Testing Correlation of Unlabeled Random Graphs

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Joint work with
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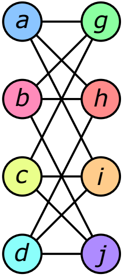
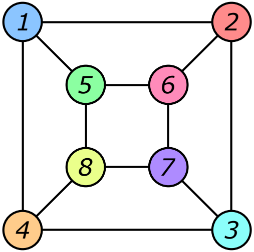
Graph isomorphism

Given two graphs A and B, decide whether $A \cong B$, i.e., there exists a bijection $\pi : V(A) \to V(B)$ such that

$$(u, v) \in E(A) \Leftrightarrow (\pi(u), \pi(v)) \in E(B)$$





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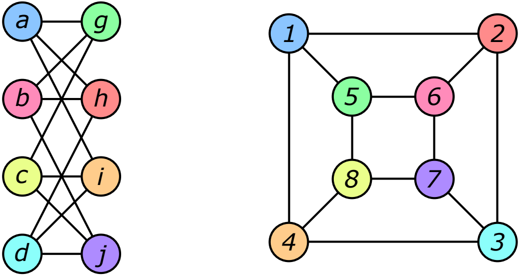

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$\pi(c) = 8$

$\pi(d) = 3$

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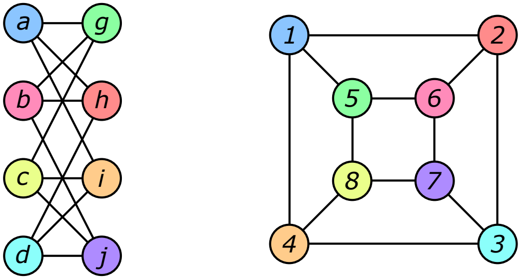

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- Not known to be solvable in polynomial time in worst case
- In practice, two graphs are often not exactly isomorphic, but still want to tell whether their topologies are similar

Beyond worst-case: Noisy graph isomorphism

Definition

Suppose we observe two random graphs A and B:

$$\mathcal{H}_0 : A \text{ and } B \text{ are independent}$$
$$\mathcal{H}_1 : A \text{ and } B^\pi = (B_{\pi(i)\pi(j)}) \text{ are edge-correlated}$$
$$\text{conditional on a uniform permutation } \pi$$

Goal: Test $\mathcal{H}_0$ versus $\mathcal{H}_1$

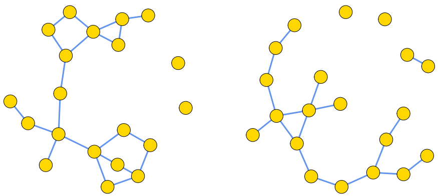
- Under $\mathcal{H}_1$, the inherent edge correlation is obscured by the latent node correspondence π
- The test needs to rely on graph invariants (invariant under graph isomorphism), such as
  - Subgraph counts (e.g. # of edges or triangles)
  - Spectrum (e.g. eigenvalues of adjacency matrices or Laplacians)

Erdős-Rényi setting

Definition (Erdős-Rényi graphs model [Barak-Chou-Lei-Schramm-Sheng'19])

$\mathcal{H}_0$: A and B are independent $\mathcal{G}(n, ps)$

$\mathcal{H}_1$: A and $B^\pi = (B_{\pi(i)\pi(j)})$ are independently edge-subsampled from a common parent graph $\mathcal{G}(n, p)$ with subsampling probability s



- Under both $\mathcal{H}_0$ and $\mathcal{H}_1$, A and B are $\mathcal{G}(n, ps)$ marginally
- Under $\mathcal{H}_1$, $\left(A_{ij}, B_{\pi(i)\pi(j)}\right)$ are correlated $\mathrm{Bern}(ps)$ with correlation coefficient $\rho \triangleq \frac{s(1-p)}{1-ps}$
- Hypothesis testing aspect of graph matching (recover π under $\mathcal{H}_1$)

Sharp threshold for detection: dense Erdős-Rényi graphs

Theorem (Wu-X.-Yu '20)

If $s^2 \geq \dfrac{2\log n}{(n-1)p\left(\log\frac{1}{p} - 1 + p\right)}$, *then* $\mathrm{TV}\left(\mathcal{P}, \mathcal{Q}\right) = 1 - o\left(1\right)$

