Sampling from the random cluster model at all temperatures on random graphs

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Phase transitions and algorithms

When are **phase transitions** barriers to **algorithms**?

Open problem in approximate counting: **#BIS**

Are phase transitions for **#BIS-hard problems on random graphs** barriers to efficient counting and sampling algorithms?

Phase transitions and algorithms

Adapt **classical techniques** from lattice spin models that complement **cavity method** / **2nd moment method**

Algorithmic approach helps us prove new **probabilistic** results

What exactly happens at the **critical temperature** for Potts on random graphs?

Potts model

Probability distribution on assignments of q colors to vertices of G:

$$\mu_G^{\text{Potts}}(\sigma) = \frac{e^{\beta\, M(G,\sigma)}}{Z_G^{\text{Potts}}(q,\beta)} \qquad Z_G^{\text{Potts}}(q,\beta) = \sum_{\sigma} e^{\beta M(G,\sigma)}$$

where $M(G, \sigma)$ is the number of **monochromatic edges**.

Inverse temperature $\beta \geq 0$ is the ferromagnetic case.

Random cluster model

Probability distribution on **subsets of edges** of G.

$$\mu_G(A) = \frac{q^{c(A)}(e^\beta - 1)^{|A|}}{Z_G(q,\beta)} \qquad Z_G(q,\beta) = \sum_{A\subseteq E} q^{c(A)}(e^\beta - 1)^{|A|}$$

where $c(A)$ is the number of **connected components** of (V,A).

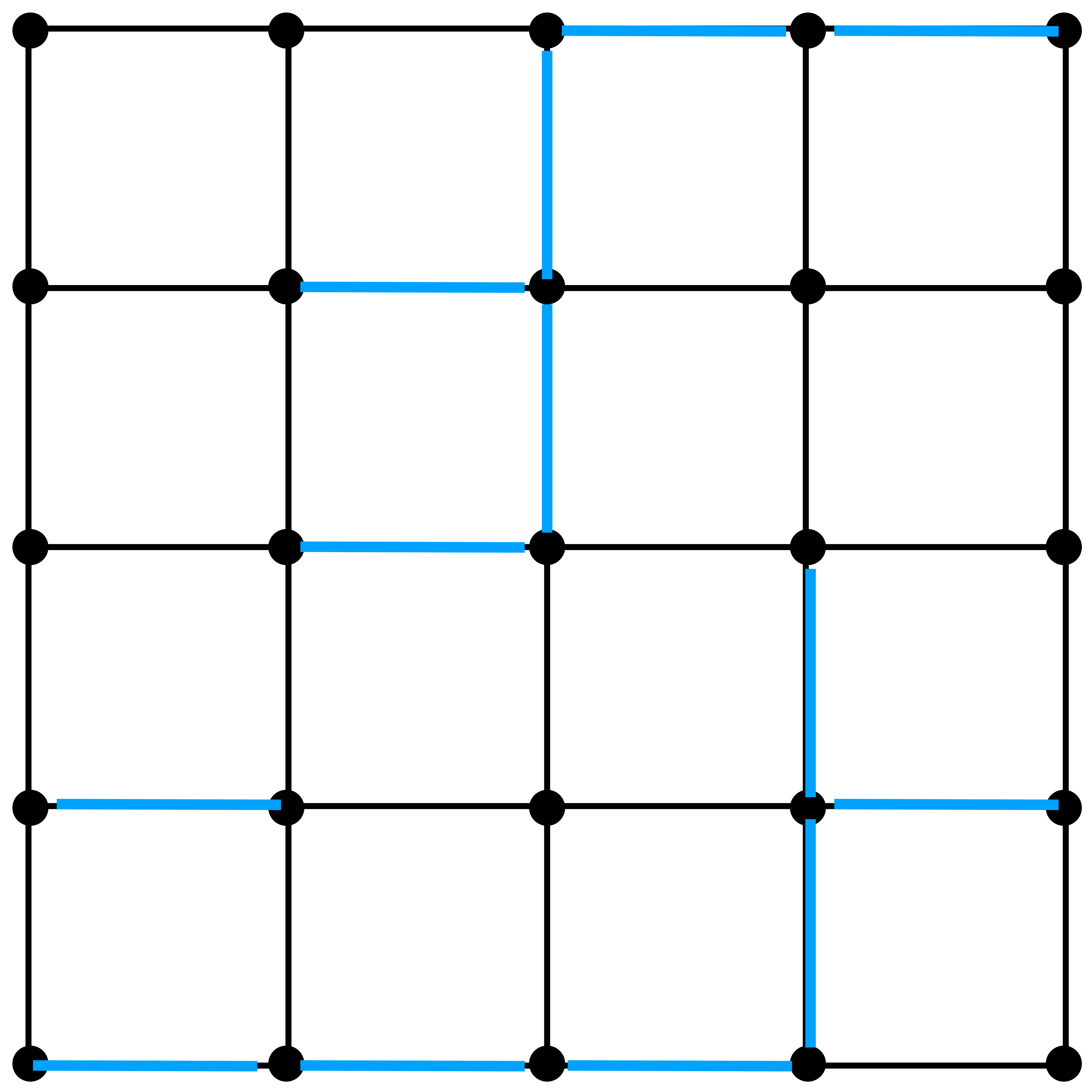
Two possible ground states: **disordered** $A = \emptyset$, **ordered** $A = E$.

