From Gaussian measure to partial colorings and linear size sparsifiers

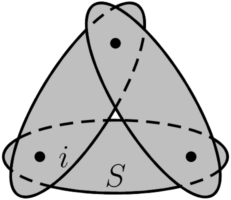
Thomas Rothvoss

Based on [R. FOCS'14], [Reis, R. SODA'20], [Reis, R. Arxiv '20]

UNIVERSITY of WASHINGTON

Discrepancy theory

- Set system $\mathcal{S} = \{S_1, \ldots, S_m\}, S_i \subseteq [n]$



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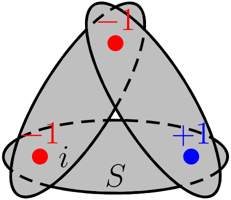

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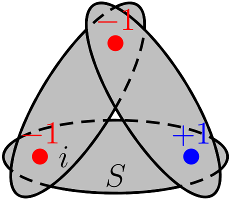

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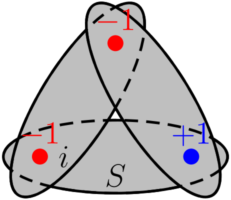

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Known results:

- n sets, n elements: $\mathrm{disc}(\mathcal{S}) = O(\sqrt{n})$ [Spencer '85]
- Every element in $\leq t$ sets: $\mathrm{disc}(\mathcal{S}) < 2t$ [Beck & Fiala '81]

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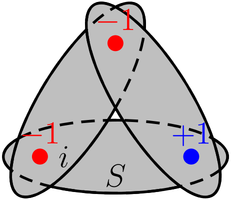

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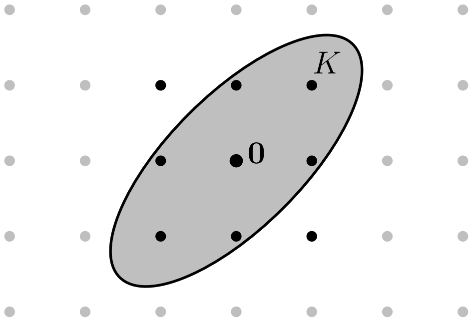
Main method: Find a **partial coloring** $\chi : [n] \to \{0, \pm 1\}$

- low discrepancy $\max_{S \in \mathcal{S}} |\chi(S)|$
- $|\mathrm{supp}(\chi)| \geq \Omega(n)$

Spencer/Gluskin/Giannopolous Thm

Theorem (1980s)

Let K be symmetric convex set with $\gamma_n(K) \geq e^{-\frac{1}{10}n}$. Then $\exists x \in K \cap \{-1, 0, 1\}^n$ with $|\text{supp}(x)| \geq n/10$.



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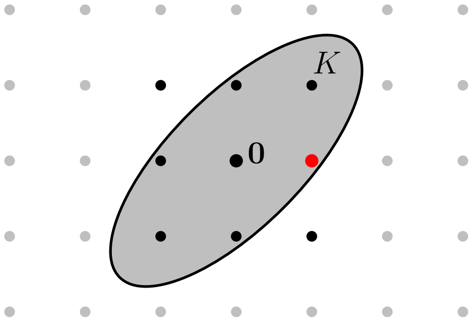

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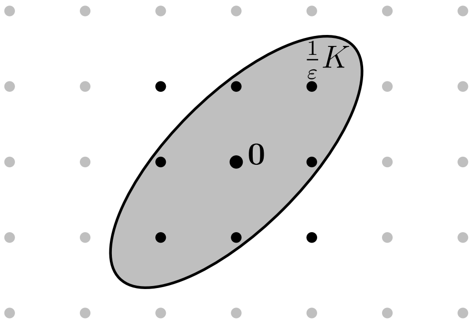
- Based on pigeonhole principle [non-algorithmic]

Algorithmic Discrepancy

Theorem

For any $\alpha > 0$, there are $\varepsilon, \delta > 0$ so that: Let K be symmetric convex set with $\gamma_n(K) \geq e^{-\alpha n}$. Can find $x \in \frac{1}{\varepsilon}K \cap [-1,1]^n$ with $|\{i : x_i \in \{-1,1\}\}| \geq \delta n$ in **poly-time**.



