SIMONS INSTITUTE for the Theory of Computing

Theory of Reinforcement Learning
Aug. 19 – Dec. 18, 2020

Part 3: Quasi Stochastic Approximation

and implications to gradient-free optimization

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COGNITION & CONTROL IN COMPLEX SYSTEMS

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Part 3: Quasi Stochastic Approximation

Outline

Special Thanks

Much of today's lecture: inspiration at NREL
collaboration with Prashant Mehta and other collaborators


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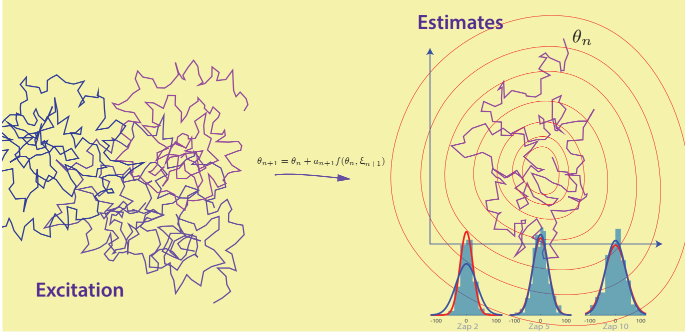
Yue Chen


Marcello Colombino



Stochastic Approximation?

What is Stochastic Approximation? $\bar{f}(\theta) = \mathsf{E}[f(\theta, W)]$

A simple goal: find solution to $\bar{f}(\theta^*) = 0$

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Under very general conditions:

the ODE, the Euler approximation, and SA are all convergent to θ^*

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Euler approximation is robust to measurement error

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[Robbins and Monro, 1951] *see Borkar's monograph [59]*

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You may have to modify the dynamics.

Newton-Raphson Flow—an approach to ensure stability:

$$\frac{d}{dt}\bar{f}(\vartheta_t) = -\bar{f}(\vartheta_t)$$

$\Longrightarrow$ Zap 

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$\alpha_{n+1} = g/(n+1)$ gives optimal convergence rate

$$\mathsf{E}[\|\theta_n - \theta^*\|^2] \approx \frac{1}{n}\mathrm{trace}\,(\Sigma_\theta)$$

Only if $\frac{1}{2}I + gA^*$ is Hurwitz, with $A^* = \partial \bar{f}\,(\theta^*)$

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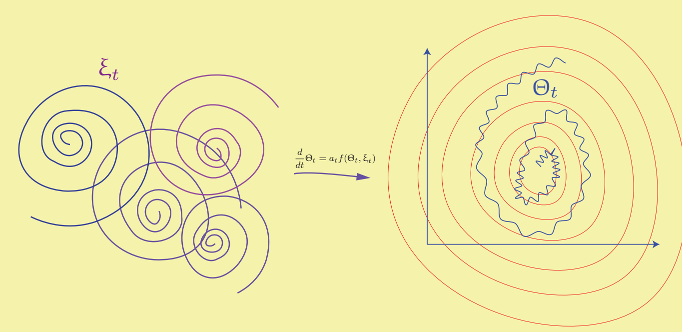
$$0 = [I + gA^*]\Sigma_\theta + \Sigma_\theta[I + gA^*]^{\mathsf{T}} + g^2\Sigma_{\text{“NOISE”}}$$

$\Longrightarrow$ CLT, etc



Quasi Stochastic Approximation

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Applications of interest:

TD, Q, gradient-free optimization, policy-gradient RL, ...

We create the noise!