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- *(Dense regime)* $p = n^{-o(1)}$*:*

$$\textit{If } s^2 \leq \frac{(2-\epsilon)\log n}{np\left(\log\frac{1}{p} - 1 + p\right)}, \textit{ then } \mathrm{TV}\left(\mathcal{P}, \mathcal{Q}\right) = o(1)$$

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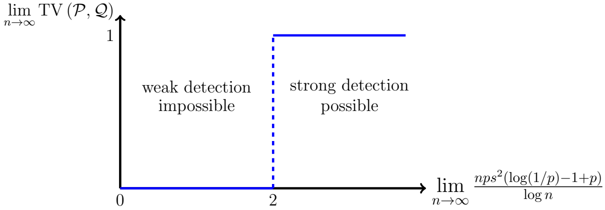

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$p = \frac{d}{n}$ for a constant d:
strong detection is possible if $s^2 > \frac{2}{d}$ and impossible if $s^2 < \frac{1}{d} \wedge 0.01$

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- Counting (weighted) trees – low-degree poly. approx. of $\frac{\mathcal{P}(A,B)}{\mathcal{Q}(A,B)}$ [Mao-Wu-X.-Yu 'forthcoming]: achieve strong detection in poly-time, if

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Polynomial-time test for $s = o(1)$ is open

Proof of positive results: QAP statistic

$$\mathcal{T}(A, B) \triangleq \max_{\pi \in \mathcal{S}_n} \sum_{i<j} A_{ij} B_{\pi(i)\pi(j)} \quad \text{(edge correlation)}$$

- This is known as Quadratic Assignment Problem
- Invariant to the node relabeling of both A and B
- Proof: First-moment calculation

Proof of negative results: second-moment method

$$\mathbb{E}_{\mathcal{Q}}\left[\left(\frac{\mathcal{P}(A,B)}{\mathcal{Q}(A,B)}\right)^2\right] = O(1) \implies \mathrm{TV}(\mathcal{P},\mathcal{Q}) \le 1-\Omega(1)$$

$$\mathbb{E}_{\mathcal{Q}}\left[\left(\frac{\mathcal{P}(A,B)}{\mathcal{Q}(A,B)}\right)^2\right] = 1+o(1) \implies \mathrm{TV}(\mathcal{P},\mathcal{Q}) = o(1)$$

Second-moment calculation: cycle (orbit) decomposition

- Node permutation σ on $[n]$
- Edge permutation σ^{E} on $\binom{[n]}{2}$: $\sigma^{\mathsf{E}}((i,j)) \triangleq (\sigma(i), \sigma(j))$

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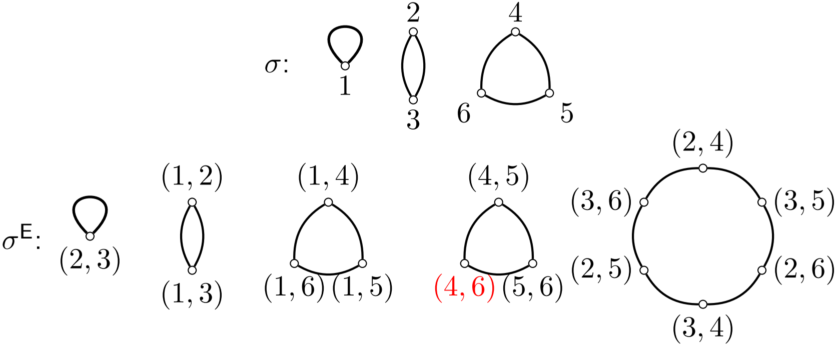
E.g. $n = 6$ and $\sigma = (1)(23)(456)$:



Cycles in node (edge) permutation are called node (edge) orbits

Second-moment calculation: cycle decomposition

$$\left(\frac{\mathcal{P}(A,B)}{\mathcal{Q}(A,B)}\right)^2 = \left(\mathbb{E}_\pi\left[\frac{\mathcal{P}(A,B|\pi)}{\mathcal{Q}(A,B)}\right]\right)^2$$

$$= \mathbb{E}_{\widetilde{\pi} \perp\!\!\!\perp \pi} \prod_{i<j} X_{ij} \quad X_{ij} \triangleq \frac{\mathcal{P}(B_{\pi(i)\pi(j)}|A_{ij})\mathcal{P}(B_{\widetilde{\pi}(i)\widetilde{\pi}(j)}|A_{ij})}{\mathcal{Q}(B_{\pi(i)\pi(j)})\mathcal{Q}(B_{\widetilde{\pi}(i)\widetilde{\pi}(j)})}$$

$$= \mathbb{E}_{\widetilde{\pi} \perp\!\!\!\perp \pi} \prod_{O \in \mathcal{O}} X_O \quad X_O \triangleq \prod_{(i,j)\in O} X_{ij}$$

