Alan J. Hoffman

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Alan was a main founding father of combinatorial optimization and other mathematics.

He was editor of the Linear Algebra Journal.

He was the first programmer of the simplex method for linear programming
and the discoverer of its cycling
at the National Bureau of Standards on one of the first computers .

He was a jolly loved person.

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Existentially Polytime Theorems. Starting in 1934 I grew up in Washington, D.C., the luckiest possible place and time for a blue-collar kid.
I was told that Great Granddad Edmonds had laid the stone point of the Washington Monument.

In 1949 at the **Naval Observatory**, I learned calculus by dividing infinitesimals from **Glen Drape**r who calculated the Naval Almanac. He thought Weierstrass and non-Euclidean geometry were foolish, and had written the book, **'What is Truth?'**
I went to a marvellous public high school, **McKinley Tech**, and then loved life at three universities.

I wanted to be a reporter and I was a copyboy in the newsroom of the **Washington Post**, but I was much too slow, so I went for being a 'mathie' in order to have a long turn-around time.

In 1959 I dropped out of university to support babies and a teenage wife. I got a $60 pad for us in the center of D.C., and a job in the **Mathematics Division of the National Bureau of Standards**, where I was assigned to the **Operations Research Section**,
newly formed by Princeton PhD, **Alan Goldman** (**effectively hired by Alan Hoffman).**
I was encouraged to study whatever might be useful.

The Marriage Theorem on 'Systems of Distinct Representatives' blew my mind, prompting me to think about NP∩coNP and to conjecture that it implies P.

This worked well enough that Alan Goldman had his Princeton mentor **Professor A.W.Tucker** arrange for me to be the junior participant in **a 1961 workshop at the RAND Corporation** for everyone prominent in combinatorics. I had a eureka moment the day before I was to preach to these high priests, with support from Alan Hoffman, **and so NP∩coNP took over my life.**

‘NP∩coNP theorems’ are sometimes called ‘good characterizations’.

The Marriage Thm is a good characterization of
when a finite set of girls can be matched to distinct boys they love.

It provides a way for the matchmaker to prove easily to the parents
<u>whenever</u> a desirable matching is not possible
by showing them a subset S of the girls who together love fewer than |S| boys.

A matching of all the girls to distinct boys they love is ‘easy’ for the parents to recognize.
A subset S of the girls which does not love enough boys is also ‘easy’ for them to recognize.
Proving that one or the other always exists can be proved by
a polynomial time algorithm which always finds one or the other.

I was mystified at first by the paper of Halmos and Vaughan saying that
the girls can be matched to distinct boys they love <u>if and only if</u>
every subset S of girls loves at least |S| boys.
Is checking every subset of girls easier that checking every possible matching?
An implementation of their proof is exponential. Amazingly it is in a book called ‘God’s Proofs’.

When mathies prove something exists they like to think that it is hard to find.
Algorithmically exponential proofs look more elegant to them.

A predicate p(x) is called NP when p(x) ≡ [∃y such that f(x,y) is true]
where the bit-size of y is at most some polynomial of the bit-size of x, and
where f(x,y) is a predicate given by a known 'polytime' algorithm, that is,
whose running time is bounded by some polynomial in the bit-sizes of x and y.

An <u>EP Theorem</u> is an NP predicate, p(x), which is true for every x.
That is: <u>∀x∃y such that f(x,y) is true</u>
where the bit-size of y is at most some polynomial of the bit-size of x, and
where f(x,y) is a predicate given by a known polytime algorithm, that is,
whose running time is bounded by some polynomial in the bit-sizes of x and y.

A theorem is an **<u>NP∩coNP Theorem</u>** when it says that
the negation of some known NP predicate $\mathbf{p^1}$ equals some other known NP predicate $\mathbf{p^2}$.
That is:

∀x (p^1 or p^2). <u>Not both.</u>

Clearly an **NP∩coNP Theorem** is a special kind of **EP Theorem**.
Most EP theorems don't have 'Not both'.

The ideal way to prove an EP Theorem is by a polytime algorithm which, for any x, **finds a y** such that f(x,y) is true.
This is not the traditional way.

Because of pigeon holes and parity arguments, I do not conjecture that for any EP Thm there is always a polytime algorithm proof. However, it seems there is one for most EP theorems. I do conjecture there is a polytime algorithm proof for any NP∩coNP theorem including factoring.

Known NP∩coNP theorems are rather rare.
Whereas EP theorems are pervasive.

'<u>Existentially Polytime Theorem</u>' is in fact a formalization of what discrete mathies often intuitively regard as beautiful.

