Obstacles to State Preparation and Variational Optimization from Symmetry Protection

Robert König

joint work with

Sergey Bravyi, Alexander Kliesch and Eugene Tang

arXiv:1910.08980

Institute for Advanced Study
Technical University of Munich

robert.koenig@tum.de

Technische Universität München

Variational methods

How well do these methods perform?

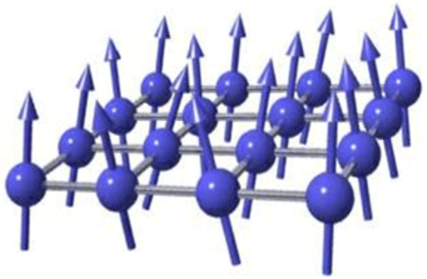
(1) What is the best energy attained by a state from S
(2) What is the best energy of a state that can be computed efficiently.

$$\min_{\psi} \langle \psi | H | \psi \rangle \quad \leq \quad \min_{\psi \in S} \langle \psi | H | \psi \rangle$$

optimization over Hilbert space of exponential dimension

A set of states that can be represented efficiently

many-body Hamiltonian H



optimize over **product states**

mean-field theory

Variational methods

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(2) What is the best energy of a state that can be computed efficiently.

$$\min_{\psi} \langle\psi|H|\psi\rangle \quad \leq \quad \min_{\psi\in S} \langle\psi|H|\psi\rangle$$

A set of states that can be represented efficiently

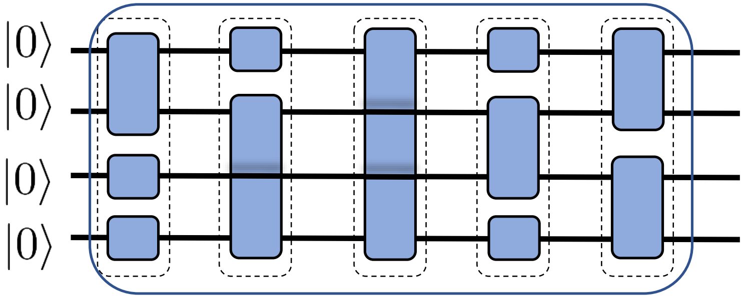
This talk:

"Classical" (diagonal) Hamiltonians

$$H = \sum_{x\epsilon\{0,1\}^n} C(x)|x\rangle\langle x|$$

Variational families of states defined by quantum circuits

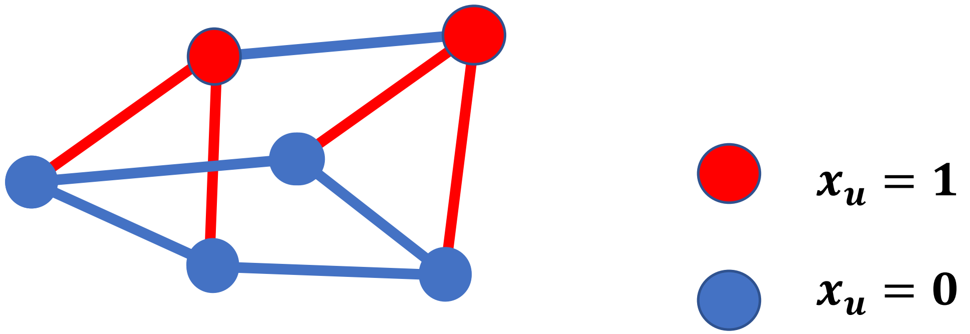
circuit U



Combinatorial optimization

Quantum approximate optimization (QAOA)

Combinatorial optimization



Given: A function $C: \{0,1\}^n \to \mathbb{R}$.

Goal: Find $x^* \in \{0,1\}^n$ such that $C(x^*)$ approximates the maximum

$$\max_{x \in \{0,1\}^n} C(x)$$

Example: MaxCUT for $G = (V, E)$

$$C_G(x) = \frac{1}{2} \sum_{(u,v) \epsilon E} (1 - (-1)^{x_u} (-1)^{x_v})$$

Computing maximum exactly Is NP-hard.

Figure of merit for an algorithm $\mathcal{A}$:

(expected) ***approximation ratio***

$$\alpha(\mathcal{A}) = \frac{\mathbb{E}_{x^* \leftarrow \mathcal{A}}[C(x^*)]}{\max_{x \in \{0,1\}^n} C(x)}$$

A polynomial-time algorithm achieving $\alpha(\mathcal{A}) \geq 0.878$ for every graph G !

Goemans and Williamson (1995)

Assuming the unique games conjecture and $P \neq NP$ there is no polynomial-time algorithm $\mathcal{A}$ satisfying $\alpha(\mathcal{A}) > 0.878$ for every graph G.

S. Khot and N. Vishnoi, FOCS (2005)

The quantum approximate optimization algorithm

E. Farhi, J. Goldstone, S. Gutmann, arXiv:1411.4028.

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$$H = \sum_{x\in\{0,1\}^n} C(x)|x\rangle\langle x|$$

$$|x\rangle = |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle$$

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Level-p QAOA algorithm

1. Prepare state ψ^* such that $\langle\psi^*|H|\psi^*\rangle$ approximates

$$\max_{\psi} \langle\psi|H|\psi\rangle$$

2. Measure in basis $\{|x\rangle\}_{\mathrm{x}}$ to obtain x

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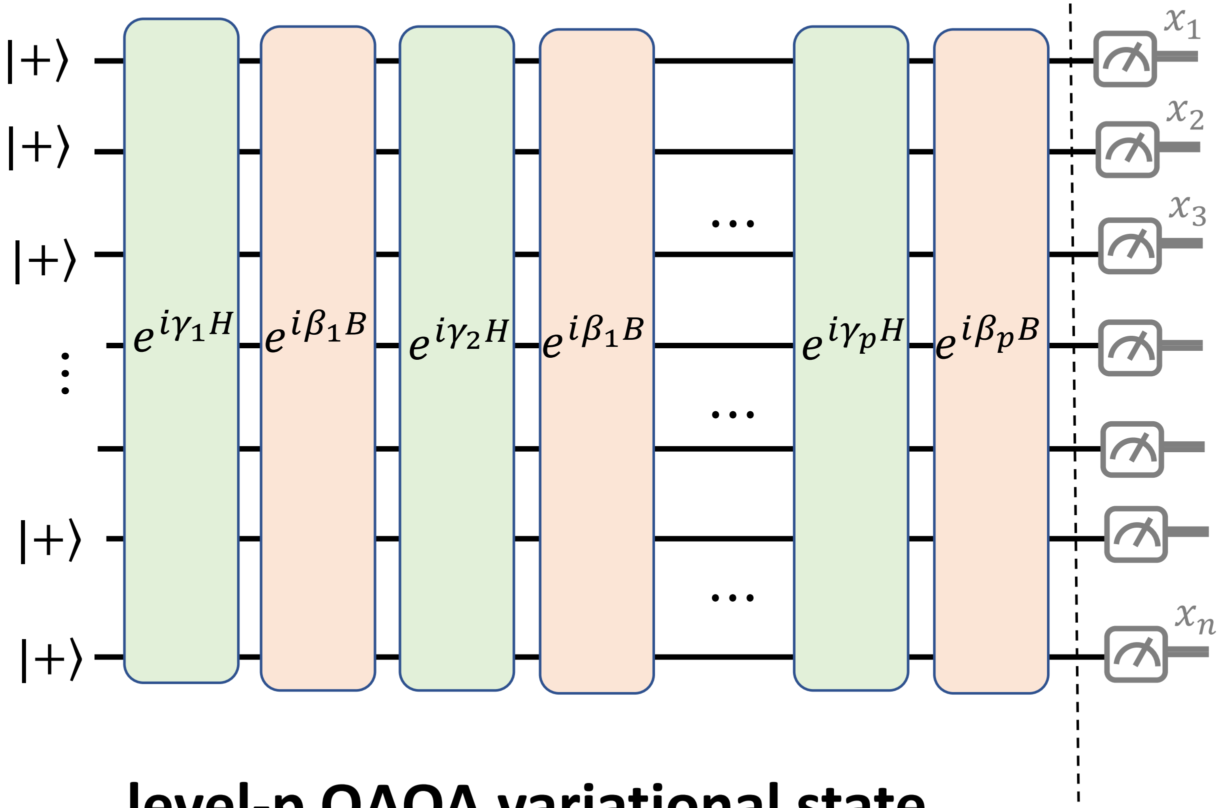

level-p QAOA variational state

$$\psi(\beta,\gamma)=\prod_{k=1}^{p} e^{i\beta_k B}\, e^{i\gamma_k H}\, |+\rangle^{\otimes n}$$

$$H = \sum_{x\epsilon\{0,1\}^n} C(x)|x\rangle\langle x| \qquad B = \sum_{j=1}^{n} X_j$$