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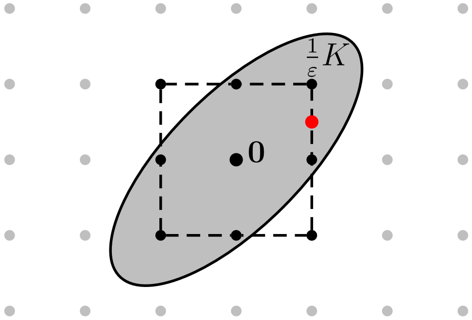

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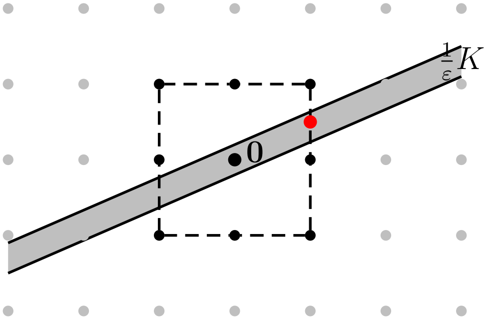

- Might not exist for $x \in \{-1,0,1\}^n$

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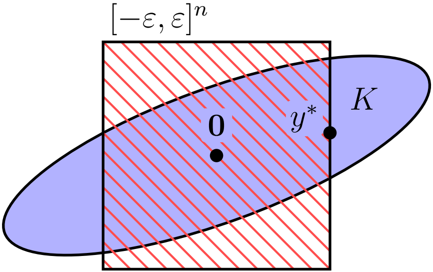

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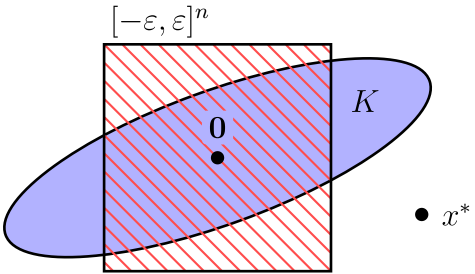

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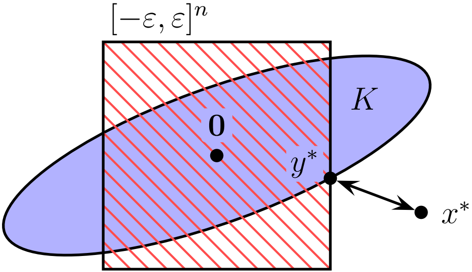

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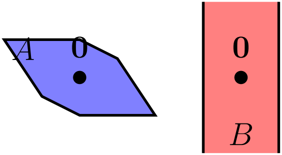

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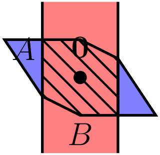

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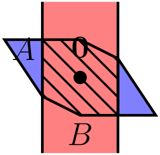
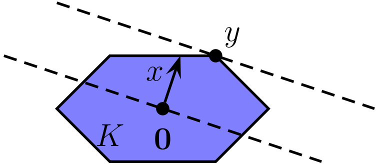
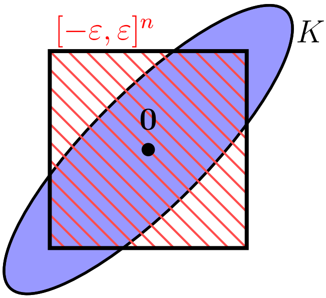


- $w(K) := \mathbb{E}_{x \sim S^{n-1}}[\max\{\langle x, y\rangle : y \in K\}]$



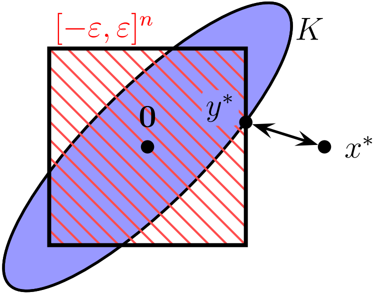
- **Urysohn:** Among convex bodies with same $\gamma_n(K)$, $\mathbf{0}$-centered Euclidean ball minimizes $w(K)$.

