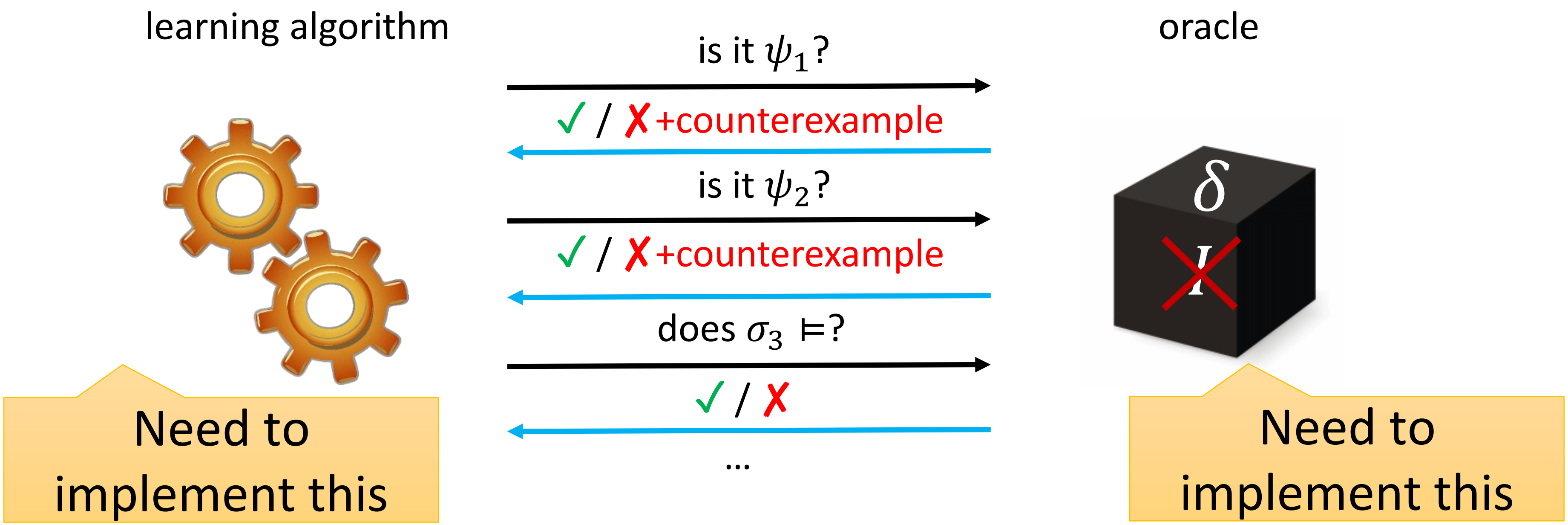
Complexity bounds from exact classification algorithms

Rich SAT queries allow exponentially faster inference

Invariant Inference with Equivalence & Membership Queries



Thm. In general, in the Hoare-query model, **no efficient way** to implement a teacher for equivalence and membership queries

[CAV'14] ICE: A Robust Framework for Learning Invariants. Garg, Löding, Madhusudan, Neider

[POPL'20] Complexity and Information in Invariant Inference. Feldman, Immerman, Shoham, Sagiv

Invariant Inference with Equivalence & Membership Queries

learning algorithm

teacher

is it ψ_1?

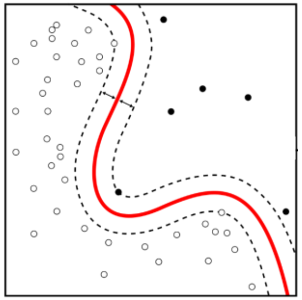
...

Sufficient conditions for

exact learning algorithms $\Longrightarrow$ invariant inference algorithms

Thm. In general, in the Hoare-query model, **no efficient way** to implement a teacher for equivalence and membership queries

From Learning to Inference



Exact **learning**
DNF formulas

ψ := false

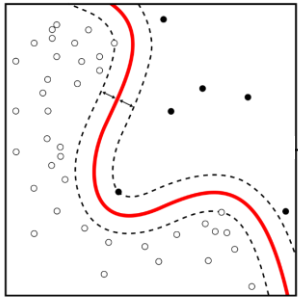
while σ' counterexample
to Equivalence(ψ):