A ‘simple graph’ G is a finite set of ‘nodes’, or ‘vertices’, and a set of different unordered pairs of those nodes called ‘edges’. ‘Degree’ of a node v in G means number of edges of G which contain v.
A matching in G is a subset of edges of G which are mutually disjoint.
A perfect matching in G is a subset of edges of G which partitions the nodes of G.

Petersen’s EP Thm (1891): A graph where each node has degree 3 either has an edge which separates the graph or a perfect matching (a subset of the edges which partitions the nodes).
(The famous ‘Petersen graph’ has no partition (‘coloring’) of the edges into 3 perfect matchings.)

Tutte’s NP∩coNP Thm (1947) is a good characterization of all G’s not having a perfect matching.
My eureka moment at RAND in 1961, supporting the conjecture that **NP∩coNP = P,**
was finding a polytime algorithm for ‘optimum matchings’ which proves Tutte’s Thm.

Vizing’s EP Thm (1965): Any simple graph has a proper k coloring of its edges or a node of degree ≥ k. (Maybe both.)

Dirac’s EP Thm (1952): Any simple graph either has a Hamiltonian cycle (a ‘t.s. tour’) or a node of degree less than half the number of nodes. (Maybe both.)
(For any G, deciding whether G has a Hamiltonian cycle is the 2nd most famous **NP complete problem, shown by Karp polytime reducing 3-SAT to it**.)

Alan Hoffman's 'Circulation' NP∩coNP Thm (1960): For any directed graph ('network') G with a non-negative integer lower bound a_e and upper bound b_e in each edge e of G, there exists an integer valued 'circulation' $y = \{y_e : e$ an edge of G$\}$ such that $a_e \leq y_e \leq b_e$ or else there exists some partition $[S,\underline{S}]$ of the nodes such that the sum of the b_e on the edges directed from S to $\underline{S}$ is smaller than the sum of the a_e on the edges directed from $\underline{S}$ to S.

Not Both. ('Circulation' means 'for each node, what goes in = what comes out'.)

Proved by linear programming theory, or alternatively by polytime 'network flow methods'.


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This presentation is in memory of Alan Hoffman. He died two weeks ago at the age of 96.

He was an important mentor to me and other people -

the mathematical computer scientist who had the most mathematical influence on me.

He took me kiting. He supported me at the 1961 RAND combinatorics workshop, after my talk and at breakfasts where, e.g., he said 'Jack, you really should study the simplex method'.

Strong Perfect Graph EP Thm.
(conjectured in 1960 and proved in 2002 after a great many papers on partial results):
For any graph G, either G has an independent set J of vertices
and a set C of cliques covering all the vertices such that |J|=|C|
or else G has an odd hole or an odd anti-hole, i.e., is <u>not</u> a 'Berge Graph'. Maybe both.

This is an unusual way to state the theorem which I hope will encourage someone
to provide a proof by a polytime algorithm which finds in any graph G
either a J and C such that |J|=|C| or an odd hole or an odd anti-hole.
I still do not know of a polytime algorithm which simply finds in any graph G
an instance of what the theorem says exists.

In 2005 some of the people involved in proving the SPGT found a polytime algorithm
for deciding whether or not any G is a Berge graph (i.e., has no odd hole or odd anti-hole)
but curiously when it is a Berge graph the algorithm does not find optimum J and C.
The algorithm which I am asking for might actually be easier
because it does not recognize when G is a Berge graph.

[Paul Seymour is central to groups who did SPGT, 4 coloring, and other fabulous achievements.
Excuse me for not clearly citing in this limited exposition his great readily known co-researchers**.]**

A (2-dimensional) <u>triangulated surface</u> S is a finite set V of 'nodes', or 'vertices',
and a set F of unordered triples of vertices, called the rooms of S, such that for every vertex v
the set of the rooms containing v is a 'disc' W(v), i.e., (W(v) – v) is a simple cycle.
A pair of vertices in a room is called a 'wall' (or an 'edge') of S.
Each wall is in 2 rooms. Each room has 3 walls.
The set E of walls and V of vertices are a kind of G embedded in surface S.

There are many purely combinatorial characterizations of when S is
a triangulated (2-dimensional) <u>sphere</u>. The simplest is Euler's formula, |V|-|E|+|F|=2.

4-Color EP Thm (conjectured in 1852 and proved in 1976)**:**
For any triangulated sphere S there is a 'proper' 4-coloring of the vertices of S
(that is, so that each edge is 2 vertices of different colors).

The only known proofs use a computer.
I am told (by Tommy Jensen) that the 1976 proof is not a polytime algorithm,
but the 1997 proof is a quadratically bounded algorithm,
and 'It is open if this complexity bound can be improved, maybe even to linear'.