Analysis



Analysis

- W.h.p. $\|x^* - y^*\|_2 \geq (1-5\varepsilon)\sqrt{n}$



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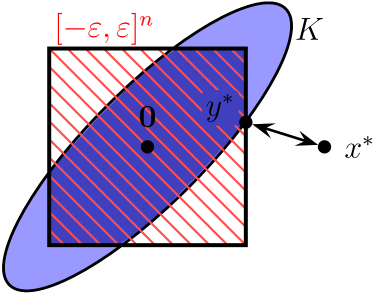

- **Key observation:**

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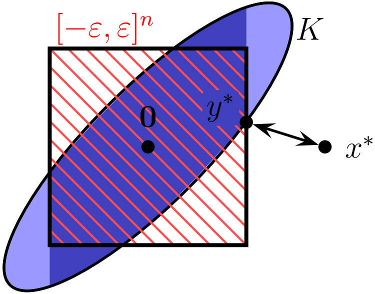

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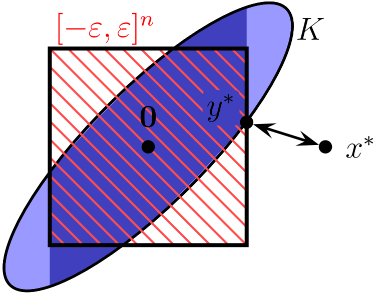

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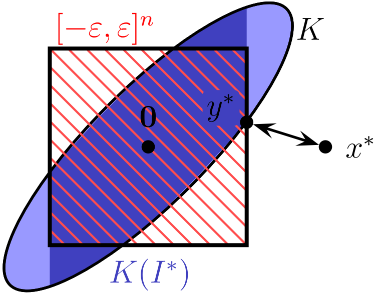

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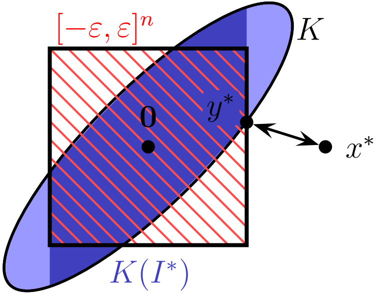

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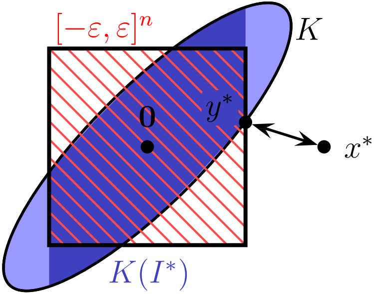

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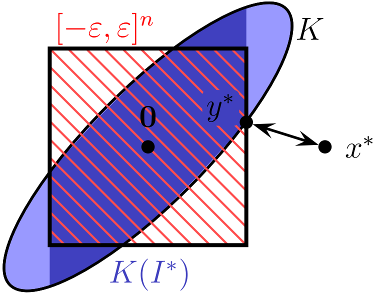

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- Union bound over all $|I| \leq \delta n$:

  $$\Pr\Big[\bigcup_{|I|\leq\delta n} d(x^*, K(I)) > (1-10\varepsilon)\sqrt{n}\Big] \leq e^{-\Omega_\varepsilon(n)} \qquad \square$$