ψ := $\psi \vee$ **generalize**(σ')

generalize(σ'):
drop literals from σ'
while Membership(σ')= ✓

[CACM'84] A Theory of the Learnable. Valiant
[ML'87] Queries and Concept Learning. Angluin
[ML'95] On the Learnability of Disjunctive Normal Form Formulas. Aizenstein and Pitt

From Learning to Inference



Exact **learning** DNF formulas $\Longrightarrow$

ψ := false

while σ' counterexample
to **Equivalence**(ψ): → **Inductive**(I)

ψ := $\psi \vee$ **generalize**(σ')

generalize(σ'):
drop literals from σ'
while **Membership**(σ')= ✓ → $\text{BMC}(\sigma', k) \cap \textbf{Bad} = \emptyset$

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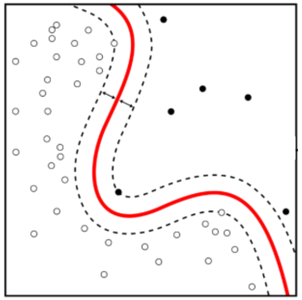
From Learning to Inference



Exact **learning** DNF formulas $\Longrightarrow$

Inferring DNF invariants

ψ := false

while σ' counterexample
    to **Equivalence**($\boldsymbol{\psi}$):

    ψ := $\psi \vee$ **generalize**(σ')

generalize(σ'):
    drop literals from σ'
    while **Membership**($\boldsymbol{\sigma'}$) = ✓

I := false

while $(\_, \sigma')$ counterexample
    to **Inductive**($\boldsymbol{I}$):

    I := $I \vee$ **generalize**(σ')

generalize(σ'):
    drop literals from σ'
    while **BMC**$(\boldsymbol{\sigma'}, \boldsymbol{k}) \cap$ **Bad** $= \emptyset$

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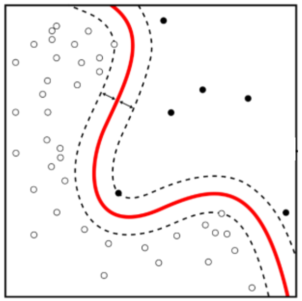
[CAV'03] Interpolation and SAT-Based Model Checking, McMillan
[HVC'12] Computing Interpolants without Proofs. Chockler, Ivrii, Matsliah

From Learning to Inference



Efficiently

Exact **learning**
DNF formulas

$\Longrightarrow$

Efficiently

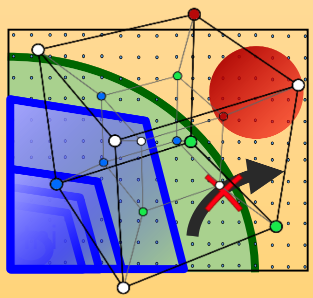
Inferring
DNF invariants

ψ := false

while σ' counte… to **Equiva**…

ψ := ψ ∨ **gen**…

generalize(σ'):
drop literals from σ'
while **Membership**(σ')= ✓



The invariant is ***k*-fenced**

…′) counterexample
Inductive(I):

∨ **generalize**(σ')