$\mathcal{O}$: disjoint orbits of edge permutation σ^{E} with $\sigma \triangleq \pi^{-1} \circ \widetilde{\pi}$

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$$\mathbb{E}_{\mathcal{Q}}[X_O] = 1+\rho^{2|O|}, \quad \rho \triangleq \frac{s(1-p)}{1-ps}$$ (use Egorychev method)

Failure of second-moment

We show

$$\mathbb{E}_{\mathcal{Q}}\left[\left(\frac{\mathcal{P}(A,B)}{\mathcal{Q}(A,B)}\right)^2\right] = \begin{cases} 1+o(1) & \text{if } \rho^2 \le \frac{(2-\epsilon)\log n}{n} \\ +\infty & \text{if } \rho^2 \ge \frac{(2+\epsilon)\log n}{n} \end{cases}$$

- Suboptimal by an unbounded factor when $p = o(1)$

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Obstruction from short orbits

$$\mathbb{E}_{(A,B)\sim\mathcal{Q}}\left[\left(\frac{\mathcal{P}(A,B)}{\mathcal{Q}(A,B)}\right)^2\right] = \mathbb{E}_{\pi \perp\!\!\!\perp \widetilde{\pi}}\left[\prod_{O\in\mathcal{O}} \mathbb{E}_{\mathcal{Q}}\left[X_O\right]\right] \overset{\widetilde{\pi}=\pi}{\ge} \frac{1}{n!}\left(1+\rho^2\right)^{\binom{n}{2}}$$

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Atypically large magnitude of $\prod_{O\in\mathcal{O}:|O|=k} X_O$ for short orbits of length $k = O(\log n) \Rightarrow$ second-moment blows up

Conditional second-moment: dense regime

It suffices to consider $k = 1$:

$$\prod_{O \in \mathcal{O}:|O|=1} X_O \approx \left(\frac{1}{p}\right)^{2e_{A \wedge B_\pi}(F)}$$

F: set of fixed points of $\sigma \triangleq \pi^{-1} \circ \widetilde{\pi}$
$A \wedge B^\pi$: Intersection graph
$e_{A \wedge B^\pi}(F)$: # of edges of subgraph of $A \wedge B^\pi$ induced by F

- Under $\mathcal{P}$: $e_{A \wedge B^\pi}(S)$ concentrates uniformly over all S when $|S|$ is large
- Conditional on this typical event $\mathcal{E}$ under $\mathcal{P}$, when $|F|$ is large,

$$\mathbb{E}_{\mathcal{Q}}\left[\prod_{O \in \mathcal{O}:|O|=1} X_O \mathbf{1}_{\{\mathcal{E}\}}\right] \lesssim \mathbb{E}_{\mathcal{Q}}\left[\left(\frac{1}{p}\right)^{2e_{A \wedge B_\pi}(F)} \mathbf{1}_{\left\{e_{A \wedge B_\pi}(F) \le \binom{|F|}{2} ps^2\right\}}\right]$$

Conditional second-moment: sparse regime

Need to consider $k = \Theta(\log n)$. It can be shown

- Long orbits:

$$\mathbb{E}_{\mathcal{Q}}\left[\prod_{|O|>k} X_O\right] \le \left(1+\rho^k\right)^{\frac{n^2}{k}} = 1 + o(1)$$

- Short incomplete orbits:

$$\mathbb{E}_{\mathcal{Q}}\left[X_O \mid O \not\subset E\left(A \wedge B^\pi\right)\right] \le 1$$

- Short complete orbits:

$$X_O = \left(\frac{1}{p}\right)^{2|O|}, \quad \forall O \subset E\left(A \wedge B^\pi\right)$$

In the subcritical regime $nps^2 < 1$, $A \wedge B^\pi \sim \mathcal{G}(n, ps^2)$ is a pseudoforest
$\Rightarrow H_k \triangleq \cup_{O:|O|\le k, O \subset E(A\wedge B^\pi)} O$ is a pseudoforest