Gaussian close to large bodies

Lemma

Let $Q \subseteq \mathbb{R}^n$ be convex symmetric with $\gamma_n(Q) \geq e^{-\alpha n}$. Then

$$\mathop{\mathbb{E}}_{x \sim \gamma_n}[d(x, Q)] \leq \sqrt{n} \cdot \left(1 - \frac{1}{512\alpha e^{4\alpha}}\right)$$

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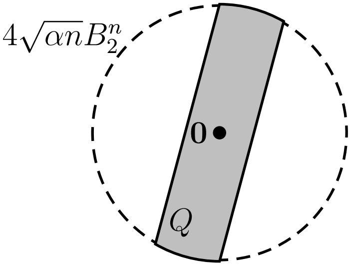

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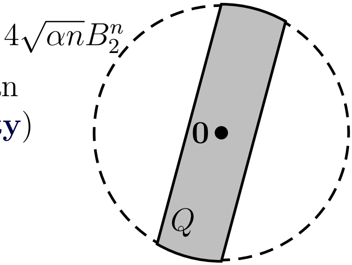

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- $z(x) := \text{argmax}\{\langle z, x\rangle : z \in Q\}$
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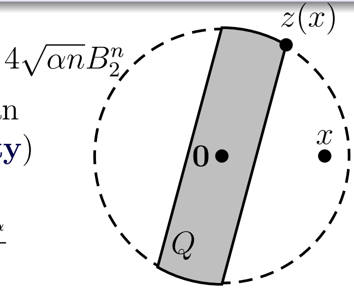

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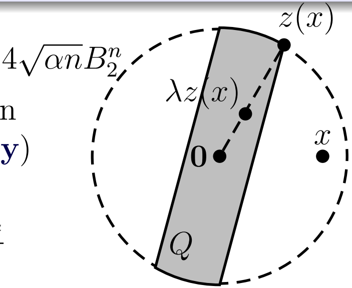

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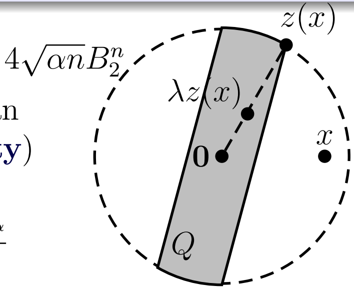

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$$\overset{\text{choose } \lambda}{\geq} n \cdot \left(1 - \frac{1}{256\alpha e^{4\alpha}}\right)$$

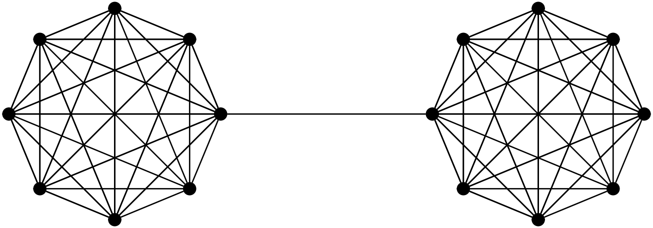
Part II

Linear size sparsifiers in graphs

Graph Sparsification

Theorem (Batson-Spielman-Srivastava '08)

For any graph $G = (V, E)$ one can find weights $s(e) \geq 0$ in poly-time with $|\text{supp}(s)| \leq O(n/\varepsilon^2)$ so that $|\delta(U)| = (1 \pm \varepsilon) \cdot |s(\delta(U))|$ for every $U \subseteq V$.

