Theory of Reinforcement Learning
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Part 2: Every Optimization Problem Is a Quadratic Program

and implications for Q Learning

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Part 2: From DP to QP to Q

Outline

Optimal Control and RL

From DP to Q-learning

$X_{k+1} = \mathrm{F}(X_k, U_k)$

Value function: $$J^\star(x) = \min_{\boldsymbol{u}} \sum_{k=0}^{\infty} c(X_k, U_k)\,, \quad X_0 = x \in \mathsf{X}$$

DP eqn: $$J^\star(X_k) = \min_{U_k} \{ \underbrace{c(X_k, U_k) + J^\star(X_{k+1})}_{Q^\star(X_k, U_k)} \}$$

A conditional expectation would appear for a Markovian model

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Model Free Error Representation for Bellman Error

$$\mathcal{E}^\theta(X_k, U_k) = -Q^\theta(X_k, U_k) + c(X_k, U_k) + \underline{Q}^\theta(X_{k+1})$$

Find θ^* among family $\{Q^\theta(x,u) : \theta \in \mathbb{R}^d\}$

Temporal Difference Methods

$X_{k+1} = \mathrm{F}(X_k, U_k)$

Dynamic Programming

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Policy Iteration: Given initial policy ϕ^0: $U_k = \phi^0(X_k)$

1. *Solve* the fixed-policy Bellman equation:

$$J^{\phi^0}(x) = \underbrace{c(x,u) + J^{\phi^0}(\mathrm{F}(x,u))}_{Q^{\phi^0}(x,u)}\Big|_{u=\phi^0(x)}$$

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2. Update policy: $\phi^1(x) = \arg\min_u Q^{\phi^0}(x,u)$ *repeat ...*

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Fixed policy Bellman equation observed:

$$Q^{\phi^n}(X_k, U_k) = c(X_k, U_k) + Q^{\phi^n}\big(X_{k+1}, \phi^n(X_{k+1})\big)$$

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Sutton et al recognized the value of the *temporal difference* in the early 80s

TD(λ): estimate value function for fixed policy $U_k = \phi(X_k)$

Modified DP equation: $Q^\phi(X_k, U_k) = c(X_k, U_k) + Q^\phi(X_{k+1}, \phi(X_{k+1}))$

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TD(λ) (or SARSA, if you like), attempts to find roots of

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$$\text{TD}(\lambda)\text{:}\quad \zeta_k^\theta = \sum_{i=0}^{k} \lambda^i \nabla_\theta Q^\theta(X_{k-i}, U_{k-i})$$

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Solution approaches: 1. ODE design: $\frac{d}{dt}\theta_t = G_t \bar{f}(\theta_t)$, and *translation:*

$$\theta_{n+1} = \theta_n + \alpha_{n+1} G_n \zeta_n \mathcal{E}^{\theta_n}(X_n, U_n)$$
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2. LSTD: Consider a linear parameterization $Q^\theta = \theta^\intercal \psi$, giving

$$0 = \frac{1}{T} \sum_{k=0}^{T-1} \zeta_k \mathcal{E}^\theta(X_k, U_k) = A_T \theta - b_T$$

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Amazing fact: $\theta_T^* = A_T^{-1} b_T$ obtained for special gain: $G_n = -A_n^{-1}$

Temporal Difference Methods

Does it work? Let's stick to $Q^\theta = \theta^{\intercal}\psi$

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Require exploration, such as $U_k = \widetilde{\phi}(X_k, \xi_k)$ $\Longleftarrow$ QSA theory to come

Persistence of excitation: $$\frac{1}{T}\sum_{k=0}^{T-1} \psi(X_k, U_k)\psi(X_k, U_k)^{\intercal} \to \Sigma_\psi > 0$$

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Some good news:

- $G_n = A_n^{-1}$ exists! (may fail for at most d values of λ)

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Well, not so fast!

This beautiful result was obtained for MDPs, in the on-policy setting:

$$U_k = \phi(X_k)$$

π is the steady-state distribution of $\{(X_k, U_k) : k \ge 0\}$

... *do you smell trouble?*

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However, is minimizing $\|Q^\theta - Q^*\|_\pi$ a compelling goal?