generalize(σ'):
drop literals from σ'
while **BMC**(σ', k) ∩ **Bad** = ∅

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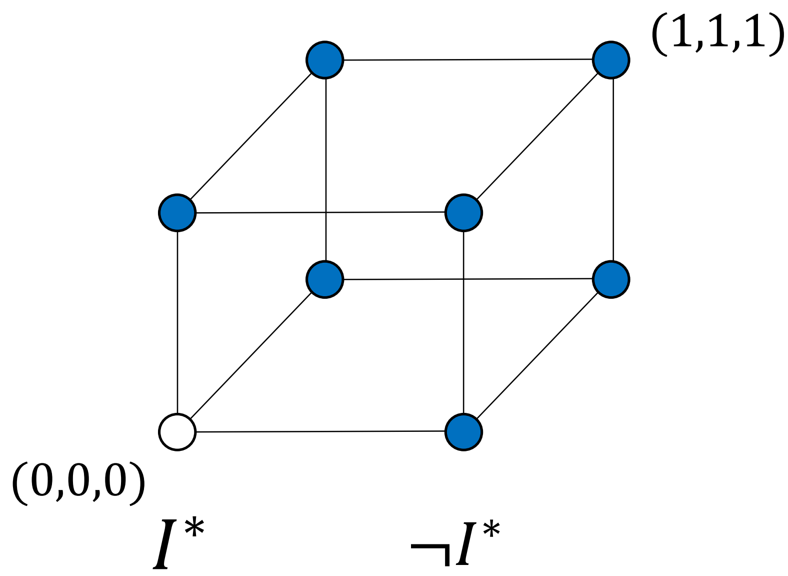
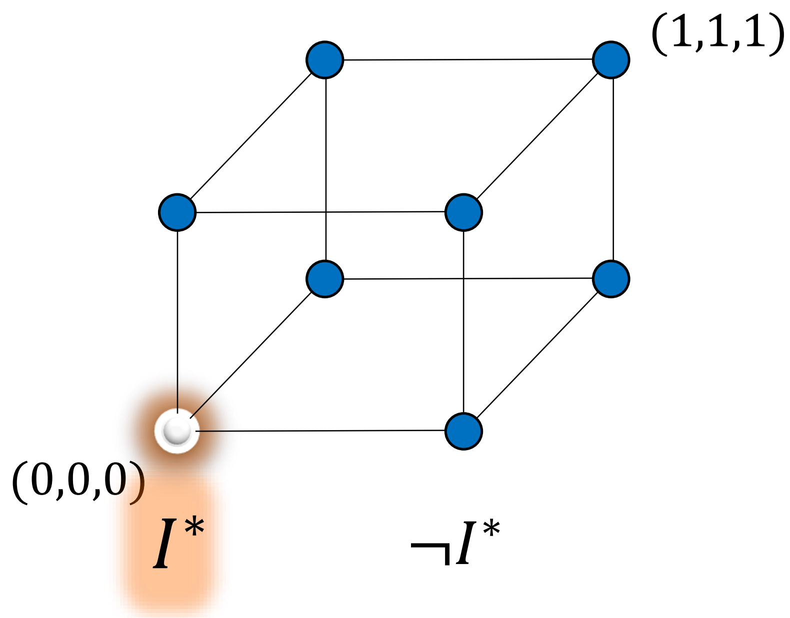
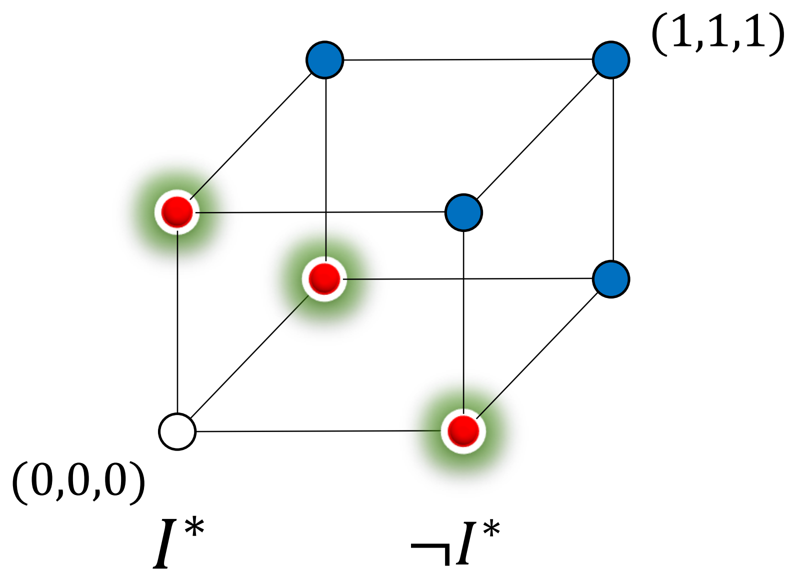
k-Fenced Invariants