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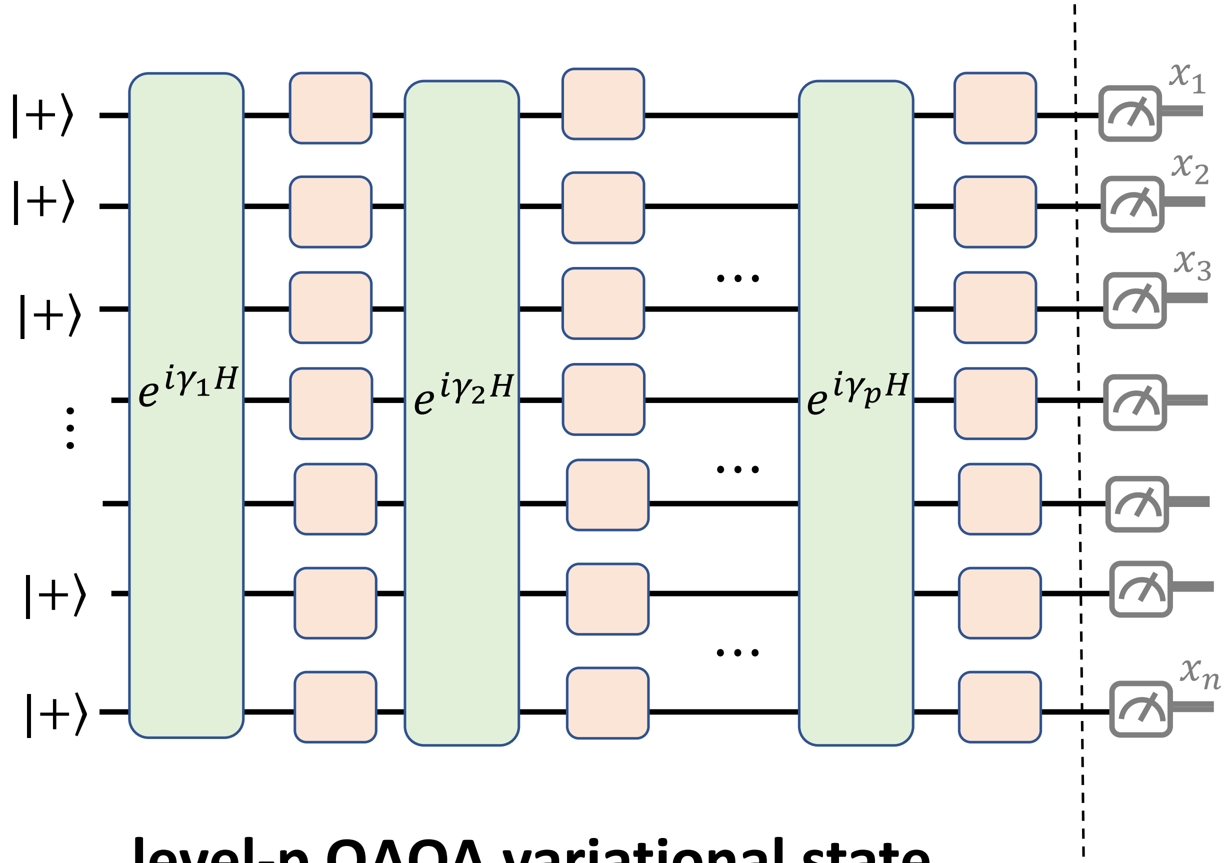

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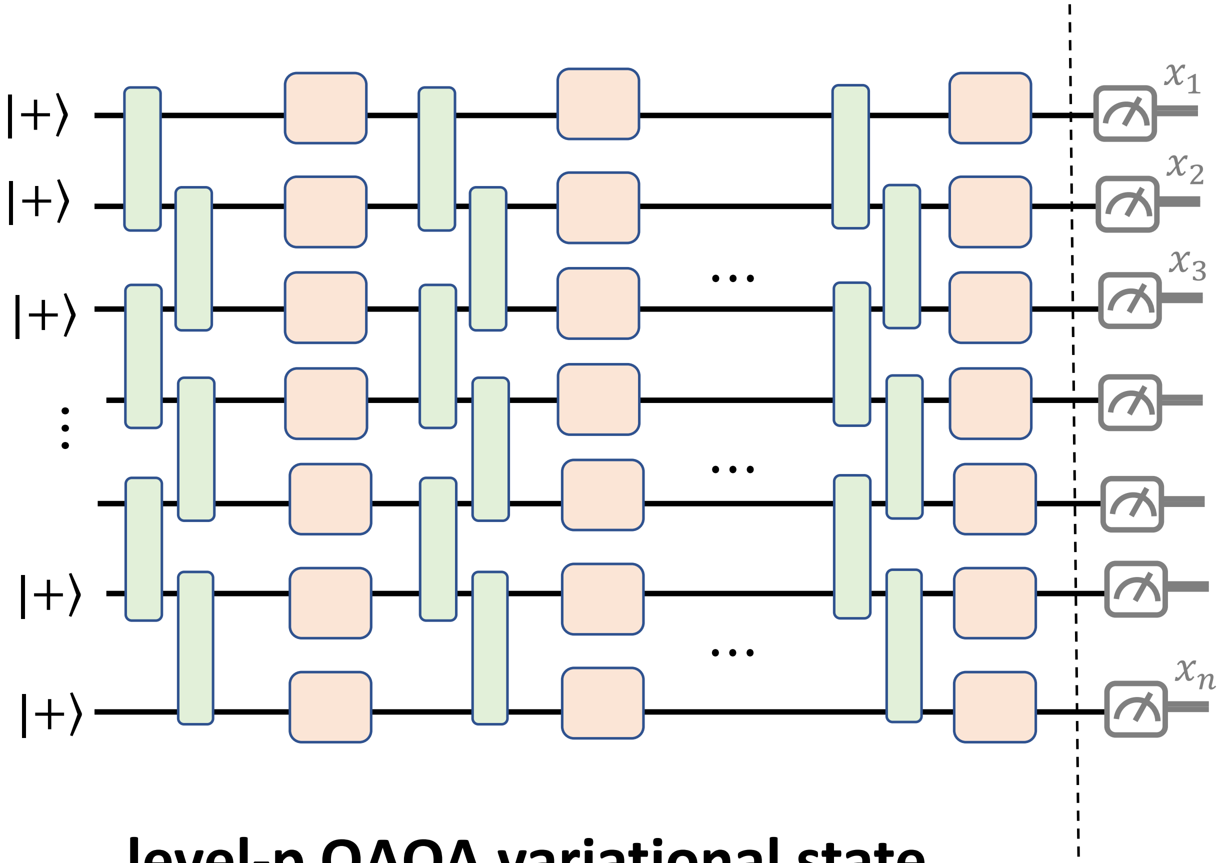

