Spectral sets and derivatives of the psd cone

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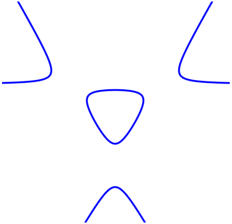

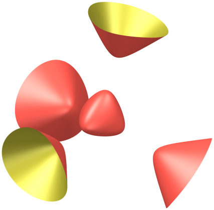

September 8, 2020

Spectrahedral cones

A spectrahedral cone is a set of the form

$$S = \{x \in \mathbb{R}^n : A(x) = x_1 A_1 + \ldots + x_n A_n \text{ is positive semidefinite}\},$$

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Question

- Which sets $K \subset \mathbb{R}^n$ are spectrahedral?

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Definition A homogeneous polynomial $h \in \mathbb{R}[x_1, \ldots, x_n]$ is hyperbolic with respect to $e \in \mathbb{R}^n$ if $h(e) \neq 0$ and if $h(te - v)$ has only real roots for all $v \in \mathbb{R}^n$.

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- $\det A(te - v) = \det(tI_d - A(v))$.
- $S = C(\det A(x), e)$.

The Generalized Lax Conjecture

Conjecture. Let $h \in \mathbb{R}[x_1, \dots, x_n]$ be hyperbolic with respect to $e \in \mathbb{R}^n$. Then $C(h, e)$ is spectrahedral.

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True if:

- $\deg h \leq 2$.
- $n \leq 3$. (Helton–Vinnikov)
- $n = 4$ and $\deg h = 3$. (Buckley–Košir)

Constructing hyperbolic polynomials

The following polynomials are hyperbolic with respect to e:

- $\det A(x)$ for $A(x)$ real symmetric matrix with linear entries and $A(e)$ positive definite.

Includes spanning tree polynomials of graphs and bases generating polynomials of regular matroids.

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Their hyperbolicity cones are clearly spectrahedral.

Renegar derivatives

Consequence of Rolle's Theorem:

- If a polynomial $p \in \mathbb{R}[t]$ has only real zeros, then its derivative p' has only real zeros.

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Let us write

$$\det(tI - X) = \sum_{k=0}^{d} (-1)^k p_k t^{d-k}$$

for suitable polynomials p_k of degree k ($p_1 = \mathrm{tr}(X), p_d = \det(X)$).

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- $p_k = \frac{1}{(d-k)!} \mathrm{D}_I^{d-k}(\det X)$.
- $p_k = \sigma_{k,d}(\lambda(X))$ where $\sigma_{k,d}$ is the elementary symmetric polynomial of degree k in d variables and $\lambda(X)$ the vector of eigenvalues of X.

Spectral hyperbolicity cones

Theorem (Bauschke–Güler–Lewis–Sendov) Let $h \in \mathbb{R}[x_1, \ldots, x_n]$ a symmetric polynomial that is hyperbolic with respect to $e = (1, \ldots, 1)$. Consider the function

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where $\lambda(X)$ is the vector of eigenvalues of X.

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b) H is hyperbolic with respect to I.

c) $C(H, I) = \{X : \lambda(X) \in C(h, e)\}$.

Renegar derivatives

Corollary

A symmetric $d \times d$ matrix A is in the hyperbolicity cone of $\mathrm{D}_I^{d-k}(\det X)$ if and only if its spectrum $\lambda(A)$ is in the hyperbolicity cone of the elementary symmetric polynomial $\sigma_{k,d}$.

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- Using this and a spectrahedral representation of the hyperbolicity cone of $\sigma_{d-1,d}$ due to Sanyal, Saunderson proved that the hyperbolicity cone of $\mathrm{D}_I^1(\det X)$ is spectrahedral.

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- Brändén constructed a spectrahedral representation of the hyperbolicity cone of $\sigma_{k,d}$ for all k.

Spectral sets

Question Let $S \subset \mathbb{R}^n$ be a spectrahedral cone which is symmetric under permuting the coordinates. Is the spectral convex set

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also spectrahedral?

- $\Lambda(S)$ is a hyperbolicity cone. (Bauschke–Güler–Lewis–Sendov)
- Yes, if S is a polyhedral cone. (Sanyal–Saunderson)

Some representation theory

Definition

A representation of $\mathfrak{S}_n$ is *short* if it consists only of such irreducible representations that correspond to partitions of length at most 2.