$q > 0$ real

Edwards-Sokal coupling

$$Z_G^{\text{Potts}}(q, \beta) = Z_G(q, \beta)$$

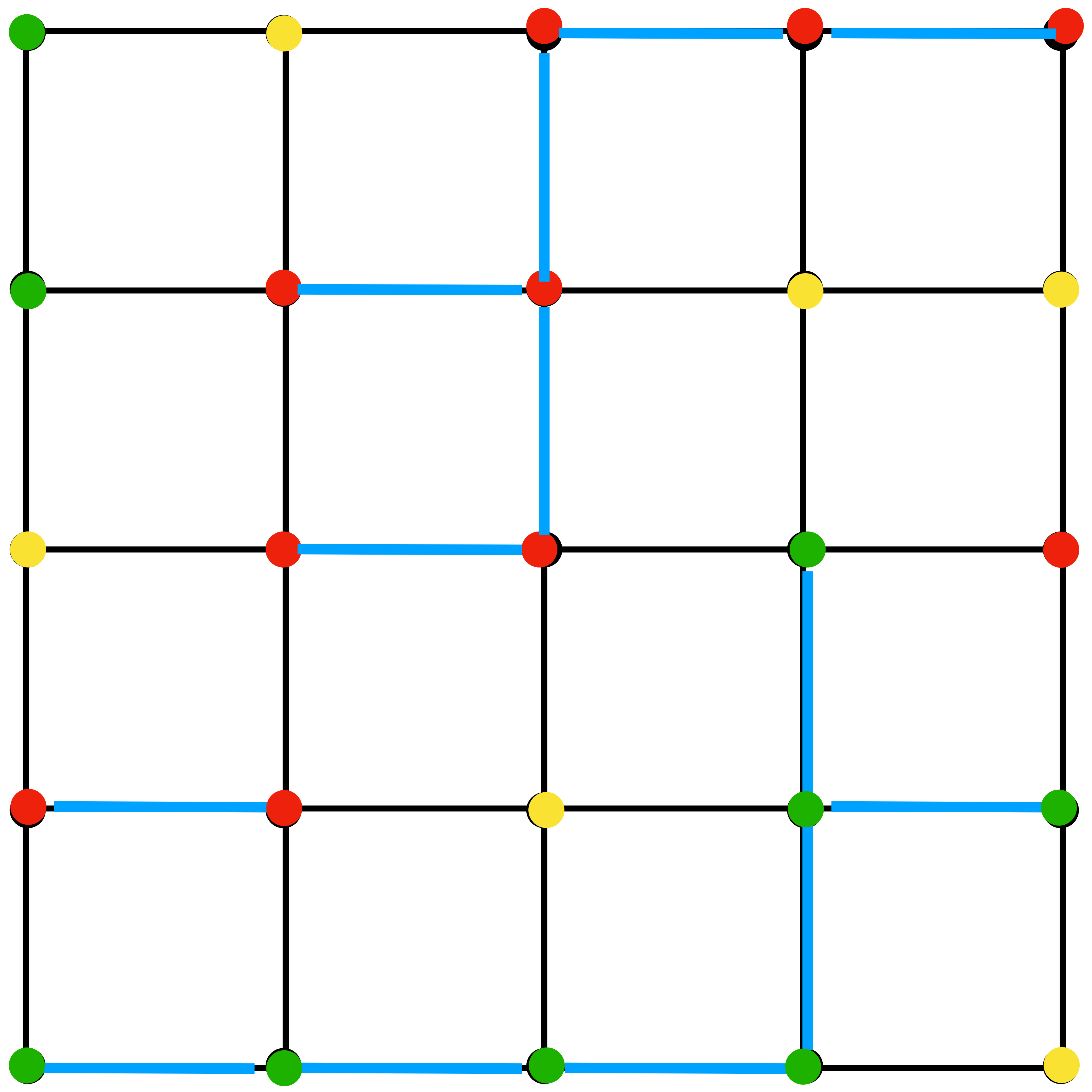
1. Pick a set of edges according to the random cluster measure
2. Determine the connected components