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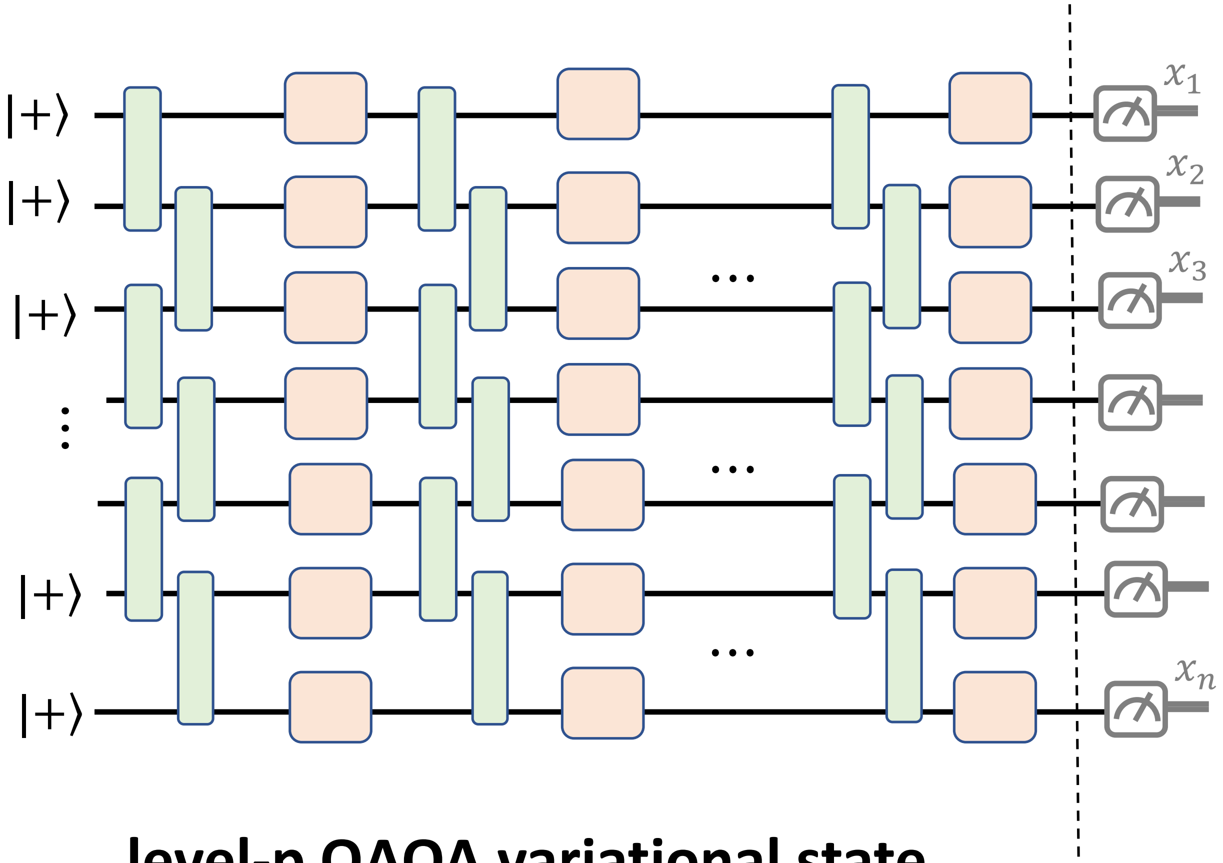

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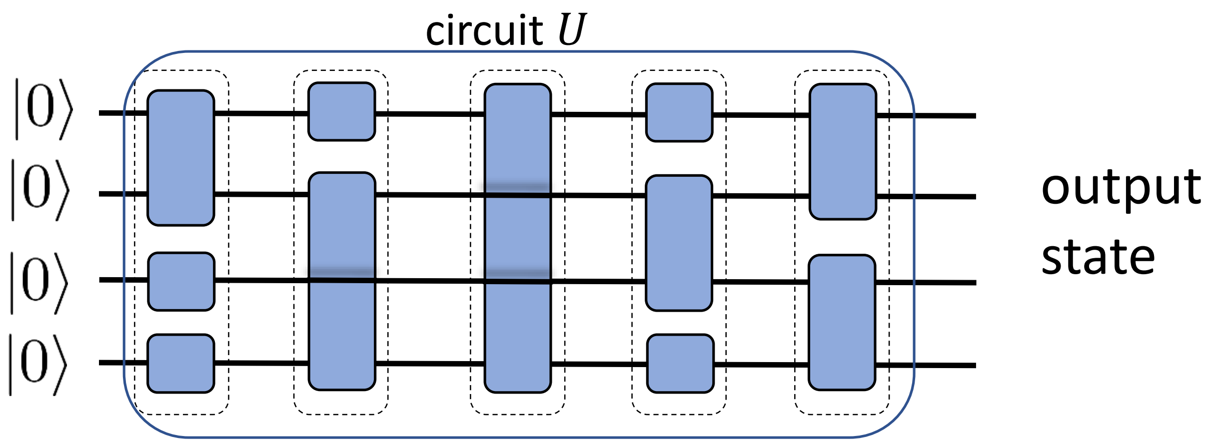
QAOA algorithm: Limitations on level p

- descriptive power of variational class of states increases with level p
- energy maximization becomes more challenging with increasing p
- NISQ implementation requires constant (small) p

Main question: Can **constant-level QAOA** outperform the best known classical algorithm (i.e., Goemans-Williamson) for MAXCUT?

Main theme: Lower bounds on *circuit-depth/circuit-range* necessary to prepare low-energy states using symmetric unitary preparation circuits



Symmetric Hamiltonians/unitaries and states

A Hamiltonian H is $\mathbb{Z}_2$-symmetric if $[H, X^{\otimes n}] = 0.$

Examples: $H_{TF} = -\sum_{k\in\mathbb{Z}_n} X_k$ $\quad H_{Ising} = -\sum_{k\in\mathbb{Z}_n} Z_k Z_{k+1}$

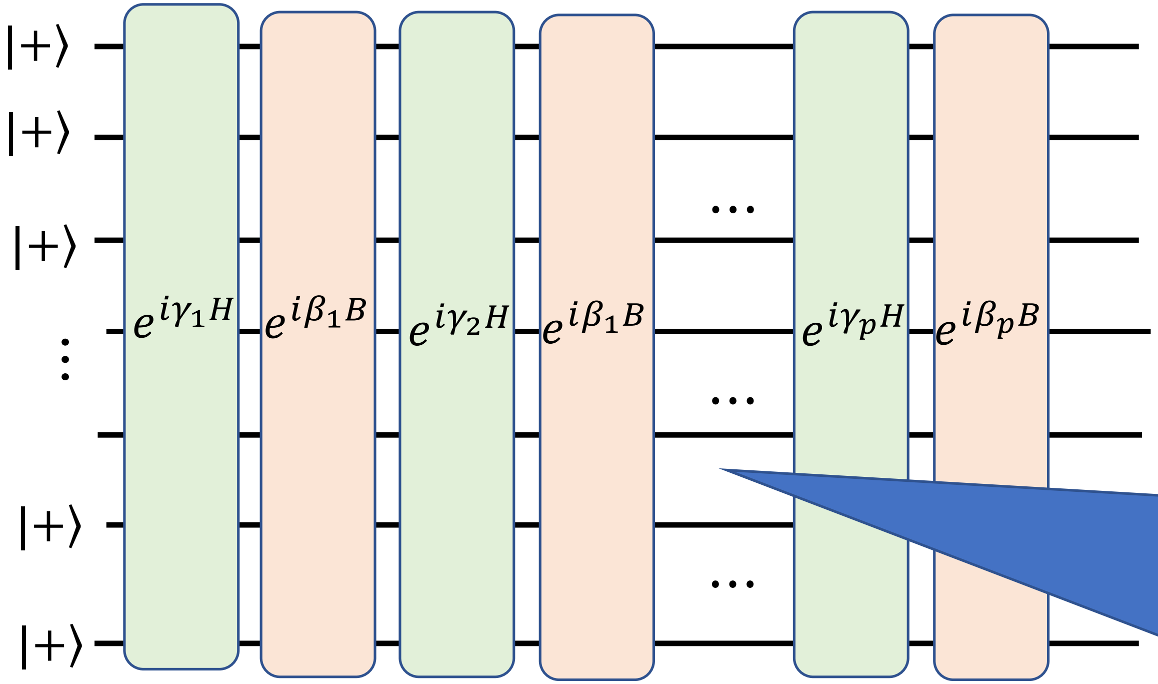
A state ψ is $\mathbb{Z}_2$-symmetric if $X^{\otimes n}\psi = \psi$ or $X^{\otimes n}\psi = -\psi.$

Examples: $|+\rangle^{\otimes n} = |+\rangle \otimes |+\rangle \otimes \cdots \otimes |+\rangle$ $\quad |GHZ_n\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n})$

A unitary U is $\mathbb{Z}_2$-symmetric if $UX^{\otimes n}U^\dagger = X^{\otimes n}.$

Examples: $U = X^{\otimes n}$ $\quad$ any circuit U composed of $\mathbb{Z}_2$-symmetric gates.

QAOA: a $\mathbb{Z}_2$-symmetric circuit



$H = \sum_{x \epsilon \{0,1\}^n} C(x)|x\rangle\langle x|$

$B = \sum_{j=1}^{n} X_j$

$|x\rangle = |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle$

This initial state is $\mathbb{Z}_2$-symmetric!

This circuit is $\mathbb{Z}_2$-symmetric if

$C(x) = C(\bar{x})$ where $\bar{x}_j = 1 - x_j$

e.g., for MAXCUT!