k-Fenced Invariants

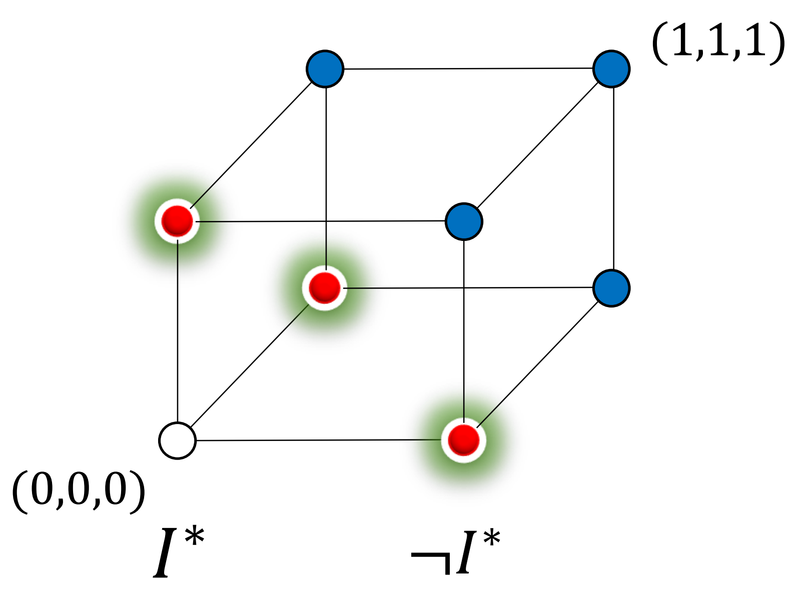


k-Fenced Invariants



$$\partial^-(I^*)$$

k-Fenced Invariants

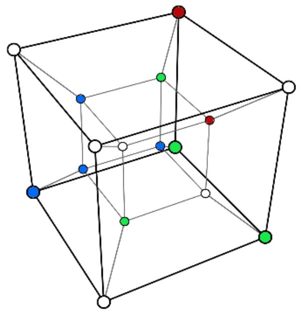


All the states in $\partial^-(I^*)$
can get to a bad state in at most k steps

Complexity Upper Bounds

Thm. Interpolation-based inference finds an invariant in a polynomial number of SAT queries when

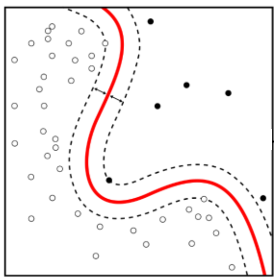
$\exists I^*.$



Fence condition: the Hamming boundary of I^* reaches bad states in k steps

No negated variables

+



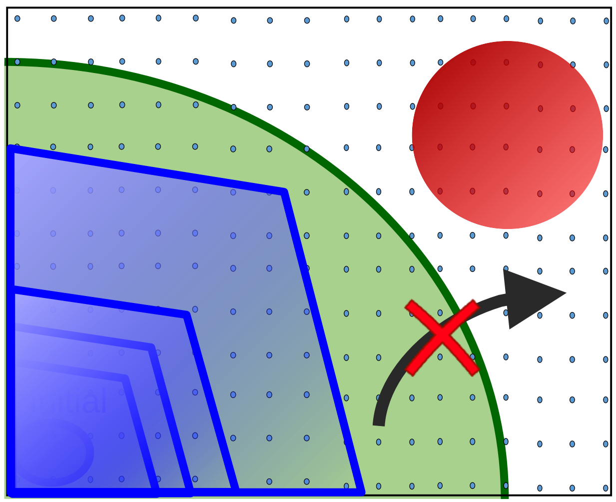
I^* is a short monotone DNF (via Angluin)
or
I^* is a short almost-monotone DNF (via Bshouty)

$O(1)$ terms with negated variables

[POPL'21] Learning the Boundary of Inductive Invariants. Feldman, Sagiv, Shoham, Wilcox

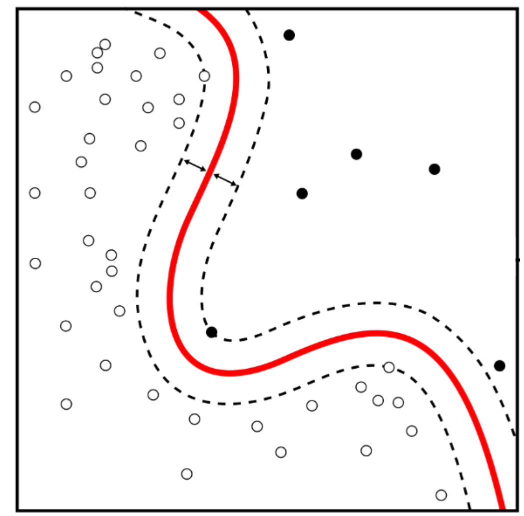
Conclusion

Invariant Inference



vs.

Exact Concept Learning



- Query-based learning models for invariant inference
- Invariant inference is harder than concept learning
- Complexity results for invariant inference algorithms from classification algorithms