Q(0) Learning and Deep Q-Learning

A generalization of Watkins' algorithm [13, 26, 10]

Model Free Error Representation:

$$\mathcal{E}^\theta(X_k, U_k) = -Q^\theta(X_k, U_k) + c(X_k, U_k) + \underline{Q}^\theta(X_{k+1})$$

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Design principle:

Step 1: consider an ODE: $\frac{d}{dt}\theta_t = -G_t \bar{f}(\theta_t)$ (matrix gain part of design)

Step 2: translate to a discrete time algorithm based on measurements.

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Goal as in the fixed-policy setting: Find roots of $\bar{f}(\theta^*) = 0$ *Why?*

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$$\bar{f}(\theta) = \lim \frac{1}{T} \sum_{k=0}^{T-1} \zeta_k^\theta \mathcal{E}^\theta(X_k, U_k) \qquad \bar{f}(\theta^*) = 0$$ *Why?*

Troubles with Q: Slow! Does a root exist? Does it have significance?

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Batch algorithms to the rescue? [28, 29, 30, 31]

DQN $\theta_{n+1} = \arg\min_\theta \Big\{ \mathcal{E}_n(\theta) + \frac{1}{\alpha_{n+1}} \|\theta - \theta_n\|^2 \Big\}$

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With a linear parameterization, this is a quadratic program!

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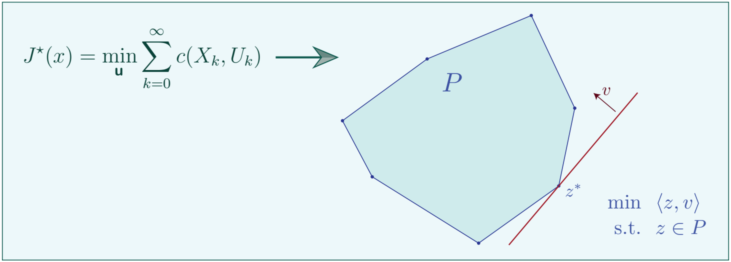
Sadly,

ODE approximation for DQN $\equiv$ Q(0) Learning

Even for neural network function approximation [M&M, 2020]



DP $\Rightarrow$ LP

Inverse Dynamic Programming

What is a good approximation? $\mathcal{E}(x) \stackrel{\text{def}}{=} -J(x) + \min_u [c(x,u) + J(\mathrm{F}(x,u))]$

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For any J, you have solved a DP equation:

$$J(x) = \min_u [c_J(x,u) + J(\mathrm{F}(x,u))]$$

$$c_J(x,u) \stackrel{\text{def}}{=} c(x,u) - \mathcal{E}(x)$$

optimal policy ϕ^J

Let J^{ϕ^J} denote *the value function under the policy* ϕ^J

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Proposition 3.7

Assume $\mathcal{E}(x) \geq -\varrho c(x,u)$, all x, u and minor assumptions on J

Then, $J^\star(x) \leq J^{\phi^J}(x) \leq (1+\varrho) J^\star(x)$

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We have our gold standard

and our first LP constraint

Every DP is an LP

Every control student knows this, starting with [Manne, 1960] [44, 45, 46]

Proposition: [Subject to mild assumptions]
$J^\star$ solves the following LP:

$$\begin{aligned} \max_J \quad & \langle \mu, J \rangle \\ \text{s.t.} \quad & J(x) \le c(x,u) + J(\mathrm{F}(x,u))\,, \qquad x \in \mathsf{X}\,,\ u \in \mathsf{U}(x) \\ & J \text{ is continuous, and } J(x^e) = 0. \end{aligned}$$

μ a probability measure on X (given)

- Applications to ADP in the thesis of de Farias (with BVR) [47, 48], and Mengdi Wang's survey on Monday, August 31
- One way to derive the SDP representation of LQR [Boyd et al]
- Applications in deterministic control every day

Every DP is an LP

Every control student knows this, starting with [Manne, 1960] [44, 45, 46]

Proposition 3.9 [Subject to mild assumptions]
The pair $(J^\star, Q^\star)$ solve the following LP:

$$\begin{aligned} \max_{J,Q} \quad & \langle \mu, J \rangle \\ \text{s.t.} \quad & Q(x,u) \leq c(x,u) + J(\mathrm{F}(x,u)) \\ & Q(x,u) \geq J(x)\,, \qquad\qquad x \in \mathsf{X}\,,\ u \in \mathsf{U}(x) \\ & J \text{ is continuous, and } J(x^e) = 0. \end{aligned}$$

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μ a probability measure on X (given)

Over-parameterization for RL more recent.

