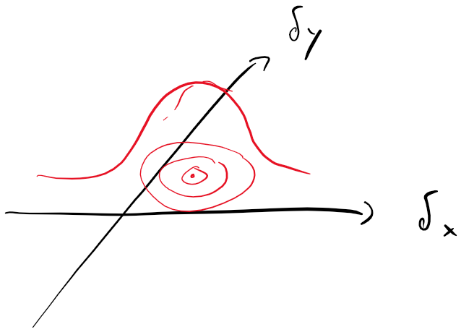
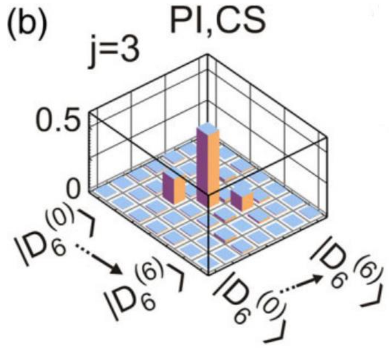
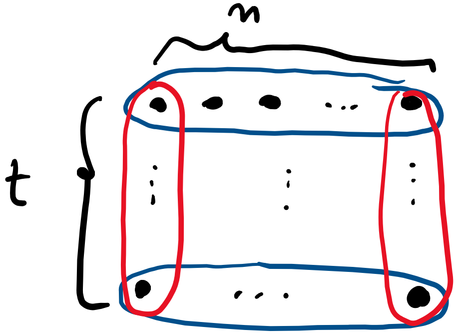
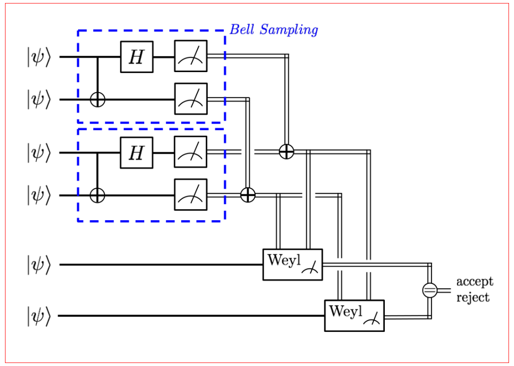
Intro to Quantum State Tomography

Testing under Clifford Symmetry







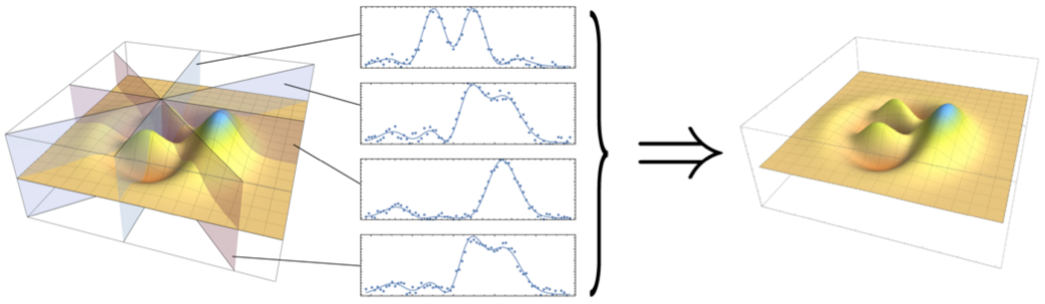


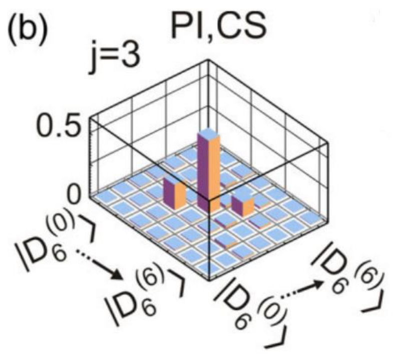
David Gross, University of Cologne

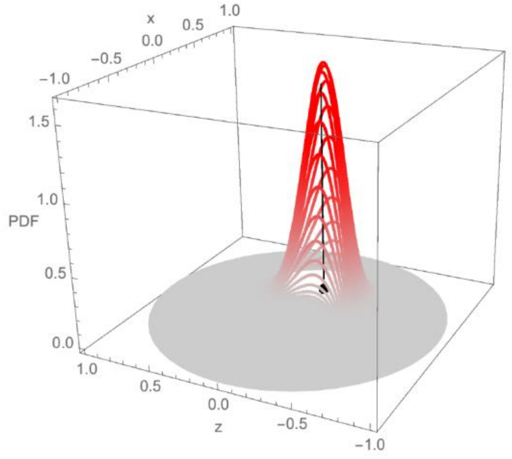
Testing & PCP Workshop, Simons Institute (kinda)