Tait's Thm (1878), not an EP Thm but beautiful:

For any triangulated (2-dim) sphere S there is a 'proper' 4-coloring of the vertices of S (that is, so that each edge is 2 vertices of different colors) **if and only if there is a 3-coloring of the edges of S so that the 3 walls of each room are the 3 different colors.**
(Tait's Theorem was a motivation for Petersen's 1891 Theory.)

And so in 1976 it was finally proved that:
For any triangulated (2-dim) sphere S there is a 3-coloring of the edges of S so that the 3 walls of each room are the 3 different colors.

A 3-dimensional triangulated **manifold** M is a finite set V of 'vertices' and a set F of unordered 4-tuples such that for any vertex v of M the set of the rooms containing v is a 'ball' W(v), i.e., (W(v) – v) is a triangulated 2-dim sphere.

A purely combinatorial meaning for when an M is a triangulated 3-dimensional sphere is problematic, having much occupied topologists. However I wonder if there are some (other) NP descriptions of a class M of manifolds which satisfies 'hyperTait':
∃ a 4-coloring of the walls so that the 4 walls of each room are the 4 different colors.
Notice that Vizing's Thm does not quite do it.

An **n-dimensional simplicial complex S** is a finite set of ‘nodes’, or ‘vertices’, and a set of unordered (n+1) tuples of vertices, **called the rooms of S**. (‘1-dim simplicial complex’ means ‘simple graph’.)

S is called **an n-dimensional <u>pseudo</u>-manifold** when each wall is in 2 rooms.

(S is called a **manifold** when for every vertex v of S, the set **W(v)** of rooms containing v is such that (W(v) – v) is an **(n-1) dimensional simplicial <u>sphere</u>**. **But what is an (n-1) dimensional sphere?)**

The next so-called **Room-Partition EP Thm** is about
any pseudo-manifold S with a given subset, say R^0, of rooms
<u>which partitions the set V of vertices</u>.
(Whether a pseudo-manifold S has such an R^0 is probably NP complete.)
Think triangulated 2-dim sphere, or plane, since that’s interesting enough.

Curiously, where the 4-Color Thm is proved by a polytime algorithm which is too complicated except for a computer, the **Room-Partition EP Thm** is proved by an algorithm simple enough for kindergarden
- but we suspect there is no polytime algorithm for it.

Room-Partition EP Thm.
For any triangulated surface S (in fact for any n-dimensional pseudo-manifold)
and any subset R^0 of the rooms of S which partitions the vertices of S,
there is another different subset R* of the rooms of S which partitions the vertices of S
(in fact R = R^0, so the number of 'room-partitions' is even).**

Algorithmic Proof: Having chosen any vertex say v^0 as special,
any room-partition R^0 determines another room-partition R* (so that R**=R^0) as follows.
Get a set of rooms R^1 from R^0 by replacing the unique room r^0 of R^0 containing vertex v^0
by the unique room r^1 which shares the wall of r^0 not containing v^0.
At the general step either set R^k of rooms is the desired room-partition R*
or there is a unique single vertex v^k contained in room r^k of R^k and also in room r^{k+1} of R^k.
Get R^{k+1} from R^k by replacing room r^{k+1} with the other room which shares the wall of r^{k+1}
not containing vertex v^k. This algorithm, since it cannot repeat, must stop.
It only can with an R*. **[INSERT MOVIE EXAMPLE]**
This algorithm for finding another room-partition is **not polytime:**
There is a sequence of inputs where the number of triangles
grows by a fixed amount while the number of steps doubles.
We suspect there is no polytime algorithm, but maybe there is.

There is **<u>a similar non-polytime</u>** algorithm for pairing the perfect matchings in any even-degree graph. **And yet there is an entirely different algorithm which is polytime for, given a perfect matching, finding another one.**

Might there be a polytime algorithm which, given a room-partition R^0 of vertices of a triangulated 2-dim sphere, finds another room-partition R* with perhaps $R^{**} \neq R^0$ even if there is no polytime pairing of the room-partitions?
Might there be a polytime algorithm for finding a Nash equilibrium even if it is not by polytime pairing something like the Lemke-Howson algorithm?

A search problem Q is called PPA complete if 'any problem of pairing' can be reduced in polytime to Q. A number of search problems which pair 'steamboats' of some kind have been shown to be PPA complete, and it is reasonable to suspect from this that there is no polytime algorithm for pairing the steamboats.

I conjecture that room-partition pairing is PPA complete.
However I think that, even if there is no polytime algorithm for finding a pairing, which always exists, there can be a polytime algorithm anyway for, given a steamboat, finding another without finding a pairing.

The Danish Alternating-Color Parity Thm (2020):

Jørgen Bang-Jensen of Southern Denmark University conjectured that
for any complete graph with edges colored red and blue, and any integer k > 1,
there is an even number of length k color-alternating paths.