Motivation: $Q(X_k, U_k) \le c(X_k, U_k) + J(X_{k+1})$ (observed)

Every DP is an LP

Explanation

Show that $J(x) \leq J^\star(x)$ for any feasible J, and all x

For any input sequence,
$J(X_k) \leq c(X_k, U_k) + J(X_{k+1})$

$$\implies J(X_0) \leq \sum_{k=0}^{T-1} c(X_k, U_k) + J(X_T)$$

$$\begin{aligned} \max_{J} \quad & \langle \mu, J \rangle \\ \text{s.t.} \quad & J(x) \leq c(x,u) + J(\mathrm{F}(x,u)) \\ & J \text{ is continuous, and } J(x^e) = 0. \end{aligned}$$

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$J(X_T) \to 0$ for policies of interest, so

$$J(x) \leq \sum_{k=0}^{\infty} c(X_k, U_k)\,, \qquad X_0 = x$$

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Take the infimum over all $\boldsymbol{U} \implies$ QED

Every DP is a QP

Proposition: [Subject to mild assumptions]
The pair $(J^\star, Q^\star)$ solve the following QP:

$$\min_{J,Q} \quad -\langle \mu, J\rangle + \kappa\langle \nu, \mathcal{E}^2\rangle$$

$$\text{s.t.} \quad 0 \leq \mathcal{E}(x,u) \overset{\text{def}}{=} -Q(x,u) + c(x,u) + J(\mathrm{F}(x,u))$$

$$Q(x,u) \geq J(x)\,, \qquad x \in \mathsf{X}\,,\ u \in \mathsf{U}(x)$$

J is continuous, and $J(x^e) = 0$.

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J is continuous, and $J(x^e) = 0$.

ν a probability measure on $\mathsf{X} \times \mathsf{U}$

The objective and constraints can be observed, without a model
$\Longrightarrow$ Long list of possible RL approximations

Convex Q-Learning

Every DP is a QP $\Longrightarrow$ Convex Q Learning

$$
\begin{aligned}
\min_{\theta} \quad & -\langle \mu, J^\theta \rangle + \kappa \langle \nu, \mathcal{E}^2(\theta) \rangle \\
\text{s.t.} \quad & 0 \le -Q^\theta(X_k, U_k) + c(X_k, U_k) + J^\theta(X_{k+1}) \\
& Q^\theta(x, u) \ge J^\theta(x) \qquad \Longleftarrow \textit{Enforce through function architecture}
\end{aligned}
$$

Every DP is a QP $\Longrightarrow$ Convex Q Learning

$$\begin{aligned} \min_{\theta} \quad & -\langle \mu, J^\theta \rangle + \kappa \langle \nu, \mathcal{E}^2(\theta) \rangle \\ \text{s.t.} \quad & 0 \le -Q^\theta(X_k, U_k) + c(X_k, U_k) + J^\theta(X_{k+1}) \qquad \Longrightarrow \qquad z_n(\theta) \ge 0 \end{aligned}$$

$$z_n(\theta) = \frac{1}{r_n} \sum_{k=T_n}^{T_{n+1}-1} \big[-Q^\theta(X_k, U_k) + c(X_k, U_k) + J^\theta(X_{k+1}) \big] \zeta_k^+$$

ζ_k^+ : vector with non-negative entries

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Convex Q Version 1.0

$$
\theta_{n+1} = \operatorname*{arg\,min}_{\theta \in \Theta} \Big\{ -\langle \mu, J^\theta \rangle + \kappa \bar{\mathcal{E}}_n^2(\theta) - \lambda_n^{\intercal} z_n(\theta) + \frac{1}{\alpha_{n+1}} \tfrac{1}{2} \|\theta - \theta_n\|^2 \Big\}
$$

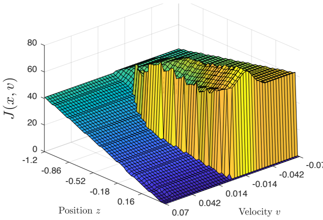
$$
\lambda_{n+1} = \big[\lambda_n - \alpha_{n+1} z_n(\theta_n) \big]_+
$$

Convex Q Learning—Does it work?