Intro to Tomography

- What's the point?
- What are the problems?
- Four technical approaches







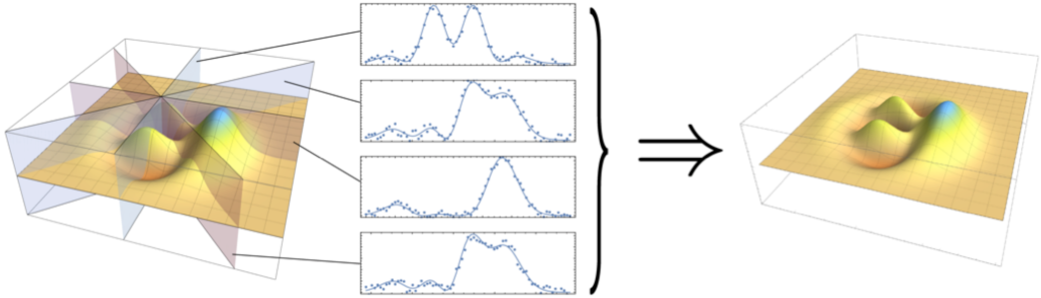
Pictures: [Shahandeh, Ringauer; Schwemmer *et al.*; Haah *et al.*]

What’s the point?

Quantum State Tomography: Estimate state ρ from measurements on n copies.

Reasons against:

- Doomed by exponential # of parameters
- Competes against efficient *certification* protocols
  [Blume-Kohout, Thursday]
- Surprisingly non-trivial



What’s the point?

Quantum State Tomography: R

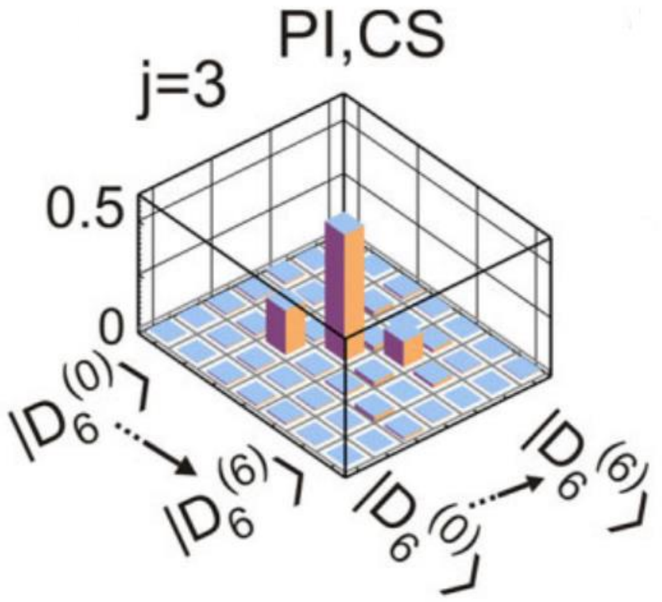
PRL **113,** 040503 (2014) PHYSICAL REVIEW LETTERS week ending 25 JULY 2014

Experimental Comparison of Efficient Tomography Schemes for a Six-Qubit State

Christian Schwemmer,[1,2] Géza Tóth,[3,4,5] Alexander Niggebaum,[6]
Tobias Moroder,[7] David Gross,[8] Otfried Gühne,[7] and Harald Weinfurter[1,2]

Reasons in favor:

- Tells you *in which way* a physical implementation deviates from its specification
- A fundamental primitive of quantum information



Symmetric Dicke state basis

What are the problems?

