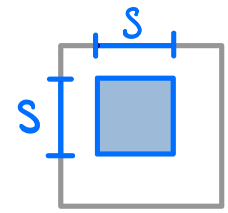
DETERMINANTAL POLYNOMIALS & THE PRINCIPAL MINOR MAP

joint work with Abeer Al Ahmadieh (UW)

Principal Minor Map: $\varphi : \mathbb{R}^{n\times n}_{sym} \to \mathbb{R}^{2^n}$

$A \mapsto (A_S)_{S\subseteq[n]}$



$A_S = \det(\square)$

Goal: Cut out image of φ with polynomial equations and inequalities.

Ex $(n=2)$ $\quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$ $\quad \varphi(A) = (A_\emptyset, A_1, A_2, A_{12})$

$= (1, a_{11}, a_{22}, a_{11}a_{22} - a_{12}^2)$

$\text{image}(\varphi) = \{(1, A_1, A_2, A_{12}) : A_1 A_2 \geq A_{12}\}$

$\to a_{11} = A_1 \quad a_{22} = A_2 \quad a_{12} = \pm\sqrt{A_1A_2 - A_{12}}$

<u>Properties of image of φ</u> (Holtz, Sturmfels 2006)

- closed, semialgebraic subset of $\mathbb{R}^{2^n}$
- dimension $= \dim(\mathbb{R}^{n\times n}_{sym}) = \binom{n+1}{2}$
- invariant under action of S_n
- invariant under action of $SL_2(\mathbb{R})^n$

Ex $(n=3)$ $\quad \dim = \binom{4}{2} = 6$ in $\{A_\emptyset = 1\} \subseteq \mathbb{R}^8$

Thm (HS '06) $(A_S) =$ princ. minors of $A \in \mathbb{R}^{3\times 3}_{sym} \implies$

$$\boxed{\begin{aligned} &A_\emptyset^2 A_{123}^2 + A_1^2 A_{23}^2 + A_2^2 A_{13}^2 + A_3^2 A_{12}^2 + 4\cdot A_\emptyset A_{12} A_{13} A_{23} + 4\cdot A_1 A_2 A_3 A_{123} \\ &\quad - 2\cdot A_\emptyset A_1 A_{23} A_{123} - 2\cdot A_\emptyset A_2 A_{13} A_{123} - 2\cdot A_\emptyset A_3 A_{12} A_{123} - 2\cdot A_1 A_2 A_{13} A_{23} \\ &\quad - 2\cdot A_1 A_3 A_{12} A_{23} - 2\cdot A_2 A_3 A_{12} A_{13}\end{aligned}} = 0.$$

HYPDET(A_S) "Cayley's Hyperdeterminant" of a $2\times2\times2$ tensor

For $n \geq 3$, HYPDET and all images under $SL_2(\mathbb{R})^n \rtimes S_n$ vanish

Thm (Oeding 2011)

$(A_S)_{S \subseteq [n]} = \varphi(A)$ for some $A \in \mathbb{C}^{n \times n}_{sym}$ $\Longleftrightarrow$ $\mathrm{HYPDET}(\gamma \cdot (A_S)) = 0$ for all $\gamma \in SL_2(\mathbb{R})^n \rtimes S_n$.

$\mathbb{C} \to \mathbb{R}$

$+ A_i A_j \geq A_{ij} A_\emptyset$

This talk: Understand $\uparrow$ through determinantal representations of multiaffine polynomials

$A \in \mathbb{R}^{n \times n}_{sym} \rightarrow$

$$f = \det\left(\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} + A\right) = \sum_{S \subseteq [n]} A_S \prod_{i \notin S} x_i$$

Ex ($n=2$)

$$f = \det\begin{pmatrix} x_1 + a_{11} & a_{12} \\ a_{12} & x_2 + a_{22} \end{pmatrix} = x_1 x_2 + a_{11} x_2 + a_{22} x_1 + a_{11} a_{22} - a_{12}^2$$

Revised (equivalent) goal: Characterize determinantal polynomials in $\mathbb{R}[x_1, \ldots, x_n]_{MA}$

Group actions on $\mathbb{R}[x_1, \ldots, x_n]_{MA}$:

$\pi \in S_n$ $\qquad \pi \cdot f = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ $\qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f = (cx_1 + d) f\left(\frac{ax_1 + b}{cx_1 + d}, x_2, \ldots, x_n\right)$

RAYLEIGH DIFFERENCES

$i, j \in [n]$
$f \in \mathbb{R}[x_1, \ldots, x_n]_{MA}$

$$\Delta_{ij}(f) = \frac{\partial f}{\partial x_i} \cdot \frac{\partial f}{\partial x_j} - f \cdot \frac{\partial^2 f}{\partial x_i \partial x_j}$$