Conditional second-moment: orbit pseudoforest

Conditional on $\mathcal{E} \triangleq \{(A, B, \pi) : A \wedge B^\pi$ is a pseudoforest$\}$ under $\mathcal{P}$:

$$\mathbb{E}_{\mathcal{Q}}\left[\prod_{O \in \mathcal{O}} X_O \mathbf{1}_{\{\mathcal{E}\}}\right] \le (1+o(1))\, \mathbb{E}_{\mathcal{Q}}\left[\left(\frac{1}{p}\right)^{2e(H_k)} \mathbf{1}_{\{H_k \text{ is a pseudoforest}\}}\right]$$

$$= (1+o(1)) \sum_{H \in \mathcal{H}_k} s^{2e(H)} \quad \text{(generating function)}$$

$\mathcal{H}_k$: pseudoforests assembled from edge orbits of length at most k

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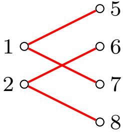
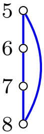
Key challenge: enumerate orbit pseudoforests $\mathcal{H}_k$ using orbit structure

Classification of edge orbits

Type	Edge orbit	Orbit graph
M	(13,24)	
B	(15,26,17,28)	
C	(56,67,78,85)	
S	(57,68)	

Assume $\sigma = (12)(34)(5678)$

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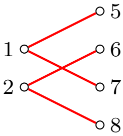
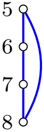
B	(15,26,17,28)	
C	(56,67,78,85)	
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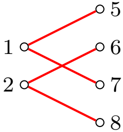
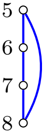
B	(15,26,17,28)	
C	(56,67,78,85)	
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Type $\mathrm{B}_{m,\ell}$ ($\ell < m$) edge orbit is cycle-free if ℓ divides m; otherwise, it contains a component with at least two cycles

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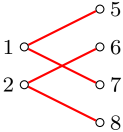
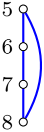
B	(15,26,17,28)	
C	(56,67,78,85)	
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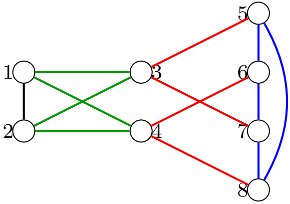
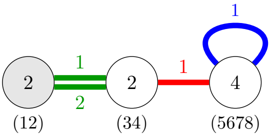
Key: identify the forbidden co-occurrence patterns under pseudoforest constraint

Backbone graph representation

- Node orbit in H (splits) $\Leftrightarrow$ giant node in Γ (shaded)
- Edge orbit in H $\Leftrightarrow$ giant edge in Γ with label



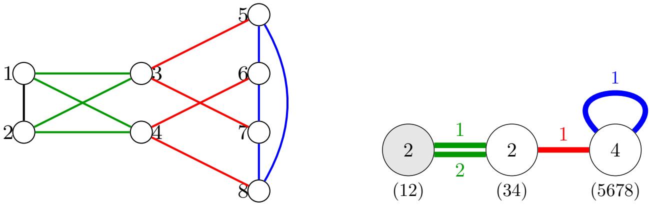
(a) Orbit graph H



(b) Backbone graph Γ

Backbone graph representation

- Node orbit in H (splits) $\Leftrightarrow$ giant node in Γ (shaded)
- Edge orbit in H $\Leftrightarrow$ giant edge in Γ with label



(a) Orbit graph H (b) Backbone graph Γ

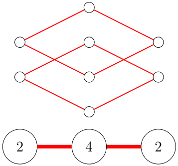
Enumeration Prototype: For $1 \le m \le k$,

1. Matching stage. Construct Γ_m induced by n_m giant nodes (node orbits) of size m
2. Splitting stage. Add splits to some components of Γ_m
3. Bridging stage. Add bridges between Γ_m and Γ_ℓ for ℓ dividing m

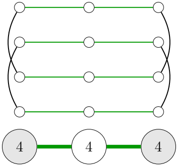
Warm-up: Enumerate orbit forest

When orbit graph H is a forest, its corresp. backbone graph Γ satisfies

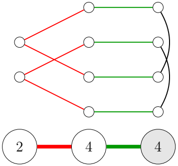
- Γ_m is a forest with no self-loop and parallel edges
- No bridge between Γ_m and Γ_ℓ unless ℓ divides m
- Each component of Γ_m contains at most 1 split or is incident to 1 bridge to Γ_ℓ, but not both.