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QSA to the rescue: $\mathsf{E}[\|\theta_n - \theta^*\|^2] \approx n^{-2} \operatorname{trace}(\bar{\Sigma}_\theta)$

$$\frac{d}{dt}\bar{\Theta}_t = a_t \bar{f}(\bar{\Theta}_t) \qquad \Longleftarrow \textit{Design for your goals}$$

$$\frac{d}{dt}\Theta_t = a_t f(\Theta_t, \xi_t) \qquad \Longleftarrow \textit{QSA (cts time is simplest)}$$

$$\theta_{n+1} = \theta_n + a_{n+1} f(\theta_n, \xi_{n+1}) \qquad \Longleftarrow \textit{Euler/Runge-Kutta}$$

Deterministic Markovian Noise $\frac{d}{dt}\Theta_t = a_t f(\Theta_t, \xi_t)$

Canonical choice is a mixture of sinusoids. Generalization:

$$\xi_t = [\exp(j\omega_1 t), \dots, \exp(j\omega_K t)]^{\intercal}$$

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Key property: partial integrals are bounded in time:

$$\xi^I_t = \int_0^t \xi_r \, dr \,, \qquad \xi^{II}_t = \int_0^t \xi^I_r \, dr \,, \quad \ldots$$

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Final generalization Deterministic Markovian probing process:

$$\frac{d}{dt}\xi_t = \mathsf{H}(\xi_t) \qquad H\colon \Omega \to \Omega \text{ continuous.}$$

ξ_t evolves on Ω (compact)

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Ergodicity: invariant measure π unique, and for continuous $g\colon \Omega \to \mathbb{R}$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T g(\xi_t)\, dt = \overline{g} \stackrel{\text{def}}{=} \int_\Omega g(z)\, \pi(dz)$$

works for complex exponentials

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Poisson's equation: center of CLT theory, and central here:

$$\hat{g}(\Xi_{t_0}) = \int_{t_0}^{t_1} [g(\Xi_t) - \overline{g}]\, dt + \hat{g}(\Xi_{t_1})$$

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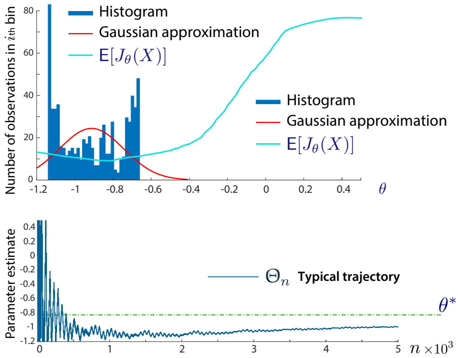
$\Longrightarrow$ optimal rate in ergodic theorem, and more



Gradient-Free Optimization

Kiefer-Wolfowitz to Extremum Seeking Control $\min_{\theta\in\mathbb{R}^d} L(\theta)$

- Kiefer-Wolfowitz: $\theta_{n+1} = \theta_n + \alpha_{n+1} f(\theta_n, W_{n+1})$ [63, 43]

Simplest formulation:

$$f(\theta_n, W_{n+1}) = -\frac{1}{2\varepsilon} G\, W_{n+1}\{L(\theta_n + \varepsilon W_{n+1}) - L(\theta_n - \varepsilon W_{n+1})\}$$

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Taylor series: (even terms cancel)

$$L(\theta + \varepsilon w) - L(\theta - \varepsilon w) = 2\varepsilon w^{\intercal} \nabla L(\theta) + \frac{\varepsilon^3}{3} \langle \partial_\theta^3 L\,(\theta), w, w, w \rangle + o(\varepsilon^3)$$

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Mean dynamics: $\frac{d}{dt}\vartheta_t = \bar{f}(\vartheta_t)$, with

$$\bar{f}(\theta) = -G\Sigma_W \nabla L(\theta) + O(\varepsilon^2)$$

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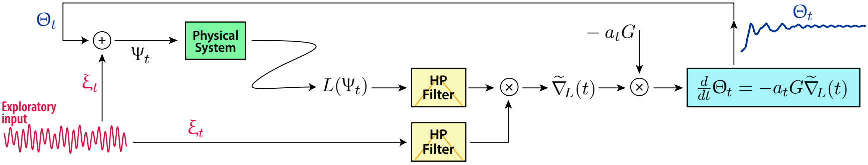
Simplest example:



Improved with LP Filter to smooth estimates, much like averaging to come

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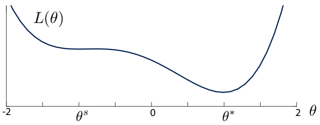
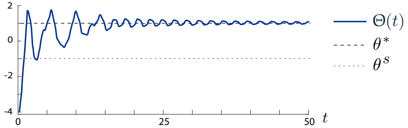
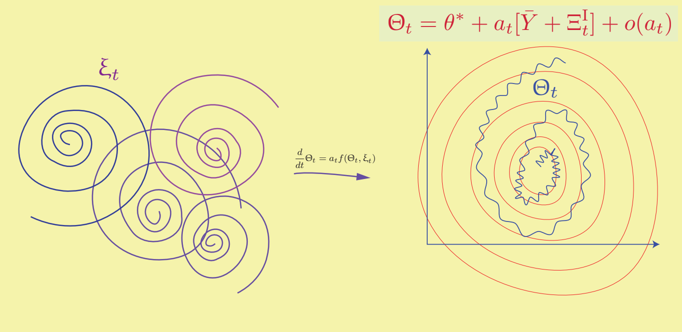




First seen in applications to finance: [77, 78]

What's new? Complete theory for convergence and convergence rate

Results today from [76]


$\Theta_t = \theta^* + a_t[\bar{Y} + \Xi^{\mathrm{I}}_t] + o(a_t)$
ξ_t
$\frac{d}{dt}\Theta_t = a_t f(\Theta_t, \xi_t)$
Θ_t

QSA Theory

Scaling and Linearization $\frac{d}{dt}\Theta_t = a_t f(\Theta_t, \Xi_t) \approx a_t\{A^*\widetilde{\Theta}_t + \Xi_t\}$

Comparison: $\frac{d}{dt}\bar{\Theta}_t = a_t \bar{f}(\bar{\Theta}_t)$, with $\bar{\Theta}_{t_0} = \Theta_{t_0}$

Step 1: Stability of ODE (by design): $\lim_{t\to\infty} \Theta_t = \lim_{t\to\infty} \bar{\Theta}_t = \theta^*$

Interesting fact: for $a_t = g/(1+t)$,

Rate of convergence of $\bar{\Theta}_t$ is $1/t$ if and only if $I + gA^*$ is Hurwitz

Emergency Exit

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$$\frac{d}{dt}Z_t = \big[r_t I + a_t A^*\big] Z_t + \widetilde{\Xi}_t\,, \qquad Z_{t_0} = 0$$

with $r_t = -\frac{d}{dt}\log(a_t) + o(a_t)$ and $\widetilde{\Xi}_t = f(\Theta_t, \xi_t) - \bar{f}(\Theta_t)$

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Stick to special case: $a_t = g/(1+t)^\rho$, giving $r_t = \rho/(1+t) + o(a_t)$:

$$\frac{d}{dt} Z_t = \begin{cases} a_t\big[A^* + o(1)\big] Z_t + \widetilde{\Xi}_t & \rho < 1 \\ a_t\big[g^{-1} I + A^* + o(1)\big] Z_t + \widetilde{\Xi}_t & \rho = 1 \end{cases}$$

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Emergency Exit

Step 4: QSA 101: $Y_t = \bar{Y} + o(1)$ *meaning ...*

Refinements and Warnings

Amazing conclusion: using $a_t = 1/(1+t)^\rho$,

$$\Theta_t = \theta^* + a_t[\bar{Y} + \Xi^{\mathrm{I}}_t + o(1)]$$

For $\rho < 1$ requires A^* Hurwitz

$$\Xi^{\mathrm{I}}_t = \Xi^{\mathrm{I}}_0 + \int_0^t f(\theta^*, \xi_r)\, dr = \widehat{f}(\theta^*, \xi_0) - \widehat{f}(\theta^*, \xi_t) \quad \textit{zero mean, bounded}$$

$$\bar{Y} = [A^*]^{-1} \int_\Omega \partial_\theta \widehat{f}(\theta^*, z) f(\theta^*, z)\, \pi(dz) \qquad !$$