$\in \mathbb{R}[x_k : k \neq i, j]_{MQ}$ $\swarrow$ deg ≤ 2 in each var

Ex: $f = x_1 x_2 + x_1 x_3 + x_2 x_3$

$$\Delta_{12}(f) = (x_2 + x_3)(x_1 + x_3) - f \cdot 1 = x_3^2$$

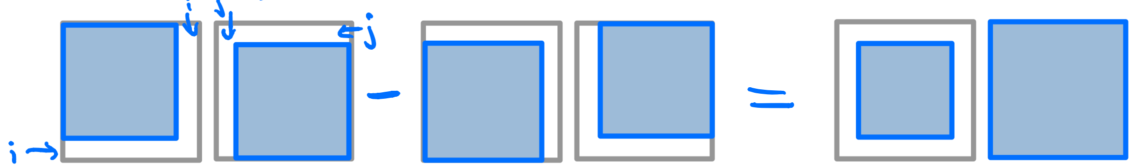
Thm (Brändén, 2007)

$f \in \mathbb{R}[x_1,\dots,x_n]_{MA}$ is stable $\iff$ $\Delta_{ij}(f) \geq 0$ on $\mathbb{R}^n$
for all $i,j \in [n]$

Thm (Kummer, Plaumann, V. 2013)

$f \in \mathbb{R}[x_1,\dots,x_n]_{MA}$ is determinantal $\iff$ $\Delta_{ij}(f)$ is a square in $\mathbb{R}[x_1,\dots,x_n]$
for all $i,j \in [n]$

IDEA for $(\Rightarrow)$: DODGSON CONDENSATION (1860's)



$$f = \det\Big(\underbrace{\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} + A}\Big)$$

$$\frac{\partial f}{\partial x_i} \cdot \frac{\partial f}{\partial x_j} - (\text{a square}) = \frac{\partial^2 f}{\partial x_i \partial x_j} \cdot f$$

Ex $(n=3)$ $\quad f = \sum_{S \subseteq [3]} A_S \prod_{i \notin S} x_i \qquad \Delta_{12}(f) \in \mathbb{R}[x_3]_{\leq 2}$
$= a x_3^2 + b x_3 + c$

$\Delta_{12}(f) = $ a square $\Rightarrow$ $\underbrace{b^2 - 4ac}_{= \mathrm{HYPDET}(A_S)} = 0 \qquad a \geq 0 \quad c \geq 0$
$\nwarrow A_1A_2 - A_{12}A_\phi$

Thm (Al Ahmadieh, V, 2020+)

$P \in \mathbb{R}[x_1,\dots,x_n]_{MQ}$ is a square

$\iff$ $\gamma \cdot P\big|_{x_2 = \dots = x_n = 0}$ is a square in $\mathbb{R}[x_1]$
for all $\gamma \in SL_2(\mathbb{R})^n \rtimes S_n$.

<u>Cor</u>: $(A_S)_{S \subseteq [n]} = \varphi(A)$ for some $A \in \mathbb{R}^{n \times n}_{sym}$ $\iff$ $\gamma \cdot \mathrm{HYPDET}(A_S) = 0$
and $\gamma \cdot (A_1A_2 - A_{12}A_\phi) \geq 0$
for all $\gamma \in SL_2(\mathbb{R})^n \rtimes S_n$.

Work in progress: characterize image of other classes of matrices under φ

Some "Geometry of Polynomials" thoughts

$$\Delta_{ij}(f) = \frac{\partial f}{\partial x_i} \cdot \frac{\partial f}{\partial x_j} - f \cdot \frac{\partial^2 f}{\partial x_i \partial x_j} = \underline{a} x_k^2 + b x_k + \underline{c}$$

f stable $\Rightarrow b^2 \leq 4ac$

$a = \Delta_{ij}\left(\frac{\partial f}{\partial x_k}\right)$

f determinantal $\Rightarrow b^2 = 4ac$

$c = \Delta_{ij}\left(f|_{x_k=0}\right)$

$$\left.\frac{\Delta_{ij}(f)}{f^2}\right|_{x=\mathbb{1}} = \mathbb{P}(i \in S)\mathbb{P}(j \in S) - \mathbb{P}(ij \in S)$$

$$= \left.\frac{a + b + c}{f^2}\right|_{x=\mathbb{1}}$$

Given $a, c|_{x=\mathbb{1}}$ maximized by determinantal poly.

Q: Are determinantal measures extremal among strongly Rayleigh measures in some meaningful way?

Thanks!