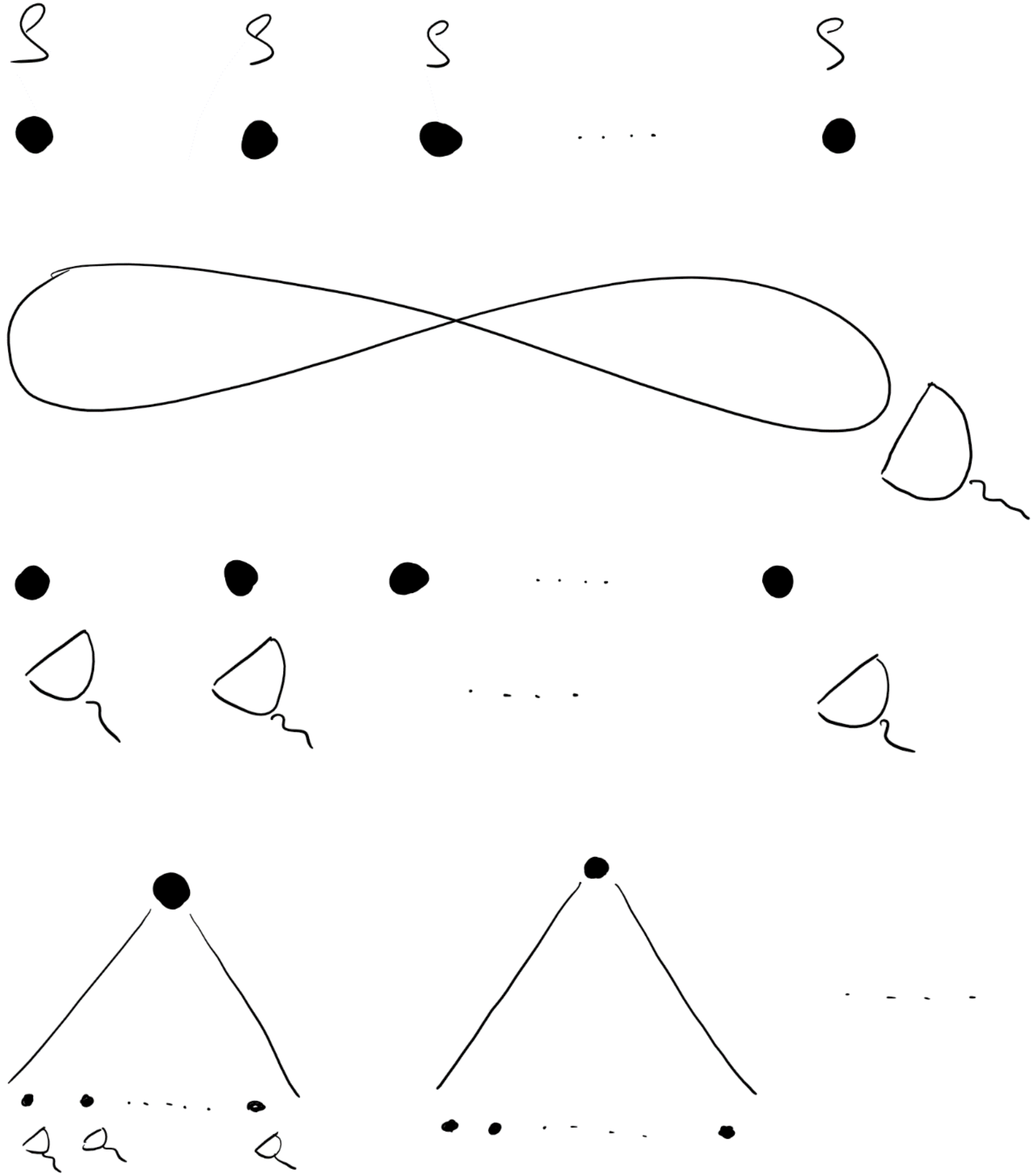
The design problem

Design problem: Which measurements should one perform?