Refinements and Warnings

Amazing conclusion: using $a_t = 1/(1+t)^\rho$,

$$\Theta_t = \theta^* + a_t[\bar{Y} + \Xi^{\mathrm{I}}_t + o(1)]$$

For $\rho < 1$ requires A^* Hurwitz, and $\rho = 1$ requires $I + A^*$ Hurwitz

Bias in qSGD is $O(\varepsilon^2)$ *I don't think anyone is worried about that*

$$\Xi^{\mathrm{I}}_t = \Xi^{\mathrm{I}}_0 + \int_0^t f(\theta^*, \xi_r)\, dr = \widehat{f}(\theta^*, \xi_0) - \widehat{f}(\theta^*, \xi_t) \quad \textit{zero mean, bounded}$$

$$\bar{Y} = [A^*]^{-1} \int_\Omega \partial_\theta \widehat{f}(\theta^*, z) f(\theta^*, z)\, \pi(dz) \qquad !$$

Refinements and Warnings

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Nope! This gives $1/T$ convergence rate if and only if $\bar{Y} = 0$

(a mysterious condition)

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Refinements and Warnings

Global convergence requires Lipschitz continuity of f

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$$\frac{d}{dt}\Theta_t = -a_t \frac{1}{\varepsilon} G \Xi_t L(\Theta_t + \varepsilon \Xi_t)$$

subject to zero-mean + symmetry assumption

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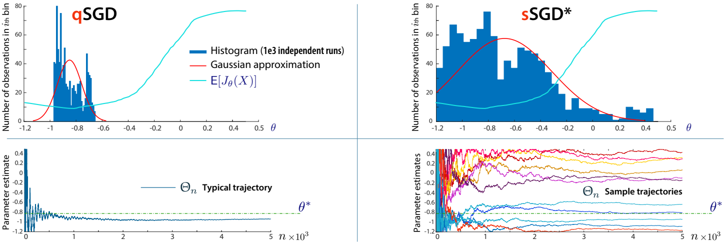

*Awful performance because f is *not Lipschitz*

Conclusions

Don't introduce volatility if you don't have to!

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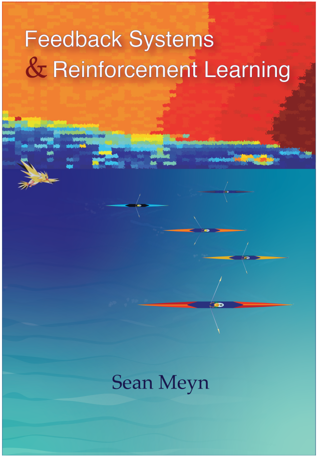
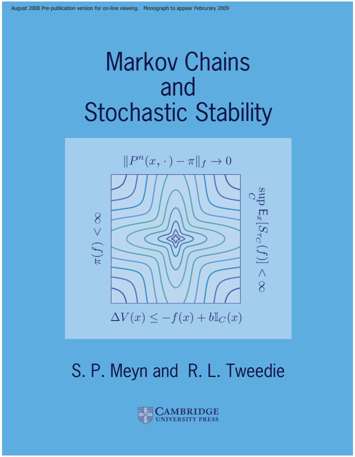
Thank you, Simons Institute and organizers of 2018 program on RTDM!
During this time at Berkeley, Panayotis Mertikopoulos@CNRS engaged me and Adithya Devraj to work on

Reinforcement learning in continuous games

with the main goal: rates of convergence for Kiefer and Wolfowitz!

The "quasi-theory" is so simple. I leave refinements of stochastic theory to others.


Feedback Systems
& Reinforcement Learning
Sean Meyn


August 2008 Pre-publication version for on-line viewing. Monograph to appear Februrary 2009
Markov Chains
and
Stochastic Stability
$\|P^n(x,\cdot\,) - \pi\|_f \to 0$
$\pi(f) < \infty$
$\sup_C E_x[S_{\tau_C}(f)] < \infty$
$\Delta V(x) \le -f(x) + b\mathbb{I}_C(x)$
S. P. Meyn and R. L. Tweedie
CAMBRIDGE
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