Edwards-Sokal coupling

$$Z_G^{\text{Potts}}(q,\beta) = Z_G(q,\beta)$$

1. Pick a set of edges according to the random cluster measure
2. Determine the connected components
3. Assign one of the q colors uniformly and independently to each connected component



Counting and sampling

Approximate counting: Given G, ϵ output a number $\hat{Z}$ so that $(1-\epsilon)\hat{Z} \le Z_G \le (1+\epsilon)\hat{Z}$ in time polynomial in n and $1/\epsilon$. FPTAS / FPRAS

Approximate sampling: Given G, ϵ output a sample σ with distribution $\hat{\mu}$ so that $||\mu_G - \hat{\mu}|| < \epsilon$ in time polynomial in n and $1/\epsilon$.

Approaches include MCMC, correlation decay method, polynomial interpolation, cluster expansion

Counting and sampling

For some models approximate counting and sampling is **NP-hard** for some range of parameters: **hard-core**, **anti-ferromagnetic Ising/Potts**, …

For these problems it's generally **NP-hard** to find a **ground state**: **max independent set**, **proper coloring**,…

For some models approximate counting and sampling is **tractable** for all graphs: **ferromagnetic Ising**, **monomer-dimer**

Counting and sampling

Then there are intermediate problems: as hard as approximating the number of independent sets in a bipartite graph (**#BIS-hard**). **Ferromagnetic Potts**, **colorings in bipartite graphs**, **stable matchings**, … Defined by **Dyer, Goldberg, Greenhill, Jerrum**

Finding a **ground state** is tractable but approximate counting/sampling is **unknown**

Hardness and random graphs

Slow mixing results for Markov chains: slow mixing for Potts known for structured families of graphs (lattices, complete graphs, random graphs)

NP-hardness for anti-ferromagnetic 2-spin systems via reductions (Sly; Sly-Sun; Galanis-Stefankovic-Vigoda); uses phase coexistence results for spin models on random bipartite graphs

#BIS-hardness via reductions (Cai-Galanis-Goldberg-Guo-Jerrum-Stefankovic-Vigoda; Galanis-Stefankovic-Vigoda-Yang) Uses phase coexistence on random graphs

For hard models, **random graphs** often provide candidate **hard instances**

But what are the **hard instances** for #BIS-hard problems?

Main results

Thm. For $d \geq 5$, $q = q(d)$ large enough, and **all** β there is an **FPTAS** and **efficient sampling algorithm** for the Potts and random cluster models on random d-regular graphs.