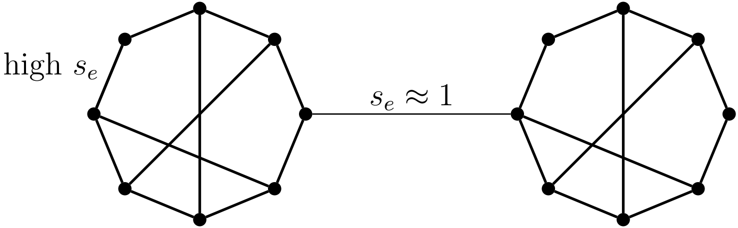


- Even stronger:
  Laplacian of weighted sparse graph $\approx$ original Laplacian

Graph Sparsification

Theorem (Batson-Spielman-Srivastava '08)

For any graph $G = (V, E)$ one can find weights $s(e) \geq 0$ in poly-time with $|\mathrm{supp}(s)| \leq O(n/\varepsilon^2)$ so that $|\delta(U)| = (1 \pm \varepsilon) \cdot |s(\delta(U))|$ for every $U \subseteq V$.



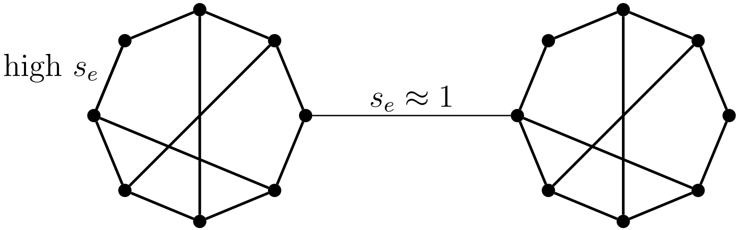
- Even stronger:
  Laplacian of weighted sparse graph $\approx$ original Laplacian

Graph Sparsification

Theorem (Batson-Spielman-Srivastava '08)

For vectors $v_1, \dots, v_m \in \mathbb{R}^n$ with $\sum_{i=1}^m v_i v_i^T = I_n$, one can find weights $s \in \mathbb{R}^m_{\geq 0}$ in poly-time with $|\text{supp}(s)| \leq O(n/\varepsilon^2)$ so that

$$(1-\varepsilon)I_n \preceq \sum_{i=1}^m s_i v_i v_i^T \preceq (1+\varepsilon)I_n$$



A new sparsification algorithm

Given: PSD matrices A_i with $\sum_{i=1}^{m} A_i = I_n$ and $\varepsilon > 0$.

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Theorem

$O(\log m)$ iterations suffice and output is $1 \pm O(\varepsilon)$ sparsifier w.h.p.

How to find partial colorings

What do we know about the set

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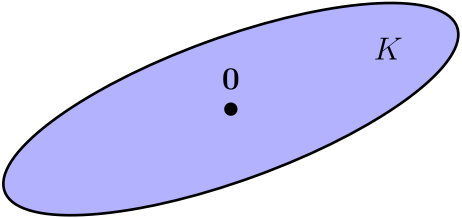

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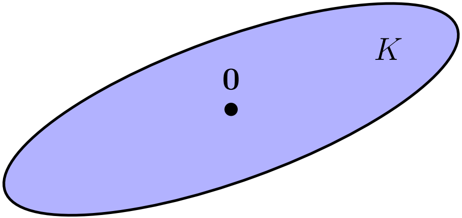

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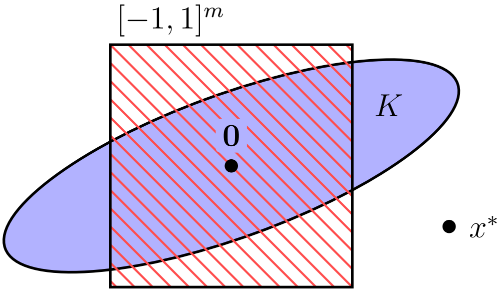

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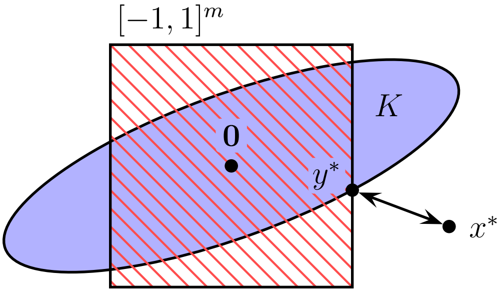