Limitations of $\mathbb{Z}_2$-symmetric circuits: a case study

$$H_n = \sum_{k \in \mathbb{Z}_n} (I - Z_k Z_{k+1})$$

Conventions throughout this talk:

- $\{H_n\}_n$ family of local Hamiltonians with n =number of qubits
- Hamiltonians are sums of local terms of strength $O(1)$
- Ground state energy zero for every Hamiltonian:

$$min_\psi \langle\psi| H_n |\psi\rangle = 0$$

Limitations of $\mathbb{Z}_2$-symmetric circuits: a case study

$$H_n = \sum_{k \in \mathbb{Z}_n} (I - Z_k Z_{k+1})$$

Goal: prepare ***a ground state*** $|\psi\rangle = U|+\rangle^{\otimes n}$ from $|+\rangle^{\otimes n}$

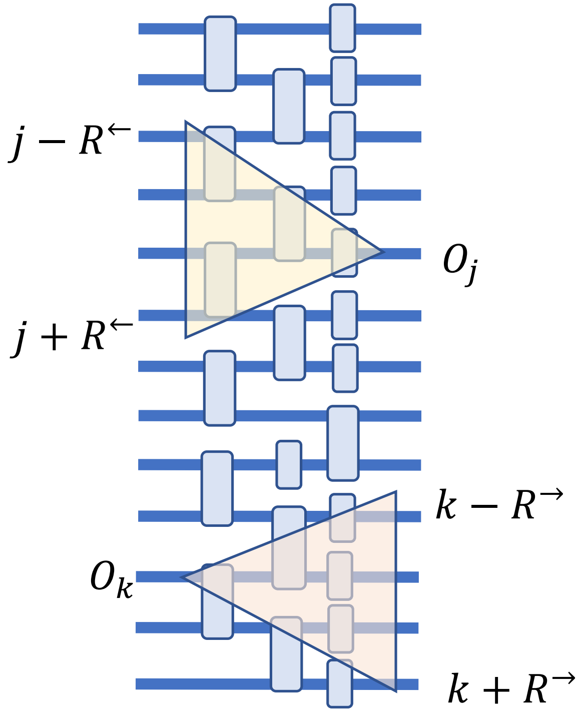
What is the required circuit range for U?

U has

(backward) ***range*** $R^{\leftarrow}$ if the backward light-cone of every output qubit j is contained in $(j - R^{\leftarrow}, j + R^{\leftarrow})$

(forward) ***range*** $R^{\rightarrow}$ if the forward light-cone of every input qubit k is contained in $(k - R^{\rightarrow}, k+R^{\rightarrow})$

range $R = \max\{R^{\leftarrow}, R^{\rightarrow}\}$.



Limitations of $\mathbb{Z}_2$-symmetric circuits: a case study

$$H_n = \sum_{k\in\mathbb{Z}_n} (I - Z_k Z_{k+1})$$

Goal: prepare ***a ground state*** $|\psi\rangle = U|+\rangle^{\otimes n}$ from $|+\rangle^{\otimes n}$

If U is arbitrary (**no symmetry**):

$|\psi\rangle = \alpha|0\rangle^{\otimes n} + \beta|1\rangle^{\otimes n}$, α, β arbitrary

$\Downarrow$

Choose $|\psi\rangle = |0\rangle^{\otimes n}$ and $U = H^{\otimes n}$

Easy! (range-1, local)

If U is **$\mathbb{Z}_2$-symmetric**:

$|\psi\rangle$ has to be

$$|GHZ_n\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n})$$

Need linear range!

This is a fundamental limitation of $\mathbb{Z}_2$-symmetric circuits!

Circuit range lower bound for preparing $|GHZ_n\rangle$

Claim: Suppose a circuit U prepares $|GHZ_n\rangle$ from a product state, i.e., $|GHZ_n\rangle = U|+\rangle^{\otimes n}$.

Then the range of U satisfies $R \geq \frac{n}{2}$.

Proof:

S. Bravyi, M. B. Hastings, and F. Verstraete, Phys. Rev. Lett. 97, 050401 (2006).

$$|GHZ_n^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n})$$

$$|GHZ_n^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} - |1\rangle^{\otimes n})$$

These two states are orthogonal, but ***locally indistinguishable***: the reduced density operators on n-1 qubits are identical.

$\xrightarrow{U^{-1}}$

$$U^{-1}|GHZ_n^+\rangle = |+\rangle^{\otimes n}$$

$$U^{-1}|GHZ_n^-\rangle$$

These states are ***locally distinguishable*** because they are orthogonal and the first is a product state

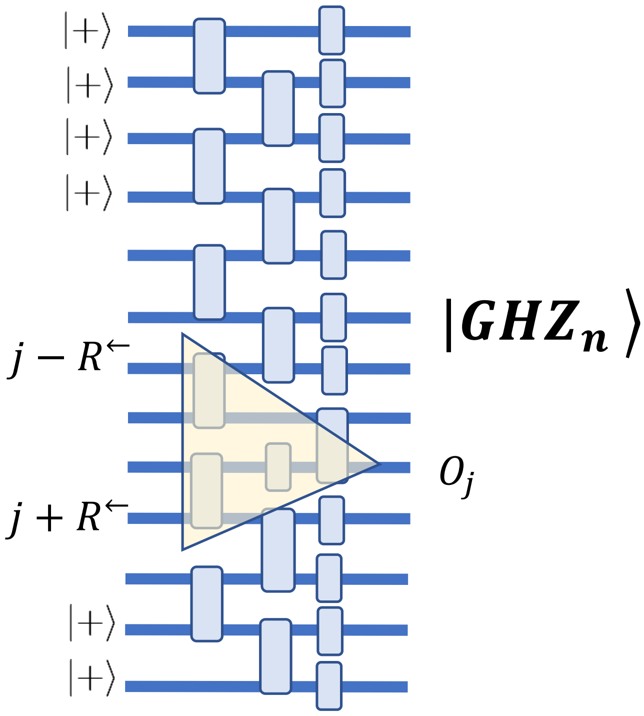
There is a single-qubit observable O_j distinguishing these two states.

$\xleftarrow{U}$

The observable

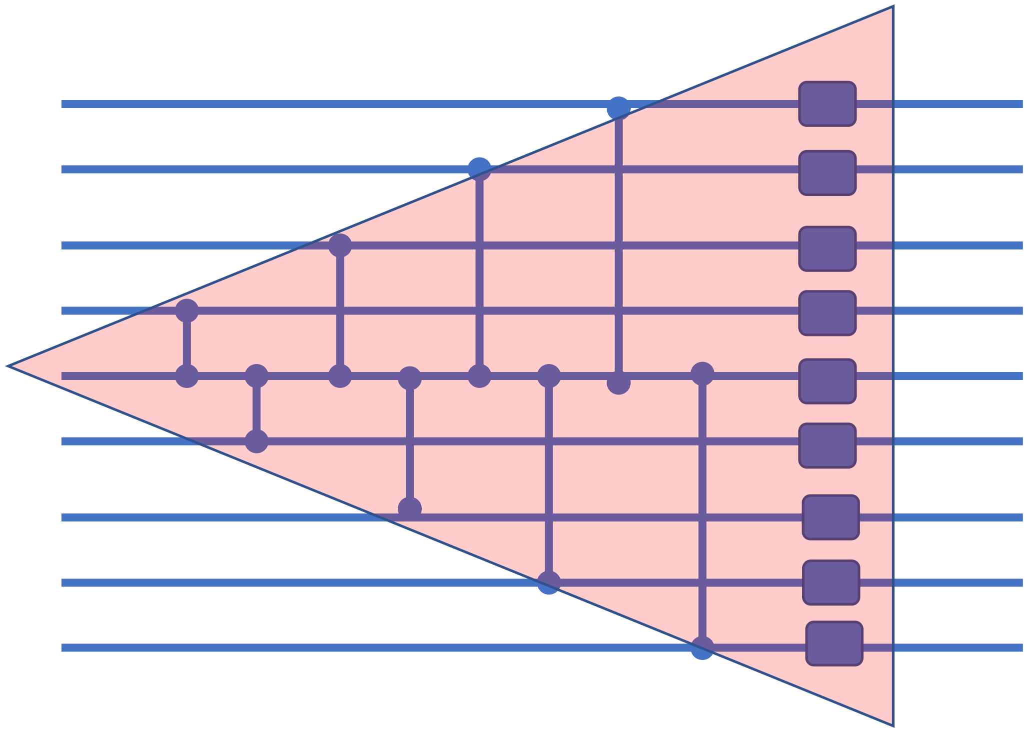
$$UO_jU^{-1}$$

distinguishes these two states.



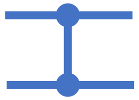
Saturating the range lower bound: GHZ-preparing circuit

n qubits $|+\rangle^{\otimes n}$



$|GHZ_n\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n})$

range $R = \left\lceil \frac{n}{2} \right\rceil$



$= \quad e^{i\pi/4} \exp(-i\frac{\pi}{4} Z \otimes Z)$

$= \quad \exp(-i\frac{\pi}{4} X)$

Each gate commutes with $X^{\otimes n}$.