Main results

Thm. For $d \geq 5$ and $q = q(d)$ large enough, there exists $\beta_c(q, d)$ so that:

1. For $\beta \neq \beta_c$ the **free energy is analytic** and μ_n exhibits **exponential decay of correlations** whp.

2. μ_n **converges locally** to μ_{free} and μ_{wire} for $\beta < \beta_c$ and $\beta > \beta_c$ respectively

3. The relative weights of the ordered and disordered states at $\beta = \beta_c$ converge to given random variables (a function of small cycle counts)

Previous results

For **Potts**, a formula for the free energy is known (Bethe formula) and

$$\beta_c = \log \frac{q-2}{(q-1)^{1-2/d} - 1}$$ for all d-regular, locally tree-like graphs

(**Galanis, Stefankovic, Vigoda,Yang**; **Dembo-Montanari-Sly-Sun**)

GSVY give a weak form of phase coexistence at β_c (inverse polynomial bound on weights); this is enough to obtain slow mixing of Swendsen-Wang at β_c. They use their results to obtain **#BIS-hardness** for Potts on bounded-degree graphs

Previous results

Local weak convergence (pick σ, pick v, look at local neighborhood) results for Ising (Dembo-Montanari, Montanari-Mossel-Sly)

Distribution of log Z via cycle counts / small subgraph conditioning (e.g Coja-Oghlan-Efthymiou-Jaafari-Kang-Kapetanopoulos,…)

Previous algorithmic results

High temperature (within the uniqueness regime): (Blanca-Galanis-Goldberg-Stefankovic-Vigoda-Yang)

Low temperature: for Potts when $\beta \gg \beta_c$ (Jenssen-Keevash-P.)

Large q R.C. model on $\mathbb{Z}^d$ at **all temperatures** (Borgs-Chayes-Helmuth-P.-Tetali)

Based on the algorithmic approach of polymer models and the cluster expansion (Helmuth-P.-Regts)

Techniques

We aim to apply the lattice techniques (**Pirogov-Sinai theory** and the **cluster expansion**) to random graphs.

Complication is that we lose some **geometry** and **transitivity**

We make up for these with **expansion**.

We use **polymer models** instead of **contour models**.

We need to deal with the **non-local random cluster interaction**.

Step 1: almost all or nothing

Use expansion properties to show that for q large, with probability $1 - \exp(-\Theta(n))$ a sample from the RC model consists of **at least .9** or **at most .1** fraction of edges. (Not hard)