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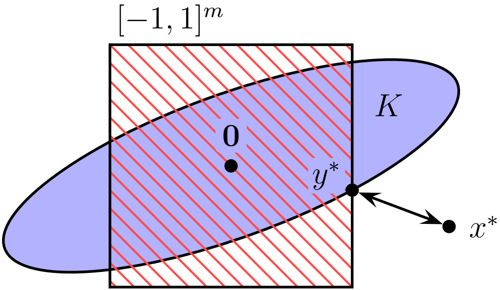

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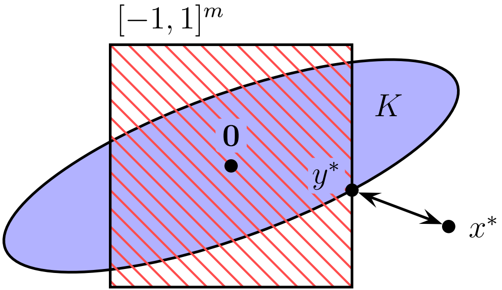

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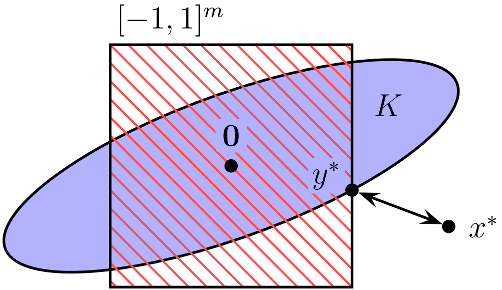

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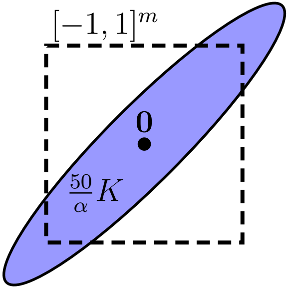

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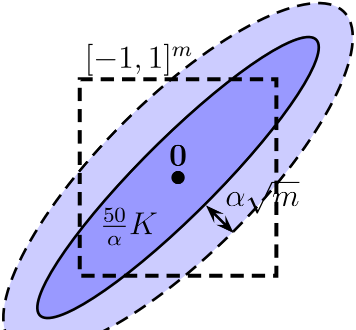

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Corollary

$w(K) \geq \Omega(\sqrt{m})$.

A coupling argument

(1) Set $\delta :=$ tiny step size and $x := \mathbf{0}, z := \mathbf{0}$

(2) FOR $T := \frac{1}{\delta^2}$ iterations DO

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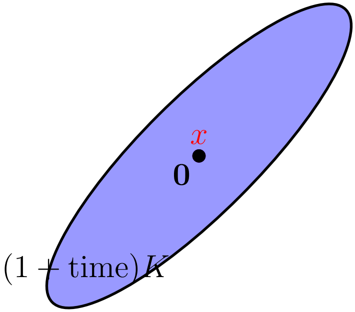

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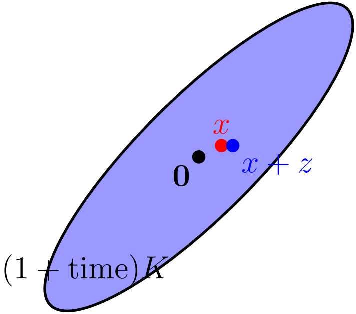

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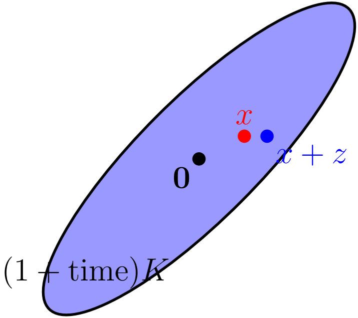