Thus the circuit is $\mathbb{Z}_2$-symmetric.

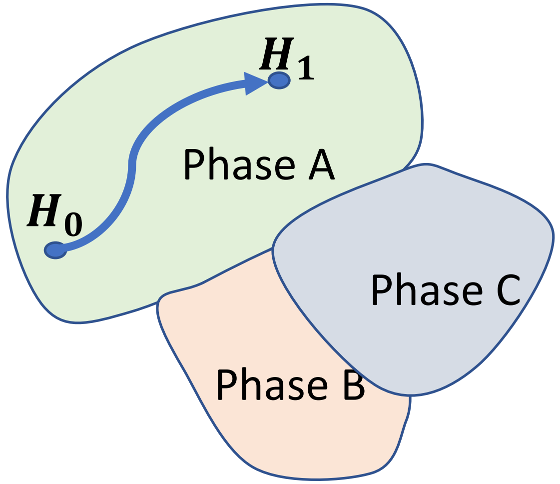
Limitations of $\mathbb{Z}_2$-symmetric circuits: a case study

$$H_n = \sum_{k\in\mathbb{Z}_n} (I - Z_k Z_{k+1})$$

Goal: prepare ***a ground state*** $|\psi\rangle = U|+\rangle^{\otimes n}$ from $|+\rangle^{\otimes n}$

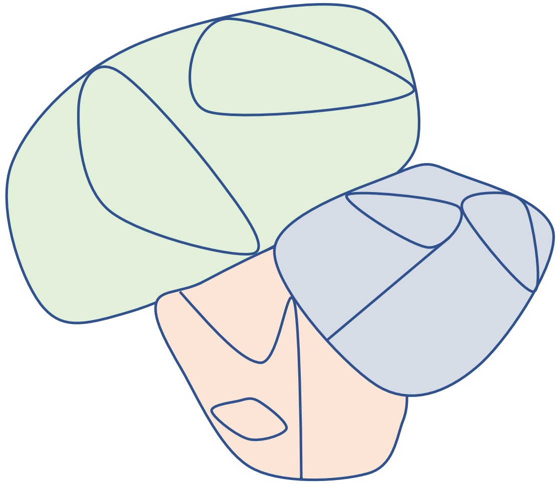
If U is arbitrary (**no symmetry**):
(range-1 suffices)

If U is $\mathbb{Z}_2$**-symmetric**:
Need linear range!



"Symmetry protection"

Haldane. PRL 50:1153-1156, 1983.
Affleck, Kennedy, Lieb, Tasaki. PRL 59:799-802, 1987.
Gu, Wen, PRB 80:155131, (2009)
Pollmann, Turner, Berg, Oshikawa. PRB 81:054439 (2010)
Haegeman, Perez-Garcia, Cirac, Schuch, PRL 102, 050402 (2012)
Chiu, Teo, Schnyder, Ryu. Rev. Mod. Phys., 88:035005,2016.



Low-energy states of Ising model: Preparation with symmetry

$$H_n = \sum_{k \in \mathbb{Z}_n} (I - Z_k Z_{k+1})$$

Theorem: Suppose $|\psi\rangle = U|+\rangle^{\otimes n}$ where U has range $R < n/4$ and is $\mathbb{Z}_2$-symmetric.

Then $\langle\psi|H_n|\psi\rangle \geq \frac{1}{2R+1} n$

Preparing any state with an energy density lower than ε density requires $R = \Omega({}^1/_\epsilon)$.

Symmetry obstructs the preparation of low-energy states!

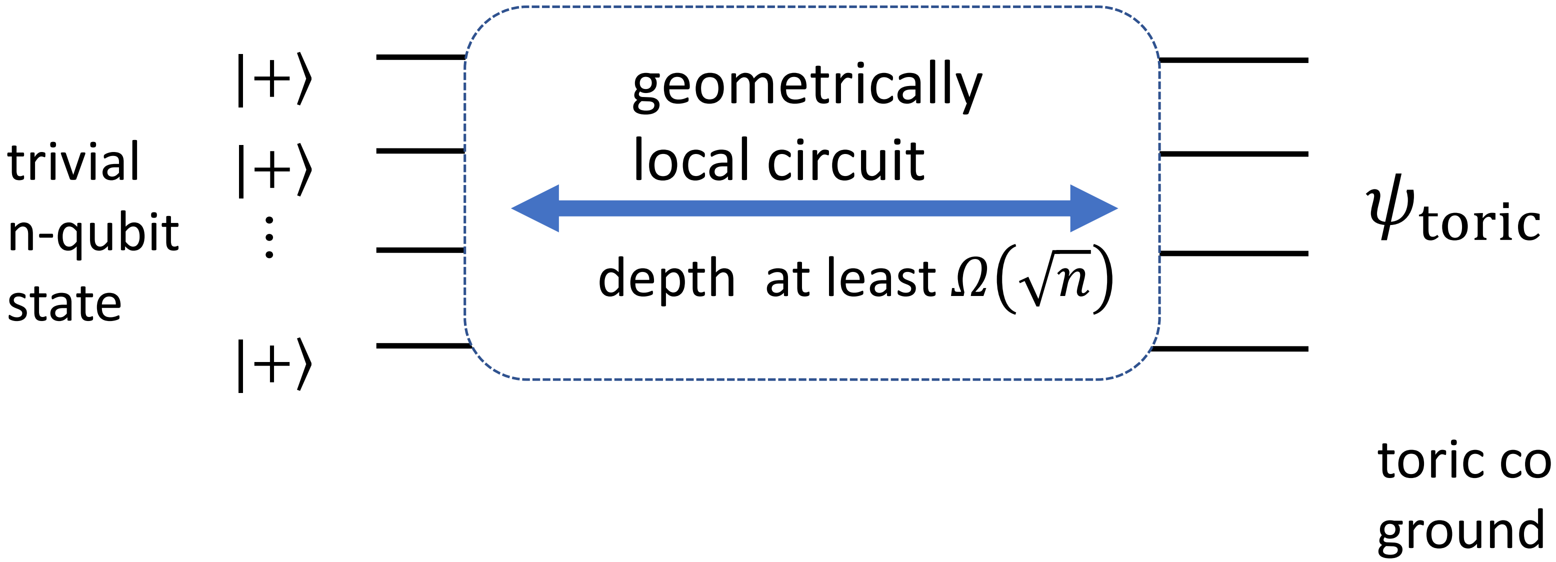
also see G. Mbeng, R. Fazio, G. Santoro, arXiv:190608948 for QAOA

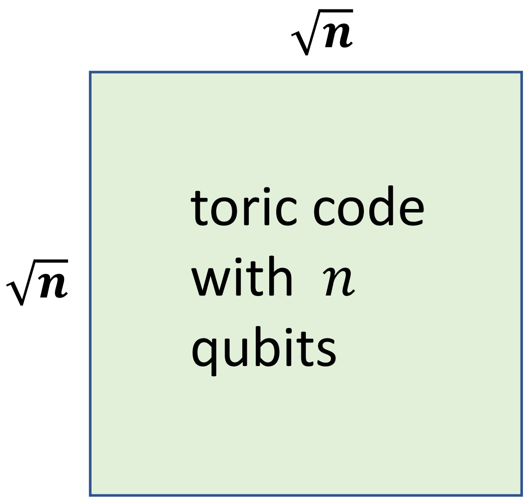
Toric code: no zero-energy trivial states

Geometrically local circuits require $\Omega(\sqrt{n})$ depth.

Bravyi, Hastings, Verstraete, PRL 97, 050401 (2006)

All toric code ***zero-energy states*** are ***non-trivial*** (topologically ordered).