Frameworks:

- Global
- Local
    - Adaptive vs identical
- "Local-local"

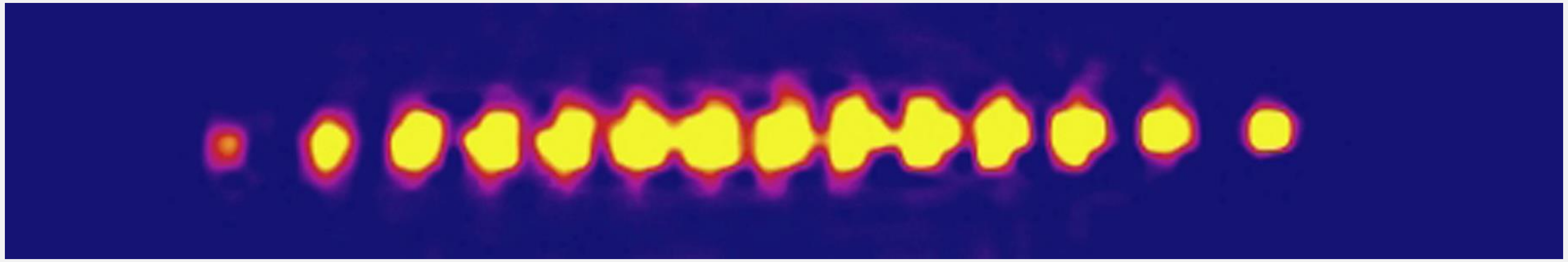
- POVMs, basis, two-outcome?

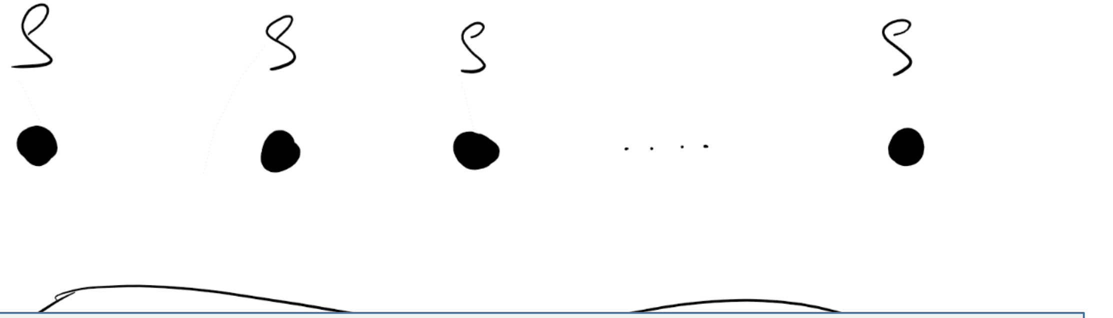


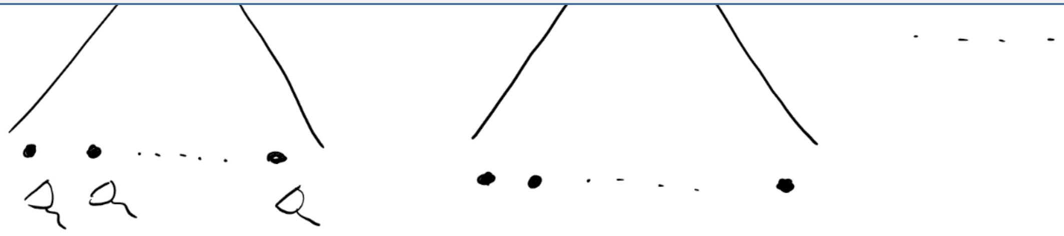
The design problem

Probably most relevant (and least understood):

- Local product basis measurements.







- POVMs, basis, two-outcome?

The estimation problem

Decide on goal:

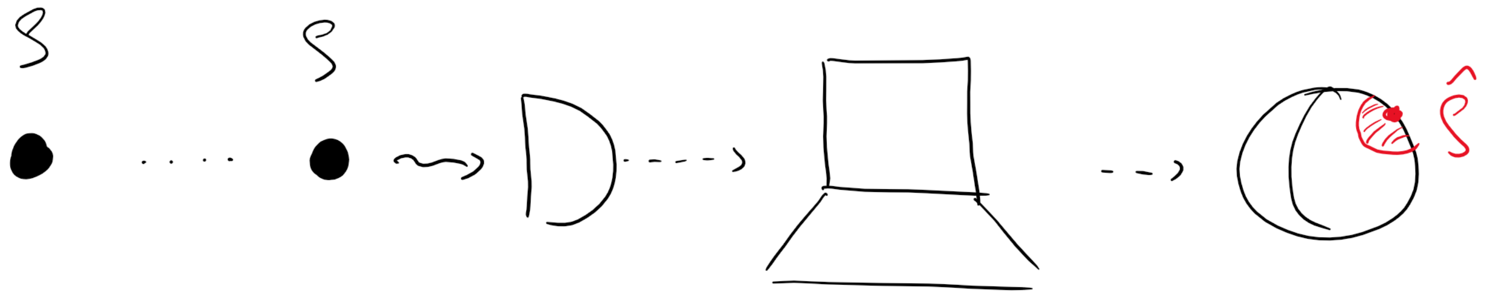
- Point estimate
- Region estimate
- Posterior distribution

Measure performance:

- Computational complexity
- Sample complexity
- In trace norm, 2-norm, fidelity…

Exploit structure:

- Low rank
- Symmetries
- MPS representation



The estimation problem

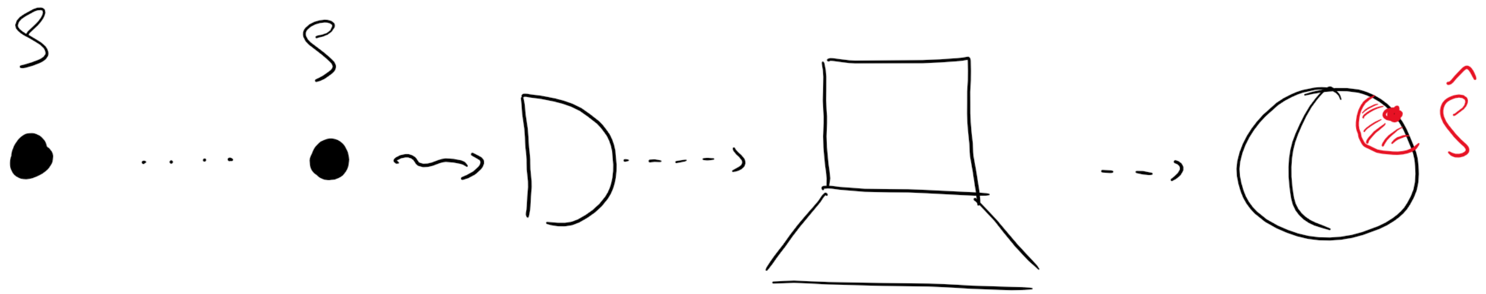
Decide on goal:

Measure performance:

Exploit structure:

This talk:

- Point estimates
- Sample complexity to reach expected trace-norm error ϵ
- Exploit low rank



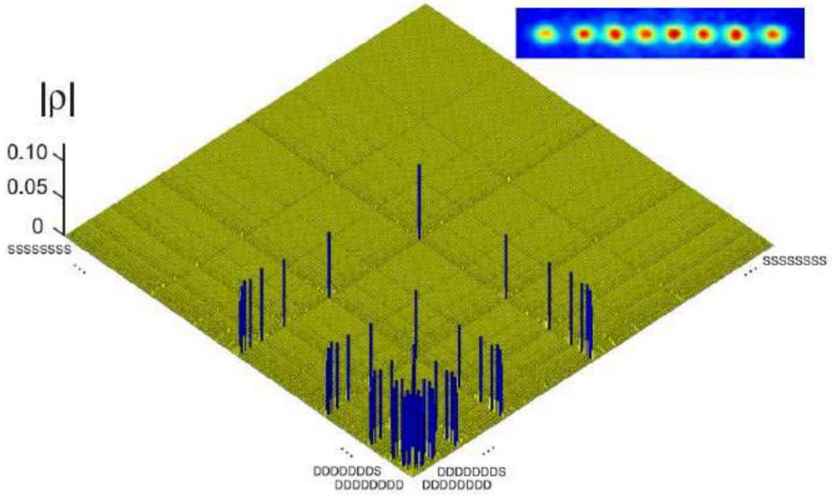
Likely?