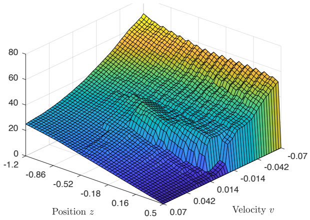
$$\theta_{n+1} = \mathop{\arg\min}_{\theta\in\Theta}\Big\{-\langle \mu, J^\theta\rangle + \kappa\bar{\mathcal{E}}_n^2(\theta) - \lambda_n^T z_n(\theta) + \frac{1}{\alpha_{n+1}}\tfrac{1}{2}\|\theta-\theta_n\|^2\Big\}$$

$$\lambda_{n+1} = \big[\lambda_n - \alpha_{n+1} z_n(\theta_n)\big]_+$$

It is only 4 weeks old! Who knows what Version 1.1 will look like.



Value function obtained from VIA



Value function approximation from convex Q

MountainCar in early August

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$$\lambda_{n+1} = \big[\lambda_n - \alpha_{n+1} z_n(\theta_n)\big]_+$$

Lessons learned from initial testing:

1. Advantage function: $A^\theta = Q^\theta - J^\theta$, with Θ chosen so
   $A^\theta(x,u) \ge 0$ all $x, u, \theta \in \Theta$
   Seems necessary for success

Convex Q Learning—Does it work?

$$\begin{aligned}\theta_{n+1} &= \operatorname*{arg\,min}_{\theta\in\Theta}\Big\{-\langle\mu, J^\theta\rangle + \kappa\bar{\mathcal{E}}_n^2(\theta) - \lambda_n^{\intercal} z_n(\theta) + \frac{1}{\alpha_{n+1}}\tfrac{1}{2}\|\theta-\theta_n\|^2\Big\}\\ \lambda_{n+1} &= \big[\lambda_n - \alpha_{n+1} z_n(\theta_n)\big]_+\end{aligned}$$

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Revisit Convex Q in continuous time [M&M 09]

Conclusions

The LP and QP characterization of DP equations gives rise to RL algorithms that are provably convergent, and for which we know what problem we are actually solving!

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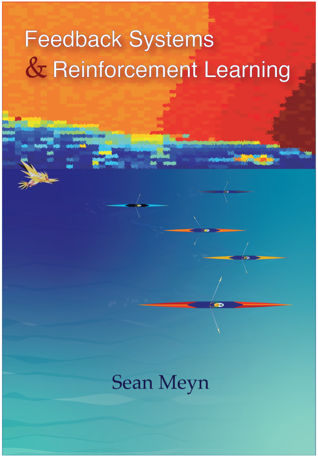
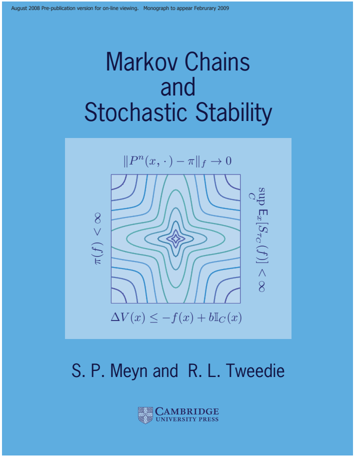
  - QSA
  - qSGD
  - qPG


Feedback Systems
& Reinforcement Learning
Sean Meyn


August 2008 Pre-publication version for on-line viewing. Monograph to appear Februrary 2009
Markov Chains
and
Stochastic Stability
$\|P^n(x,\cdot)-\pi\|_f \to 0$
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$\sup_C \mathsf{E}_x[S_{\tau_C}(f)] < \infty$
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S. P. Meyn and R. L. Tweedie
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