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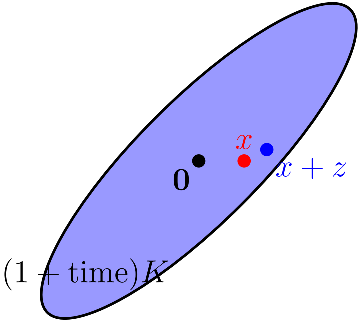

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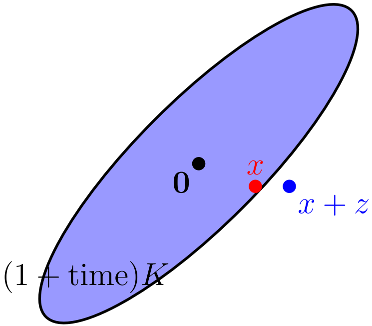

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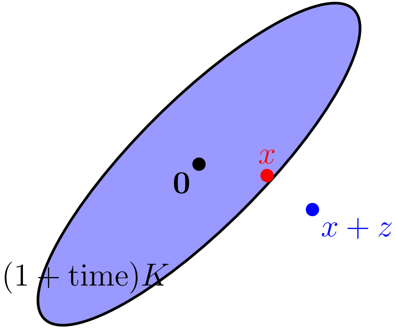

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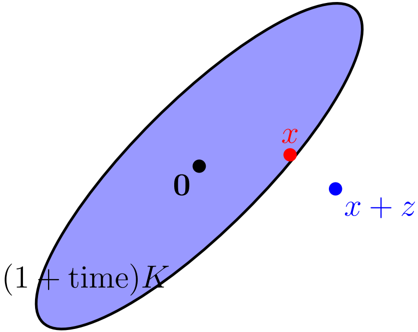

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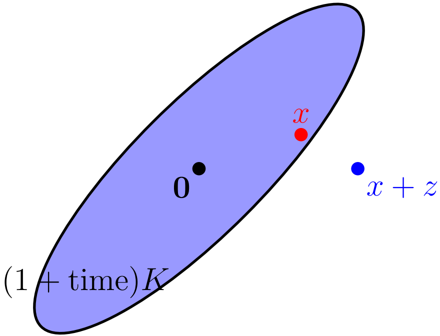

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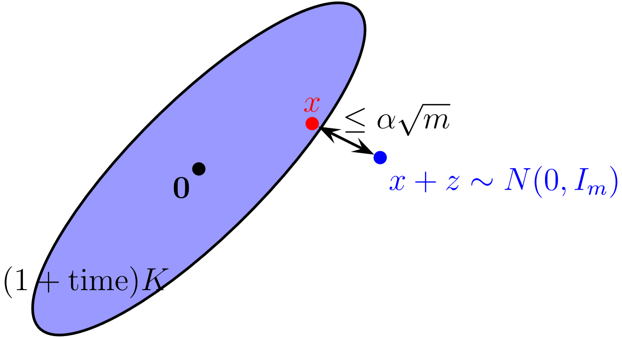

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Main Lemma

If $\Phi(x) \leq \frac{Dm^2\alpha^2}{10}$, then there is $X \preceq I_m$ with $\mathrm{Tr}[X] \geq (1-\alpha^2)m$ so that $y \sim N(0, X)$ one has $\mathbb{E}[\Phi(x + \delta y)] \leq \Phi(x)$.

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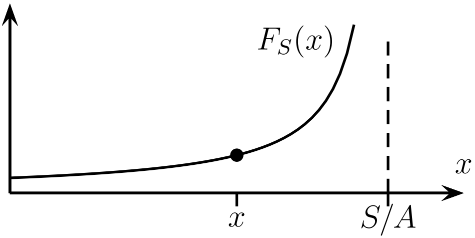
- As $\|x\|_2^2 = \Theta(m)$ one has $\lambda_{\max}(\sum_{i=1}^{m} x_i A_i) \leq C + D\|x\|_2^2 = O(\frac{\varepsilon}{\alpha})$