Toric code: existence of low-energy trivial states

If $n \geq d^2$ the output state is NOT a ground state of H_n^{toric}

Bravyi, Hastings, Verstraete, PRL 97, 050401 (2006)

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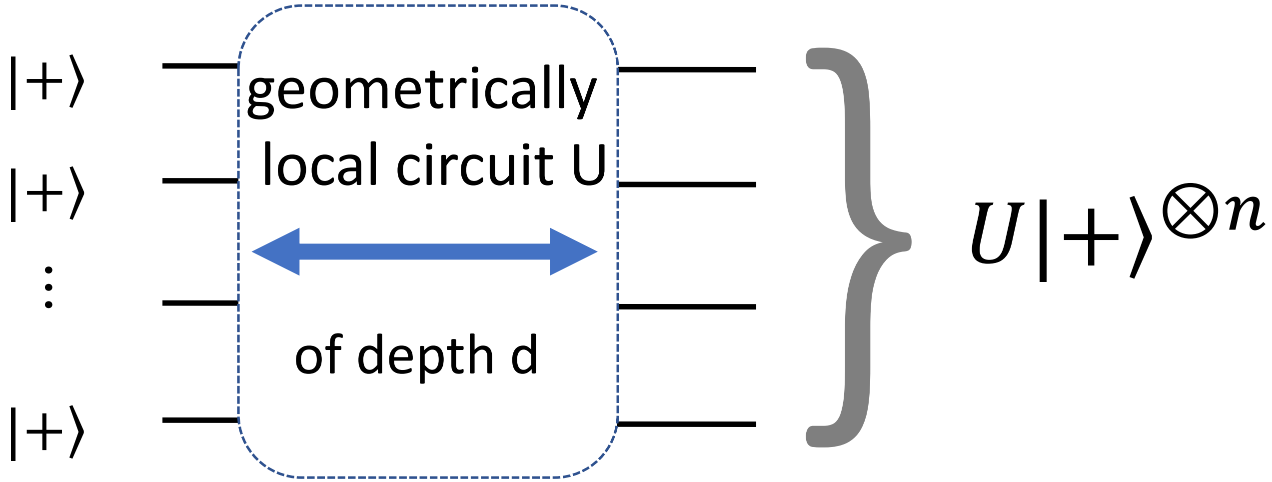
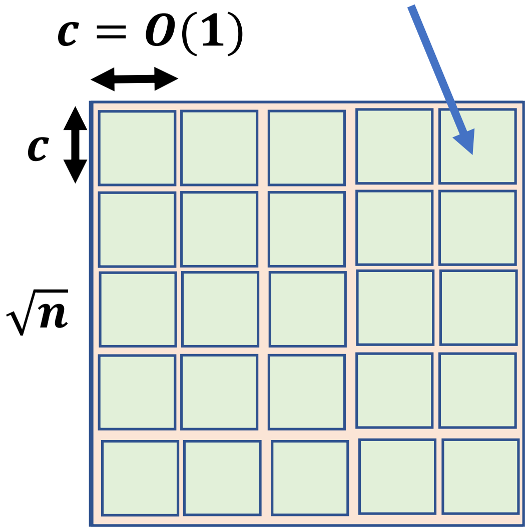


constant-size patches of local ground states (can be created in parallel)



For every constant $\varepsilon > 0$ there is a **constant-depth** circuit U such that $\langle +|^{\otimes n} U^\dagger H_n^{toric} U |+\rangle^{\otimes n} \leq \varepsilon n$

The toric code has ***low-energy states that are trivial.***

The NLTS conjecture

Freedman and Hastings, Quant. Inf. Comp. 14 (2014)

No low-energy trivial states (NLTS) property:

There is $\varepsilon > 0$ and a function $f: \mathbb{N} \to \mathbb{N}$ such that for any depth-d (local) circuit U

$$\langle +|^{\otimes n} U^{\dagger} H_n U |+\rangle^{\otimes n} > \varepsilon n \quad \text{for any} \quad n \geq f(d)$$

Conjecture: There is a family $\{H_n\}_n$ of local Hamiltonians that has the NLTS property.

The following families $\{H_n\}_n$ **do not** satisfy the NLTS property:

Hamiltonian family	Reference
toric code Hamiltonians	Freedman & Hastings 2014
2-local Hamiltonians on non-expanding graphs	Brandao and Harrow 2013
2-local Hamiltonians with commuting terms	Bravyi and Vyalyi 2005
3-qubit Hamiltonian with commuting terms	Aharonov and Eldar 2011
O(1)-local Hamiltonians with commuting terms with high local expansion	Aharonov and Eldar 2015
Sparse commuting O(1)-local Hamiltonians corresponding to graphs with high girth	Hastings 2012

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Evidence for the NLTS conjecture:

- There is a family of toric-code like (CSS-stabilizer) Hamiltonians on simplicial complexes such that an NLTS-like statement holds ***when one restricts to a certain subset of excited states***. (Freedman and Hastings)
- There is a family of Hamiltonians satisfying a related "***no low-error trivial states property***" (Harrow and Eldar, FOCS 2017)

Main result: NLTS with symmetry protection

for a family

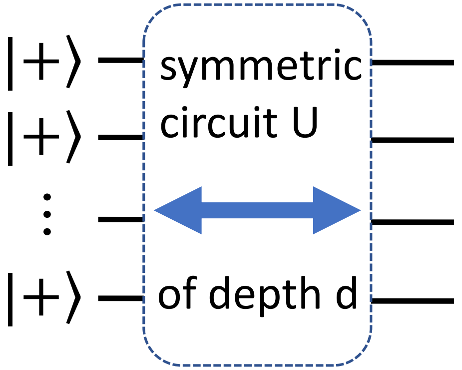
$\{H_n\}_n$ of local $\mathbb{Z}_2$-symmetric Hamiltonians

No low-energy $\mathbb{Z}_2$ -trivial states property:

There is $\varepsilon > 0$ and a function $f: \mathbb{N} \to \mathbb{N}$ such that for any $\mathbb{Z}_2$-symmetric depth-d (local) circuit U

$$\langle +|^{\otimes n} U^\dagger H_n U |+\rangle^{\otimes n} > \varepsilon n \quad \text{for any} \quad n \geq f(d)$$

Main result: Construction of a family $\{H_n\}_n$ of local Hamiltonians that has the NL$\mathbb{Z}_2$TS property.



$U|+\rangle^{\otimes n}$

Symmetry-protected NLTS

NL $\mathbb{Z}_2$TS

Main result: Ising models on expander graphs satisfy NL$\mathbb{Z}_2$TS

Let $\{G_n\}_{n\in I}$ be an infinite family of D-regular graphs such that $h(G_n) \geq h$ for all $n \in I$

Graph $G = (V, E)$ given $S \subset V$

(edge) boundary $\partial(S) = \{e \in E \mid |e \cap S| = 1\}$

Cheeger constant of G: $h(G) = \min_{\substack{S \subseteq V \\ 0<|S|\leq|V|/2}} \frac{|\partial(S)|}{|S|}$

We need infinite families of D-regular graphs with $h = \Omega(1)$.

Ramanujan graphs:

- connected
- satisfy $h(G) \geq \frac{1}{2}(D - 2\sqrt{D-1})$

There is an infinite family of D-regular Ramanujan graphs for every $D \geq 3$.

Marcus, Spielman, Srivastava, Annals of Mathematics 182, 307 (2015)

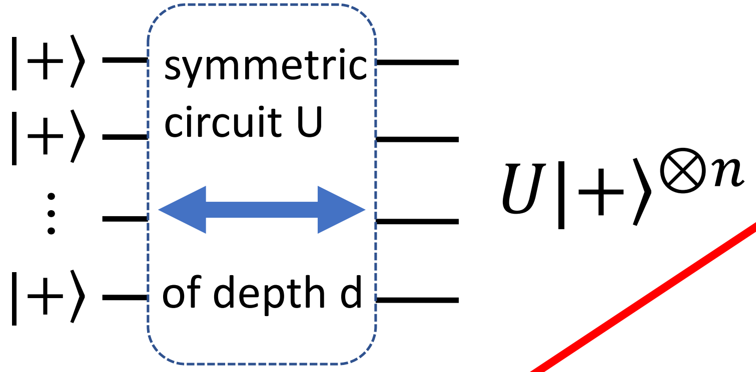
Main result: Ising models on expander graphs satisfy NL$\mathbb{Z}_2$TS

Let $\{G_n\}_{n\in I}$ be an infinite family of D-regular graphs such that $h(G_n) \geq h$ for all $n \in I$

Let $H_n = \frac{1}{2}\sum_{(u,v)\in E_n}(I - Z_u Z_v)$

Theorem: $\langle + |^{\otimes n} U^\dagger H_n U | + \rangle^{\otimes n} > \left(\frac{h}{6}\right) n$ for any $n > 24^2 2^{4d/3}$

and any $\mathbb{Z}_2$-symmetric depth- d (local) circuit U



Corollary: Unless $d = \Omega(\log n)$, no low energy (density) state can be prepared.