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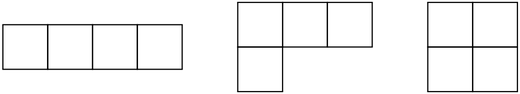
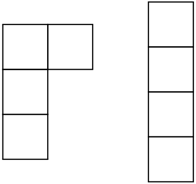
Short: 

Not short: 

Some representation theory

Example

Let $\mathrm{Ma}_{d,n} \subset \mathbb{R}[x_1, \dots, x_n]$ be the vector space of all homogeneous *multiaffine* polynomials of degree d. Then $\mathrm{Ma}_{d,n}$ is a short representation:

- $\mathrm{Ma}_{d,n} = \mathrm{Ind}_{\mathfrak{S}_d \times \mathfrak{S}_{n-d}}^{\mathfrak{S}_n}(\mathrm{Trv})$
- Young's rule: $\mathrm{Ma}_{d,n} = \oplus_{i=0}^{\min(d,n-d)} V_{n-i,i}$

The main result

Theorem

Let V be a short representation of $\mathfrak{S}_n$ and $\varphi : \mathbb{R}^n \to \mathrm{Sym}_2(V)$ an $\mathfrak{S}_n$-linear map. Let $S \subset \mathbb{R}^n$ be the preimage of the positive semidefinite cone in $\mathrm{Sym}_2(V)$ under φ. Then $\Lambda(S) \subset \mathrm{Sym}_2(\mathbb{R}^n)$ is a spectrahedral cone.

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Corollary

The hyperbolicity cone of $\mathrm{D}_I^k(\det A(x))$ spectrahedral.

- For any fixed k, the size of this spectrahedral representation is $\mathcal{O}(n^{2\cdot(\min(k,n-k)+1)})$ when the size n of $A(x)$ grows.

Idea of the proof

Theorem Let V be a short representation of $\mathfrak{S}_n$ and

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an $\mathfrak{S}_n$-linear map. Then there is a representation W of $\mathrm{O}(n)$ and an $\mathrm{O}(n)$-linear map map

$$\Phi : \mathrm{Sym}_2(\mathbb{R}^n) \to \mathrm{Sym}_2(W)$$

such that $\Phi(A)$ is positive semidefinite if and only $\varphi(\lambda(A))$ is positive semidefinite.

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Let $0 \leq 2d \leq n$. We have $\mathrm{Ma}_{d,n} = \oplus_{i=0}^{d} V_{n-i,i}$. More precisely:

- $V_{n-i,i} = \ker(\mathrm{D}_e^{d-i+1}) \cap \ker(\mathrm{D}_e^{d-i})^{\perp}$

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- The decomposition of the $\mathrm{O}(n)$-module $\mathrm{Min}_{d,n}$ into irreducibles is $\mathrm{Min}_{d,n} = \oplus_{i=0}^{d} E^{(i,i)'}$.

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- Here $E^{(i,i)'} = \ker(\mathrm{D}_I^{d-i+1}) \cap \ker(\mathrm{D}_I^{d-i})^{\perp}$.

To obtain W replace each $V_{n-i,i}$ in V by $E^{(i,i)'}$.

Idea of the proof

Let V be a short representation and $V = V^1 \oplus \cdots \oplus V^r$ its decomposition into irreducibles. What are the $\mathfrak{S}_n$-linear maps $\varphi : \mathbb{R}^n \to \mathrm{Sym}_2(V)$?

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Have to understand $\mathfrak{S}_n$-linear maps:

- $\mathbb{R}^n \to \mathrm{Sym}_2(V^i)$
- $\mathbb{R}^n \to \mathrm{Hom}(V^i, V^j)$

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Lemma Let $0 \leq 2d < 2d' \leq n$. There is a nonzero $\mathfrak{S}_n$-linear map $\mathbb{R}^n \to \mathrm{Hom}(V_{n-d',d'}, V_{n-d,d})$ if and only if $d' = d + 1$. In that case it is unique up to a scalar factor and given by $a \mapsto \mathrm{D}_a$.

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- The corresponding $\mathrm{O}(n)$-linear map $\mathrm{Sym}_2(\mathbb{R}^n) \to \mathrm{Hom}(E^{(d',d')'}, E^{(d,d)'})$ is then just $A \mapsto \mathrm{D}_A$.

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- Similar procedure for $\mathbb{R}^n \to \mathrm{Sym}_2(V_{n-d,d})$.

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Thanks!