(a) A component in Γ_4 is incident to 2 bridges



(b) A component in Γ_4 contains 2 splits



(c) A component in Γ_4 contains 1 split and is incident to 1 bridge

For $1 \le m \le k$,

1. Matching stage. Construct a rooted forest Γ_m consisting of n_m giant nodes and a_m giant edges
2. Splitting stage. Choose b_m components from $n_m - a_m$ tree components of Γ_m and add a split to the root
3. Bridging stage. Choose c_m out of the remaining $n_m - a_m - b_m$ tree components of Γ_m, add a bridge connecting its root to Γ_ℓ for some ℓ dividing m

$$\sum_{H \in \mathcal{F}_k} s^{2e(H)} \le \prod_{1 \le m \le k} \left(1 + \underbrace{s^{2m} m n_m}_{\text{matching}} + \underbrace{s^m \mathbf{1}_{\{m:\text{even}\}}}_{\text{splitting}} + \underbrace{s^{2m} \sum_{\ell < m} \ell n_\ell}_{\text{bridging}} \right)^{n_m}$$

Enumerate orbit pseudoforest

When orbit graph H is a forest, its corresp. backbone graph Γ satisfies

- Γ_m is a psedudoforest with self-loops and parallel edges counted as cycles
- No bridge between Γ_m and Γ_ℓ unless ℓ divides m
- Each unicyclic component of Γ_m is plain (contains no split and is not incident to bridge in Γ_ℓ)
- Each tree component of Γ_m contains at most 2 splits

Enumerate orbit pseudoforest

When orbit graph H is a forest, its corresp. backbone graph Γ satisfies

- Γ_m is a psedudoforest with self-loops and parallel edges counted as cycles
- No bridge between Γ_m and Γ_ℓ unless ℓ divides m
- Each unicyclic component of Γ_m is plain (contains no split and is not incident to bridge in Γ_ℓ)
- Each tree component of Γ_m contains at most 2 splits
- If there are two bridges incident to a common tree component in Γ_m, then the ending points of the two bridges must belong to distinct plain tree components in $\Gamma_{m/2}$
- If there is a bridge incident to a tree component that contains a split in Γ_m, then the ending point of the bridge must belong to a plain tree component in $\Gamma_{m/2}$

Enumerate orbit pseudoforest

For $1 \le m \le k$,

1. Matching stage. Construct a rooted pseudoforest Γ_m consisting of n_m giant nodes and a_m giant edges
2. Splitting stage. Choose b_m components from $n_m - a_m$ tree components of Γ_m and add either 1 or 2 splits
3. Forward bridging stage. Choose c_m out of the remaining $n_m - a_m - b_m$ tree components of Γ_m, add a bridge connecting its root to Γ_ℓ for some ℓ dividing m
4. Backward bridging stage. Choose d_m from the remaining $n_m - a_m - b_m - c_m$ tree components of Γ_m, add a bridge connecting its root to Γ_{2m}

$$\sum_{H \in \mathcal{H}_k} s^{2e(H)} \le \prod_{m=1}^{k} \Bigg(1 + \underbrace{s^{2m} 2mn_m}_{\text{matching}} + \underbrace{s^m n_m}_{\text{splitting}} + \underbrace{s^{2m} \sum_{\ell<m} \ell n_\ell}_{\text{FB}} + \underbrace{s^{4m} mn_{2m}}_{\text{BB}}\Bigg)^{n_m}$$

Averaging over orbit lengths

Theorem (Wu-X.-Yu '20)

Suppose $k(\log k)^4 = o(n)$. *If* $s \le 0.1$,

$$\mathbb{E}_{\pi \perp\!\!\!\perp \widetilde{\pi}} \left[\prod_{m=1}^{k} \left(1 + s^m n_m + 2s^{2m} \sum_{\ell \le m} \ell n_\ell + s^{4m} m n_{2m} \right)^{n_m} \right] = O(1),$$

where n_m *is the number of* m*-node orbits in* $\sigma = \pi^{-1} \circ \widetilde{\pi}$

- Poisson approximation: n_m's are approximately i.i.d. $\mathrm{Pois}(\frac{1}{m})$
- Partition the product into disjoint parts and recursively peel off the expectation backwards

Concluding remarks

- Formulate the problem of testing network correlation and characterize the statistical detection limit
- The impossibility proof applies conditional second-moment method
- The sparse setting leverages the pseudoforest structure of subcritical Erdős-Rényi graphs
- A large computational gap may exist between the statistical and computational limits

Open problem

- Sharp detection threshold in the sparse regime
- Prove the existence of or close the computational gaps

Reference

- Y. Wu, J. X, & S. H. Yu *Testing correlation of unlabeled random graphs*. arXiv:2008.10097.