Write $Z = Z_{dis} + Z_{ord} + Z_{err}$

Suffices to approximate Z_{dis} and Z_{ord}

Polymer models and cluster expansion

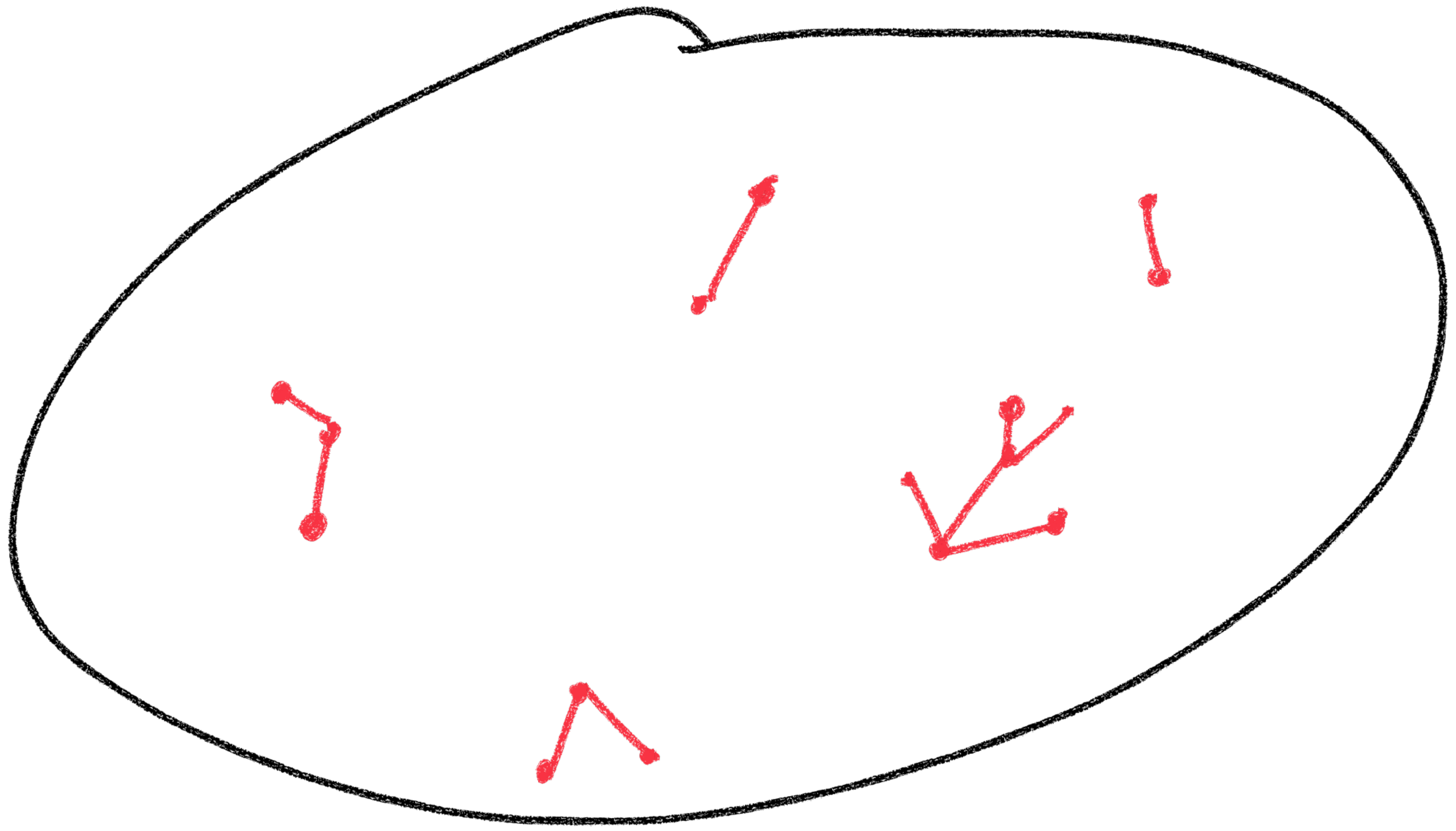
Rewrite partition function as a sum over collections of disjoint geometric objects (polymers) of product of polymer weights

If **weights decay fast enough** as a function of size, then the cluster expansion, a power series for log Z, converges

Weights must **factorize** and **decay**

Step 2: disordered

Express **disordered** configurations in terms of deviations from the **empty configuration**