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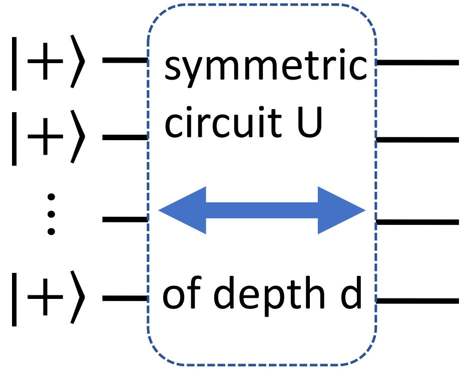
Proof:



$U|+\rangle^{\otimes n}$

Suppose $\langle+|^{\otimes n} U^\dagger H_n U |+\rangle^{\otimes n} < \left(\frac{h}{6}\right) n$

Main result: Ising models on expander graphs satisfy NL$\mathbb{Z}_2$TS

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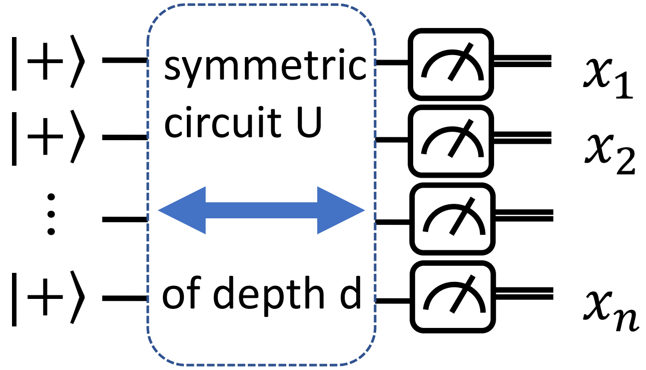
and any $\mathbb{Z}_2$-symmetric depth- d (local) circuit U

Proof:



Suppose $\langle + |^{\otimes n} U^\dagger H_n U | + \rangle^{\otimes n} < \left(\frac{h}{6}\right) n$

Consider the distribution

$p(x) = |\langle x | U | + \rangle^{\otimes n}|^2$ where $x \in \{0,1\}^n$

By Markov's inequality $p(S_{low}) \geq 1/2$ where $S_{low} := \{x \in \{0,1\}^n \mid \langle x | H | x \rangle < \frac{h}{3} n\}$

"low energy configurations"

Main result: Ising models on expander graphs satisfy NL$\mathbb{Z}_2$TS

Let $\{G_n\}_{n\in I}$ be an infinite family of D-regular graphs such that $h(G_n) \geq h$ for all $n \in I$

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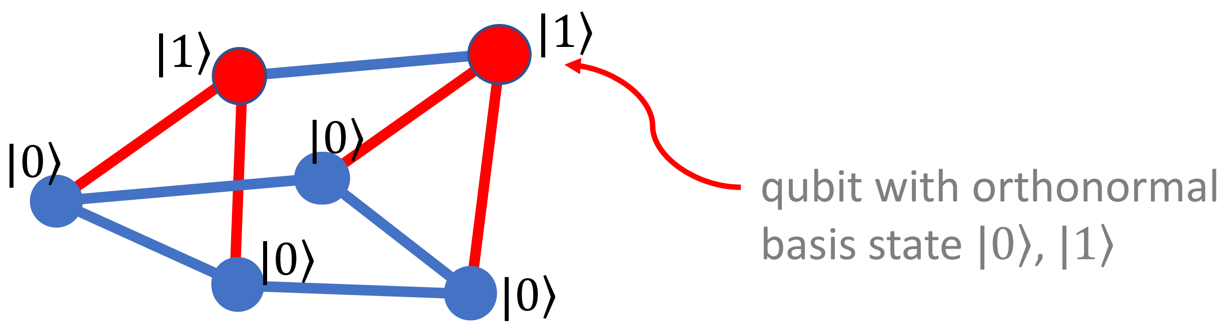

A classical configuration $|x\rangle = |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle$ has energy

$\langle x|H|x\rangle$ = **cut size** for the bipartition

$V_0 := \{u : x_u = 0\}$ $\quad$ $V_1 := \{u : x_u = 1\}$

$\geq h \cdot \min\{|V_0|, |V_1|\}$

$S_{low} := \{x \in \{0,1\}^n \mid \langle x|H|x\rangle < \frac{h}{3}n\}$

“low energy configurations”

$S_{low} \subset S_0 \cup S_1$

$S_0 := \{x \in \{0,1\}^n \mid |x| < \frac{n}{3}\}$ “low weight strings”

$S_1 := \{x \in \{0,1\}^n \mid |x| > \frac{2n}{3}\}$ “high weight strings”

Main result: Ising models on expander graphs satisfy NL$\mathbb{Z}_2$TS

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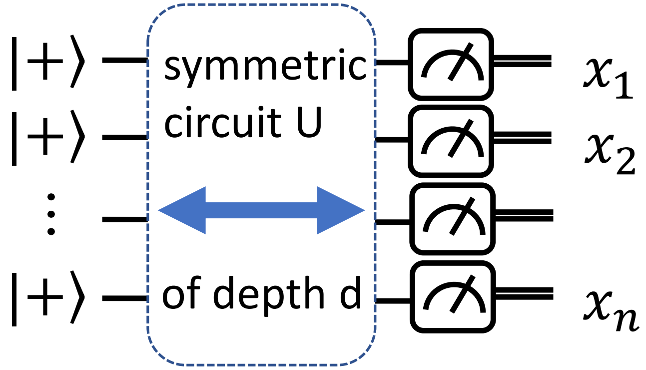

Suppose $\langle+|^{\otimes n} U^\dagger H_n U |+\rangle^{\otimes n} < \left(\frac{h}{6}\right) n$

Consider the distribution

$p(x) = |\langle x|U|+\rangle^{\otimes n}|^2$ where $x \in \{0,1\}^n$

$p(S_0 \cup S_1) \geq 1/2$

This is only possible if $d = \Omega(\log n)$

(Eldar and Harrow, 2017)

by $\mathbb{Z}_2$-symmetry: $p(S_0) \geq 1/4$ and $p(S_1) \geq 1/4$

$S_0 := \{x \in \{0,1\}^n \mid |x| < \frac{n}{3}\}$ "low weight strings"

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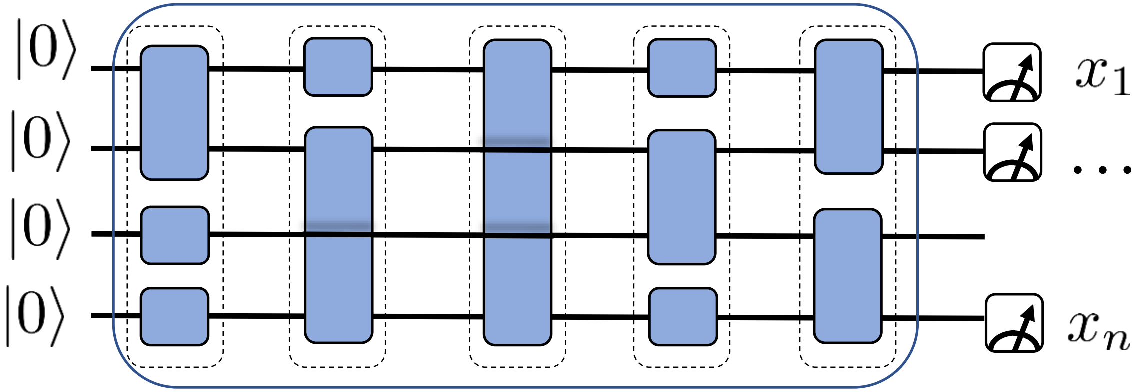
Circuit depth lower bound for sampling from bimodal distributions

Theorem:
(Corollary 43,
Eldar & Harrow, 2017)

Let $p(x)$ denote the output distribution of a depth-d quantum circuit U.
Let $S_0, S_1 \subset \{0,1\}^n$ be such that $p(S_0) > 0$ and $p(S_1) > 0$. Then

$$\mathrm{dist}\,(S_0, S_1) \leq \frac{4n^{1/2}2^{3d/2}}{\min\{p(S_0),p(S_1)\}}$$

A distribution produced by a shallow quantum circuit does not have large support on any two distant subsets of strings at the same time.