Practitioners use

- Maximum Likelihood point estimates

$$\rho_{\mathrm{MLE}} = \max_{\rho} p(\text{ data} \mid \rho)$$

- *Bootstrap* for uncertainty quantification



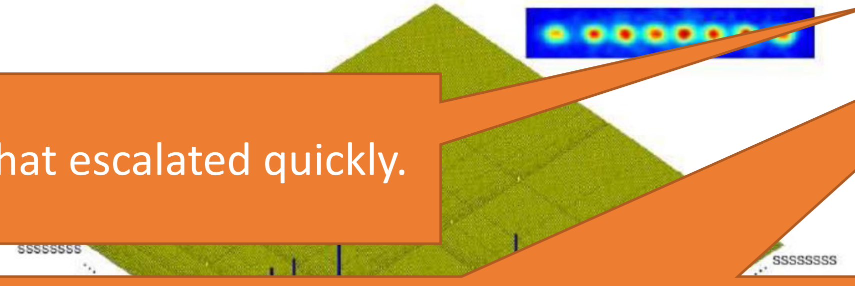
Is this sound?

Yup, it’s OK.

I hear MLE is *optimal*!!

Only asymptotically and away from the boundary (=full rank states). Not terribly relevant.

es



But it’s the *most likely* state given the data!

Ronald Fisher settled this in the 1920s! He’s a **knight**.

Stop wasting your time thinking of new estimators!

Wow. That escalated quickly.

Likelihood has no operational meaning.
“Most likely” vacuous ≠ “most probable” or something

• *Bootstrap* for uncertainty quantificatio

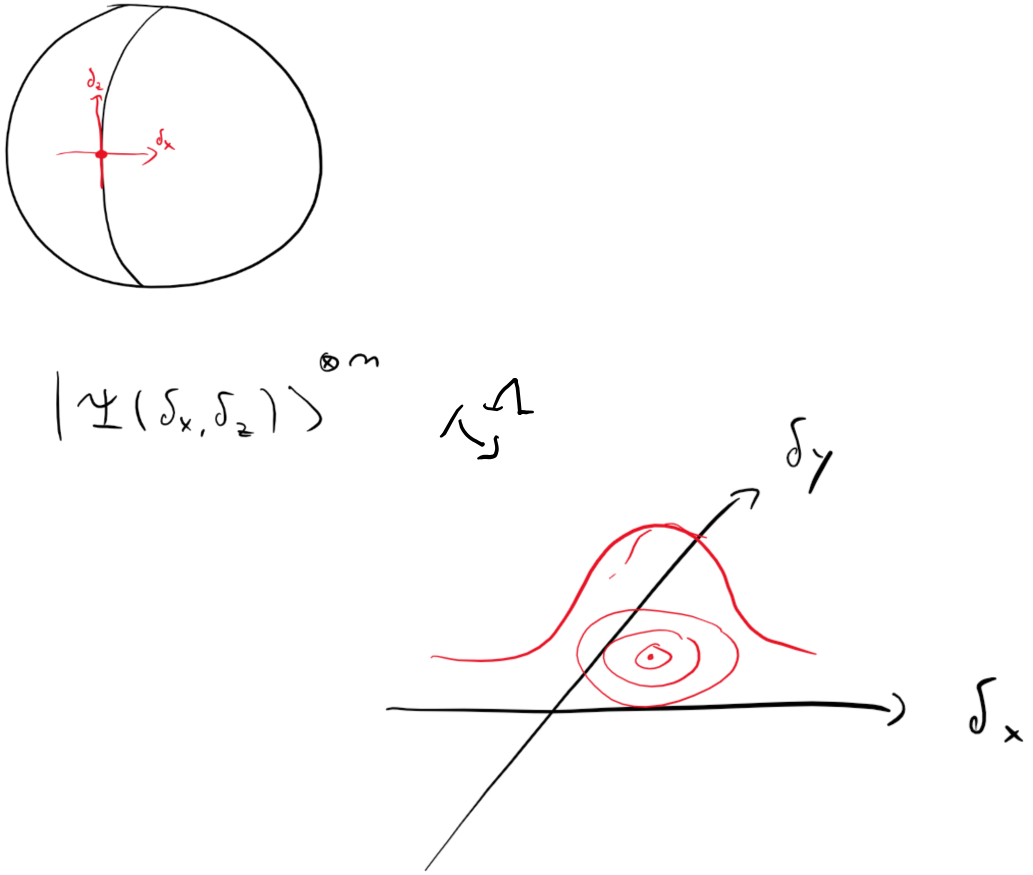
Look:

- MLE is OK
- Not as optimal or canonic as some think
- Performance still needs to be analyzed

Four approaches

(very rough exposition)

1 / 4 Local asymptotic normality



Primakoff-Holstein:

- Consider states $\psi(\delta_x, \delta_z)$ close to reference state $\psi(0,0)$.
- There is channel Λ that sends $|\psi(\delta_x, \delta_z)\rangle^{\otimes n}$ to Gaussian state with first moments δ_x, δ_y.
- Tr-norm isometry for large *n*.

Idea:

1. Find rough estimate, use as $\psi(0,0)$
2. Implement Λ
3. Use heterodyning to find first moments

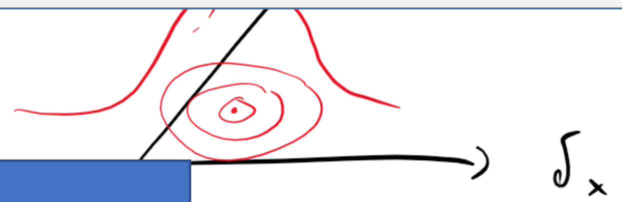
[Madalin Guta, Jonas Kahn]

1 / 4 Local asymptotic normality

Primakoff-Holstein:

LAN:

- Optimal sample complexity for fixed dimension
- Non-optimal scaling in dimension [Haah *et al*.]



Completely ludicrous!

1. Find rough estimate, use as $\psi(0,0)$
2. Implement Λ
3. Use heterodyning to find first moments

[Madalin Guta, Jonas Kahn]

2 / 4 **Keyl, Werner**, Schur, and Weyl

- Write

$$\rho = U \,\mathrm{diag}(p)\, U^*$$

- Estimate spectrum p and eigenbasis U separately.

Spectrum estimation problem: From $\rho^{\otimes n}$, estimate λ.

Ansatz:

- Problem invariant under $U(d)$ and S_n
- ⇒ Try POVMs commuting with both symmetries.

2 / 4 Keyl, Werner, **Schur, and Weyl**

- Local basis changes commute with permutations of systems:

$$U(d) \ni U \mapsto U \otimes \cdots \otimes U,$$
$$S_n \ni \pi\colon |\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle \mapsto |\psi_{\pi_1}\rangle \otimes \cdots \otimes |\psi_{\pi_n}\rangle.$$

- Operator A commutes with $U^{\otimes n}$ iff

$$A = \sum_{\pi \in S_n} c_\pi \, \pi$$

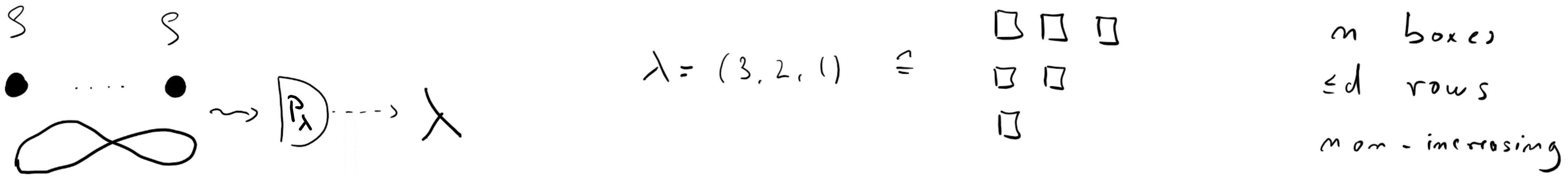
and vice versa.