For k even, Bjarne Toft proved it with a ridiculously easy pairing algorithm.

Other Danes, Tommy Jensen and Carsten Thomassen, independently, eventually found proofs that there is an even number of length k color-alternating paths
when k > 1 is odd
but <u>not yet</u> by an algorithm which pairs such paths
(though of course a pairing algorithm exists).

I do not know of any reductions between Room-Partitioning, the Danish Alternating Color Thm, and other known parity theorems, except of course for known PPA Complete problems.

'Large SAT is Easy' is an EP Thm about SAT which is nicely proved by a counting argument.
For a given conjunctive-normal-form formula F, let 2^{-k} be 'the weight' of a clause with k literals.
Let the weight of F be the sum of the weights of its clauses.

'Large SAT is Easy' EP Thm: For any CNF formula F,
either there is a valuation of its variables which satisfies F
or else the weight of F is at least 1.

I was thinking I would describe a polytime algorithm which finds
a satisfying valuation whenever F has weight less than 1
but what the heck.
Instead, I'll award a gold-painted trophy to anyone who sends me
an enjoyable exposition of EP theorems which includes their personally
designed polytime algorithm for, given any CNF formula with weight < 1,
find a satisfying valuation of its variables.

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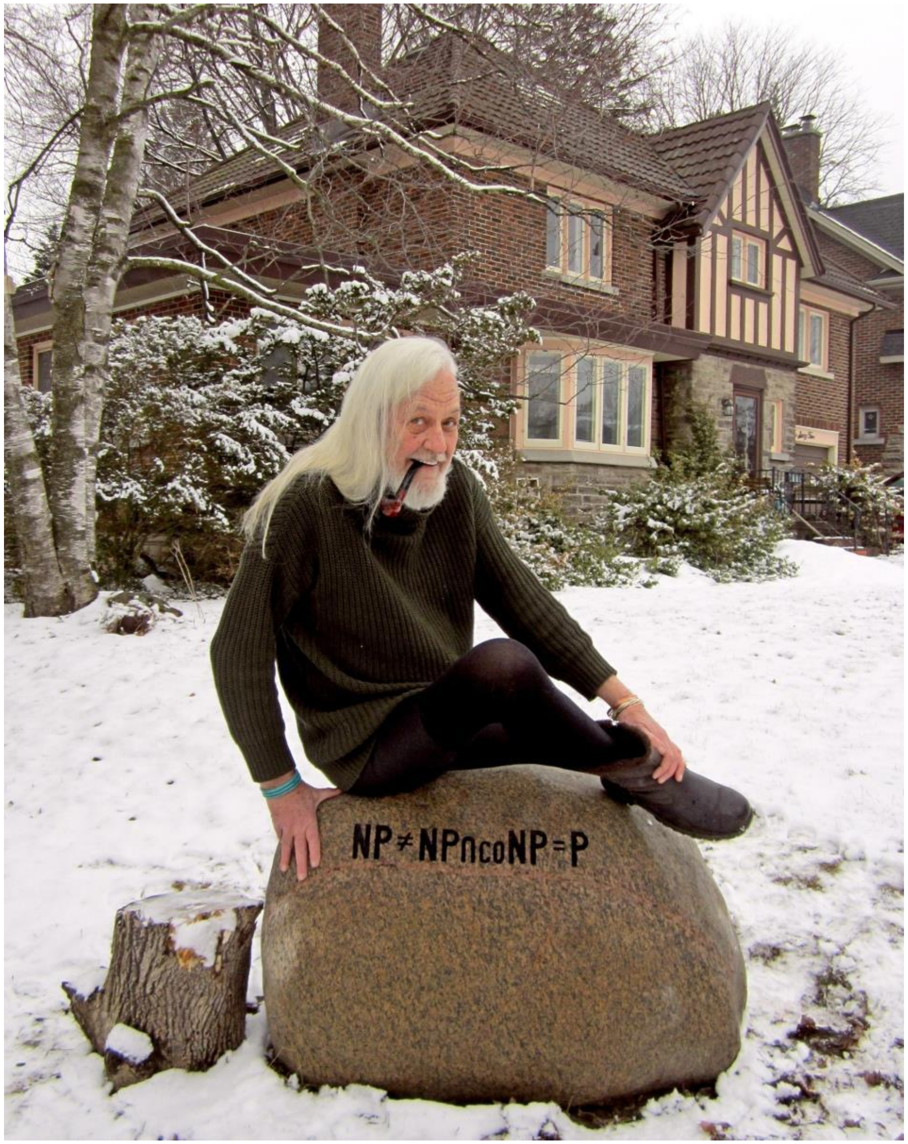

Thanks for reading.