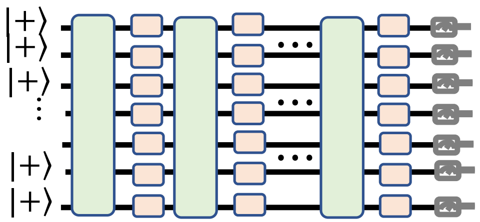


$$p(x) = |\langle x|U|0\rangle^{\otimes n}|^2$$

Classical vs Quantum

level-p QAOA variational state

$$\psi(\beta,\gamma) = \prod_{k=1}^{p} e^{i\beta_k B}\, e^{i\gamma_k H}\, |+\rangle^{\otimes n}$$



MAXCUT on graph	approximation ratio to classical algorithm	approximation ratio achieved by QAOA	required QAOA level p
any		1	$p \to \infty$
triangle-free D-regular graphs		$\frac{1}{2} + \frac{1}{2\sqrt{D}}(1 - \frac{1}{D})^{(D-1)/2}$	$p = 1$

Farhi et al. 2014
Lloyd 2018

Wang, Hadfield, Jiang, Rieffel, PRA 97, 022304 (2018)
Ryan-Anderson, arXiv:1812.04735 (2018).

Sampling from the output distribution of $(p=1)$ −QAOA cannot be efficiently simulated classically unless the polynomial hierarchy collapses (Farhi & Harrow 2016)

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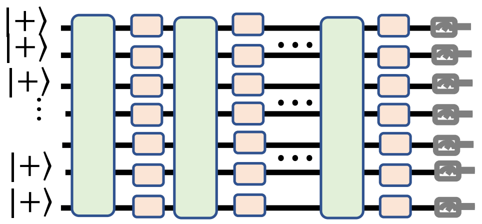

MAXCUT on graph	approximation ratio to classical algorithm	approximation ratio achieved by QAOA	required QAOA level p
any		1	$p \to \infty$
triangle-free D-regular graphs $D \leq 1000$	numerically optimized local algorithm	$\frac{1}{2} + \frac{1}{2\sqrt{D}}\left(1 - \frac{1}{D}\right)^{(D-1)/2}$	$p = 1$

D	classical algorithm	QAOA
2	0.2500	0.2500
3	0.1875	0.1925
4	0.1406	0.1624
5	0.1562	0.1431
6	0.1221	0.1294
7	0.1282	0.1190
8	0.1166	0.1108
9	0.1077	0.1040
10	0.1077	0.0984
11	0.0925	0.0936
12	0.0987	0.0894
13	0.0886	0.0858
14	0.0905	0.0825
15	0.0853	0.0796
16	0.0833	0.0770
17	0.0816	0.0747
18	0.0771	0.0725
19	0.0778	0.0705

MAXCUT on D-regular graphs, for $D \leq 1000$

Classical and Quantum Bounded Depth Approximation Algorithms

Matthew B. Hastings[1,2]

[1] *Station Q, Microsoft Research, Santa Barbara, CA 93106-6105, USA*
[2] *Quantum Architectures and Computation Group, Microsoft Research, Redmond, WA 98052, USA*

We consider some classical and quantum approximate optimization algorithms with bounded depth. First, we define a class of "local" classical optimization algorithms and show that a single step version of these algorithms can achieve the same performance as the single step QAOA on MAX-3-LIN-2. Second, we show that this class of classical algorithms generalizes a class previously considered in the literature[1], and also that a single step of the classical algorithm will outperform the single-step QAOA on all triangle-free MAX-CUT instances. In fact, for all but 4 choices of degree, existing single-step classical algorithms already outperform the QAOA on these graphs, while for the remaining 4 choices we show that the generalization here outperforms it. Finally, we consider the QAOA and provide strong evidence that, for any fixed number of steps, its performance on MAX-3-LIN-2 on bounded degree graphs cannot achieve the same scaling as can be done by a class of "global" classical algorithms. These results suggest that such local classical algorithms are likely to be at least as promising as the QAOA for approximate optimization.

Classical vs Quantum

level-p QAOA variational state

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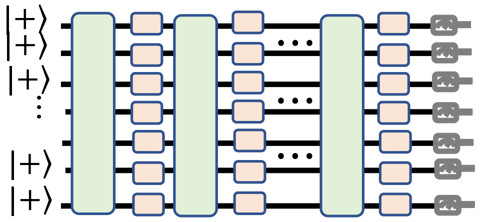

MAXCUT on graph	approximation ratio to classical algorithm	approximation ratio achieved by QAOA	required QAOA level p
any		1	$p \to \infty$
triangle-free D-regular graphs $D \leq 1000$	numerically optimized local algorithm	$\frac{1}{2} + \frac{1}{2\sqrt{D}}(1 - \frac{1}{D})^{(D-1)/2}$	$p = 1$
triangle-free bipartite 3-regular graphs, o(n) squares	0.87856	0.756	$p = 2$
	0.87856	**?**	WHAT ABOUT $p > 1$? (constant)

Farhi et al. 2014
Lloyd 2018

Wang, Hadfield, Jiang, Rieffel, PRA 97, 022304 (2018)
Ryan-Anderson, arXiv:1812.04735 (2018).
Hastings 2019
(based on Hirvonen et al. 2014)

Farhi et al. 2014

Goemans and Williamson, 1995

Main result for MAXCUT-QAOA with $p > 1$

Theorem: For every $D \geq 3$ there is an infinite family of D-regular bipartite graphs $\{G_n\}_{n \in I}$ such that

$$\alpha(QAOA_p) \leq \frac{5}{6} + \frac{\sqrt{D-1}}{3D} \qquad \text{if} \qquad p \leq D^{-1}\left(\frac{1}{3}\log_2 n - 4\right)$$

In particular:

$\boldsymbol{\alpha(QAOA_p) < 0.87856 = \alpha}$**(Goemans-Williamson)** if $D \geq 54$

The **best classical polynomial-time algorithm** (Goemans-Williamson) beats QAOA ***for any constant* level p**

Main result for MAXCUT-QAOA with $p > 1$

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Proof: Take $\{G_n\}_n$ to be family of D-regular bipartite Ramanujan graphs. (Marcus, Spielman, Srivastava 2015)

$$\max_{\psi} \langle\psi|H_n|\psi\rangle = |E_n| \qquad H_n = \frac{1}{2}\Sigma_{(\mathrm{u,v})\in E_n}(I - Z_u Z_v)$$

$$\max_{(\beta,\gamma)} \langle\psi(\beta,\gamma)|H_n|\psi(\beta,\gamma)\rangle = \frac{|E_n|}{2} + \max_{(\beta,\gamma)} \langle\hat{\Psi}(\beta,\gamma)|\hat{H}_n|\hat{\Psi}(\beta,\gamma)\rangle \qquad \hat{H}_n = \frac{1}{2}\Sigma_{(\mathrm{u,v})\in E_n} Z_u Z_v$$

because G_n is bipartite.

NL $\mathbb{Z}_2$TS: $\langle\hat{\Psi}(\beta,\gamma)|\hat{H}_n|\hat{\Psi}(\beta,\gamma)\rangle < \frac{|E_n|}{2} - \frac{hn}{6}$ because $\prod_{k=1}^{p} e^{i\beta_k B}\, e^{i\gamma_k H_n}$ is $\mathbb{Z}_2$-symmetric depth, depth $d \leq p\,D$

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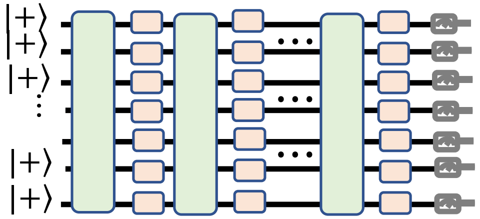

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***D*-regular bipartite expander graphs**	0.87856	$\leq \frac{5}{6} + \frac{\text{const}}{\sqrt{D}} \to 0.8333$ $(D \to \infty)$	$1 < p < \frac{const}{D} \log(n)$	Goemans and Williamson, 1995 THIS WORK

The **best classical polynomial-time algorithm** (Goemans-Williamson) beats QAOA ***for any constant* level p**

Conclusions and open problems

- **$\mathbb{Z}_2$-symmetric No Low Energy Trivial States (NLTS) property** for a family of Ising models on expander graphs

  - Other symmetries?
  - General NLTS conjecture still open

- **Limitations to quantum approximate optimization algorithm** (QAOA):

  Efficient (i.e., constant-level) **QAOA *underperforms*** compared to the **best classical polynomial-time algorithm** (Goemans-Williamson)

  - Comparison for generic instances (instead of worst-case)?

    Finding independent sets in random graphs:

The Quantum Approximate Optimization Algorithm Needs to See the Whole Graph: A Typical Case

Edward Farhi[1], David Gamarnik[2], and Sam Gutmann

[1]Google Inc., Venice CA 90291 and Center for Theoretical Physics, MIT, Cambridge MA, 02139
[2]Operations Research Center and Sloan School of Management MIT, Cambridge MA, 02140

April 21, 2020

- **Non-local modifications of QAOA/RQAOA**: some evidence for their suitability:

  - More extensive benchmarks/case studies?