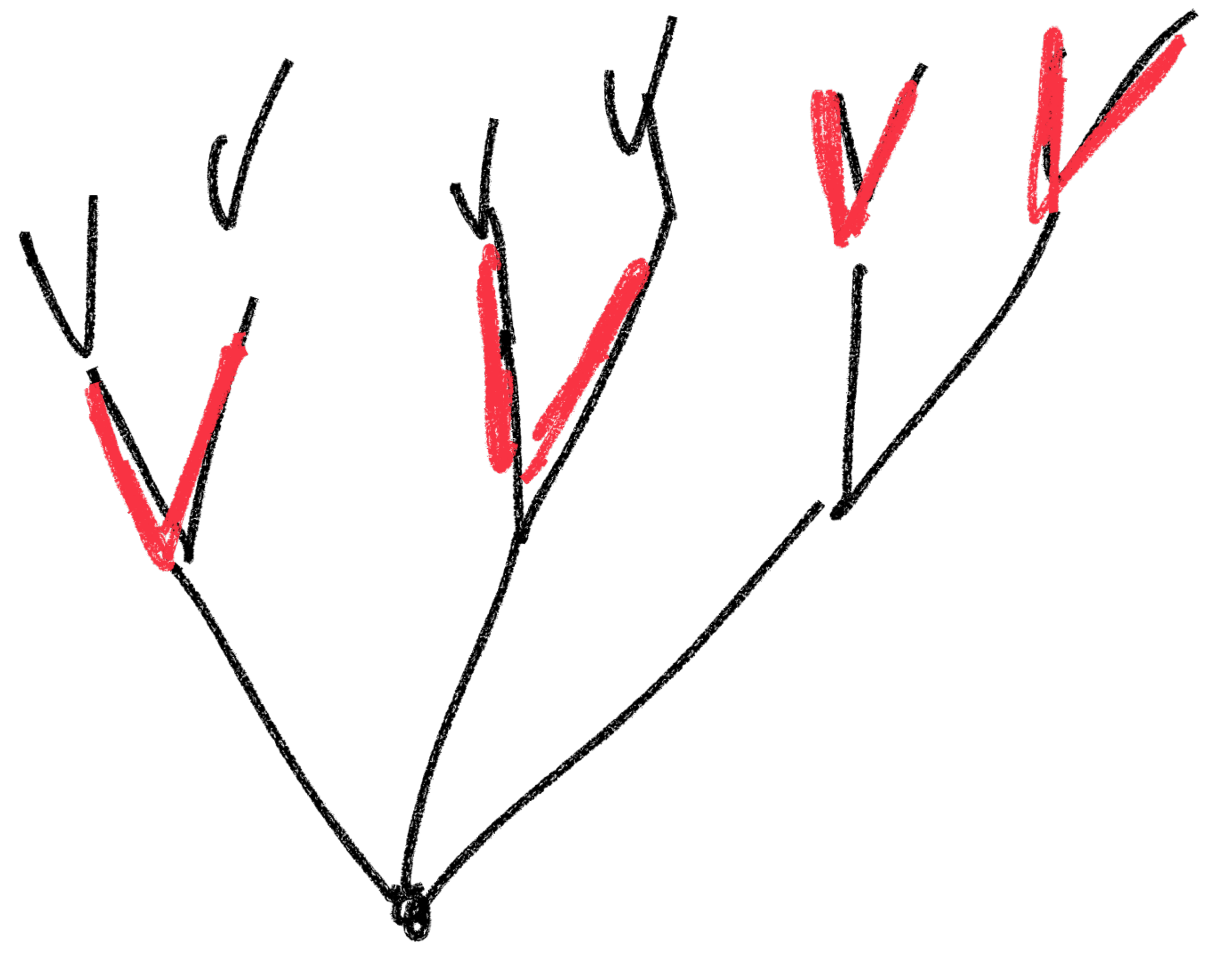


Polymers are **connected components of occupied edges**

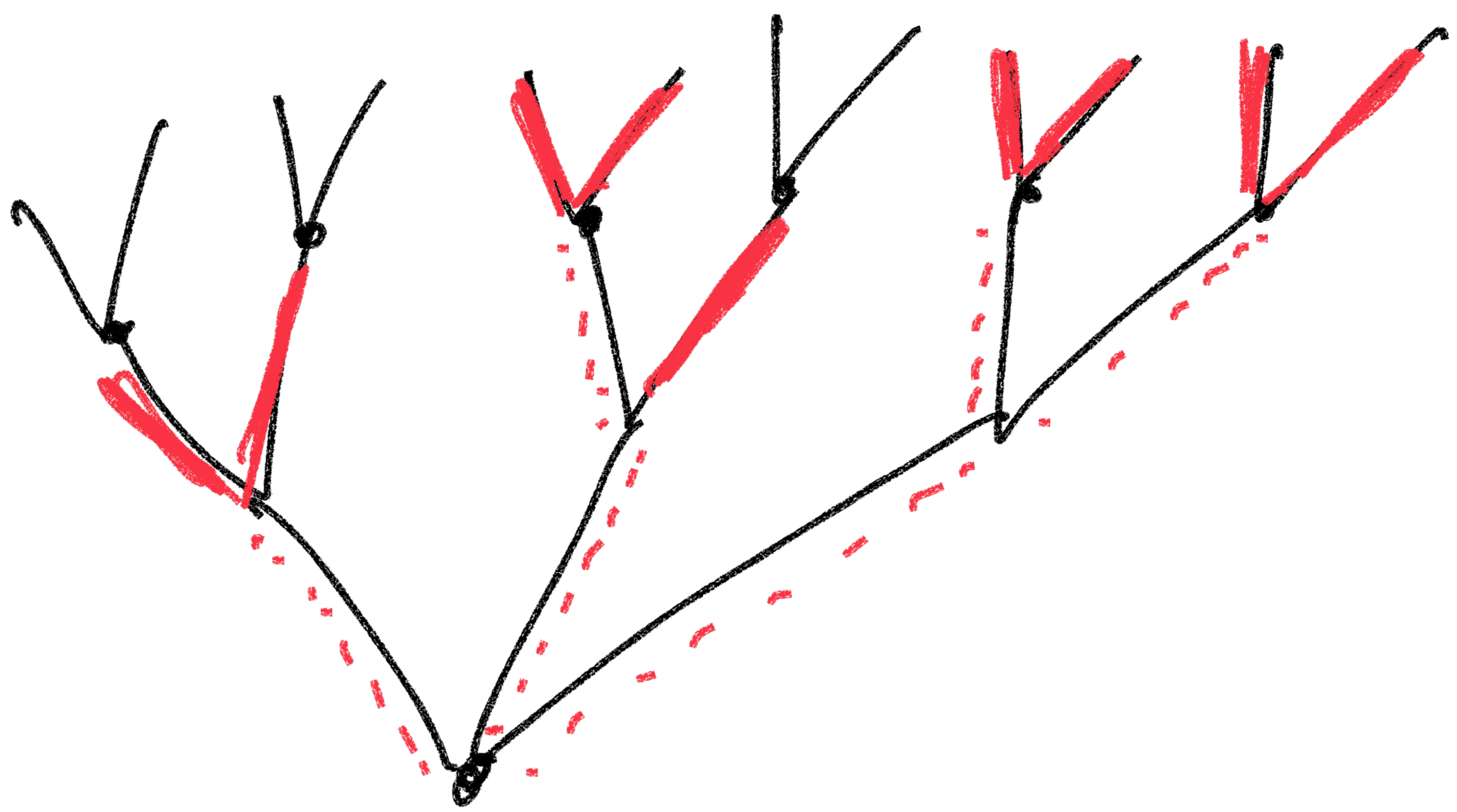
$$w(\gamma) = q^{1-|\gamma|}(e^{\beta} - 1)^{|E(\gamma)|}$$

Step 3: ordered

Express **ordered** configurations in terms of defects from the **all-occupied configuration**



Step 3: ordered



Define **boundary** by starting with unoccupied edges; inductively add all edges incident to any vertex with at least **5/9-fraction of its edges in boundary**.

Polymers are **connected components of the boundary**

$$w(\gamma) = q^{c'(\gamma)}(e^{\beta} - 1)^{-|E_u(\gamma)|}$$

Consequences

For q large, ordered and disordered cluster expansions converge in **overlapping range of β**.

This gives algorithms at **all temperatures**.

Convergent cluster expansion gives properties like exponential decay of correlations, large deviation bounds, CLT's…

Open questions

Prove that for the random cluster model on random d-regular graphs,

$$\beta_c = \log \frac{q-2}{(q-1)^{1-2/d} - 1}$$

Extend the current results to all $d \geq 3$ (more refined def of ordered polymers)

Apply the second-moment method / cavity method to the **random cluster model**

Give sampling/counting algorithms for **hard-core on random bipartite graphs** for all λ

Make other probabilistic techniques **algorithmic**

Thank you!