One-dimensional intuition



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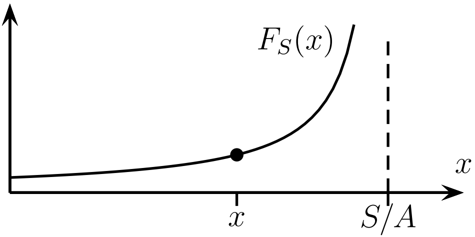

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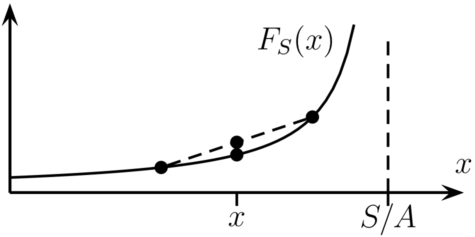

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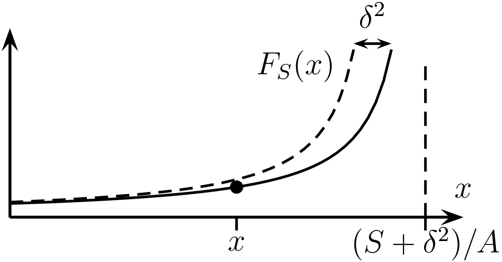

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- Shift gives

$$F_{S+\delta^2}(x) - F_S(x) \approx -\delta^2 F_S'(x) = -\delta^2 \frac{A}{(S-Ax)^2}$$

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- Pick Gaussian y orthogonal to x and any linear term in analysis

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- Pick Gaussian y orthogonal to x and any linear term in analysis
- Set $B := \sum_{i=1}^{m} y_i A_i - \delta\|y\|_2^2 I_n$.
- Change of potential function is

$$\mathrm{tr}[(A - \delta B)^{-1}] - \mathrm{tr}[A^{-1}]$$

One update step (2)

- Recall

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$$\begin{aligned} & \mathrm{tr}[(A-\delta B)^{-1}] - \mathrm{tr}[A^{-1}] \\ \overset{\text{Taylor}}{=} \quad & \delta \cdot \mathrm{tr}[A^{-1}BA^{-1}] + \Theta(\delta^2)\cdot \mathrm{tr}[A^{-1}BA^{-1}BA^{-1}] \end{aligned}$$

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One update step (3)

- **Side calculation:**

$$\mathbb{E}\left[\operatorname{tr}\left[A^{-1}\left(\sum_{i=1}^{m} y_i A_i\right)A^{-1}\left(\sum_{j=1}^{m} y_j A_j\right)A^{-1}\right]\right]$$

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- Putting everything together..

$$\mathbb{E}[\mathrm{tr}[(A-\delta B)^{-1}]] - \mathrm{tr}[A^{-1}]$$

$$\approx -\delta^2 D\,\mathbb{E}[\|y\|_2^2]\mathrm{tr}[A^{-2}] + \Theta(\delta^2)\mathrm{tr}\Big[A^{-1}\Big(\sum_{i=1}^{m} y_i A_i\Big)A^{-1}\Big(\sum_{j=1}^{m} y_j A_j\Big)A^{-1}\Big]$$

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- y is Gaussian from $(1-\Theta(\alpha^2))m$ dimensional subspace

Open problems

Let $A_1, \ldots, A_n \in \mathbb{R}^{n \times n}$ symmetric with $\|A_i\|_{\mathrm{op}} \leq 1$. Consider

$$K := \Big\{ x \in \mathbb{R}^n : \Big\| \sum_{i=1}^n x_i A_i \Big\|_{\mathrm{op}} \leq \sqrt{n} \Big\}$$

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- Is it true that $\gamma_n(K) \ge 2^{-\Theta(n)}$? [stronger]
- Is it true that $w(K) \ge \Omega(\sqrt{n})$? [weaker]
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Thanks for your attention