- Under action of $S_n \times U(d)$:

$$(\mathbb{C}^d)^{\otimes n} \simeq \bigoplus_\lambda S_\lambda \otimes U_\lambda$$

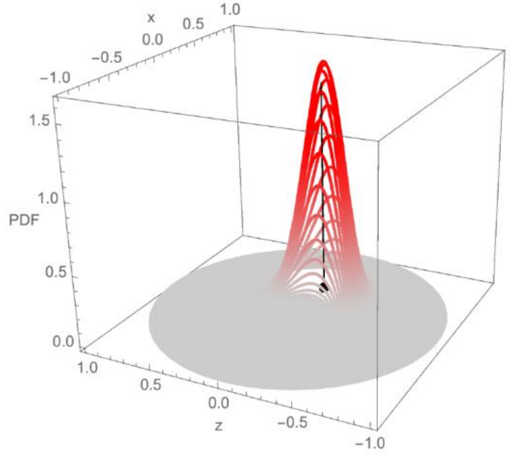
- S_λ irrep of S_t, U_λ irrep of $U(d)$
- **Projections P_λ form POVM commuting with both!**

2 / 4 **Keyl, Werner**, Schur, and Weyl



1. Perform collective measurement $\{P_\lambda\}$, obtain outcome λ
2. Representation spaces are labeled by partitions, visualized as *Young frames*
3. Renormalized λ/n is probability distribution → guess for spectrum!

Turns out to be near-optimal estimator!



2 / 4 **Keyl, Werner**, Schur, and Weyl

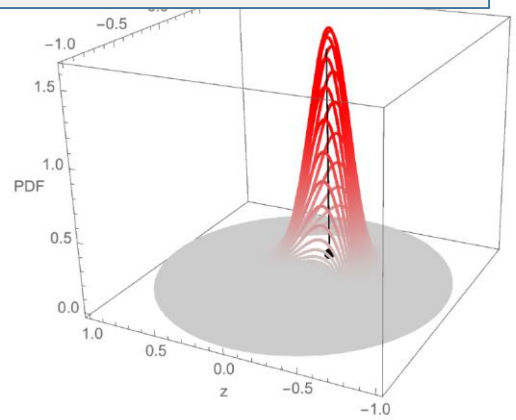
$\lambda = (3, 2, 1)$ n boxes ≤ d rows

KW:

Haah *et al*.; O’Donnell, Wright 2015:

- Complexity of eigenbasis given spectrum $\simeq$ complexity of estimating spectrum
- $n = O(\frac{r\, d}{\epsilon^2})$, which is optimal
- Measurements non-local, but efficient circuits exists (quantum Schur transform)

Turns out to be near-optimal estimator!



3 / 4 Compressed Sensing

- Write

$$\rho = \sum_{i=1}^{r} \lambda_i \, |\psi_i\rangle\langle\psi_i|$$

- depends on $O(rd) \leq O(d^2)$ parameters.

Compressed sensing: Can one recover rank-*r* matrix from $O(rd)$ expectation values

$$y_i = \mathrm{Tr}(\rho\, A_i)?$$

- Naïve:

$$\arg\min_{\rho'} \operatorname{rank} \rho' , \qquad \text{s.t.} \qquad \mathrm{Tr}(\rho'\, A_i) = \mathrm{Tr}(\rho\, A_i)$$

...numerically unstable, NP-hard in general. ☹

- But SDP relaxation...

$$\arg\min_{\rho'} \|\rho'\|_{tr} \qquad \text{s.t.} \qquad \mathrm{Tr}(\rho'\, A_i) = \mathrm{Tr}(\rho\, A_i)$$

...works efficiently for almost all measurements! ☺

3 / 4 Compressed Sensing

- Write

Compressed sensing: Can one recover

CS:

- Can recover from $O(rd)$ *observables*, which is optimal
- Works in local-local model
- $n = O(\frac{r^2\, d}{\epsilon^2})$, which is optimal in local model

[DG, Flammia, Liu, Eisert; Kueng, Rauhut, Terstiege]

- But SDP relaxation…

$$\arg\min_{\rho'} \|\rho'\|_{tr} \quad \text{s.t.} \quad \mathrm{Tr}(\rho'\, A_i) = \mathrm{Tr}(\rho\, A_i)$$

…works efficiently for almost all measurements! ☺

4 / 4 Projected Least Squares

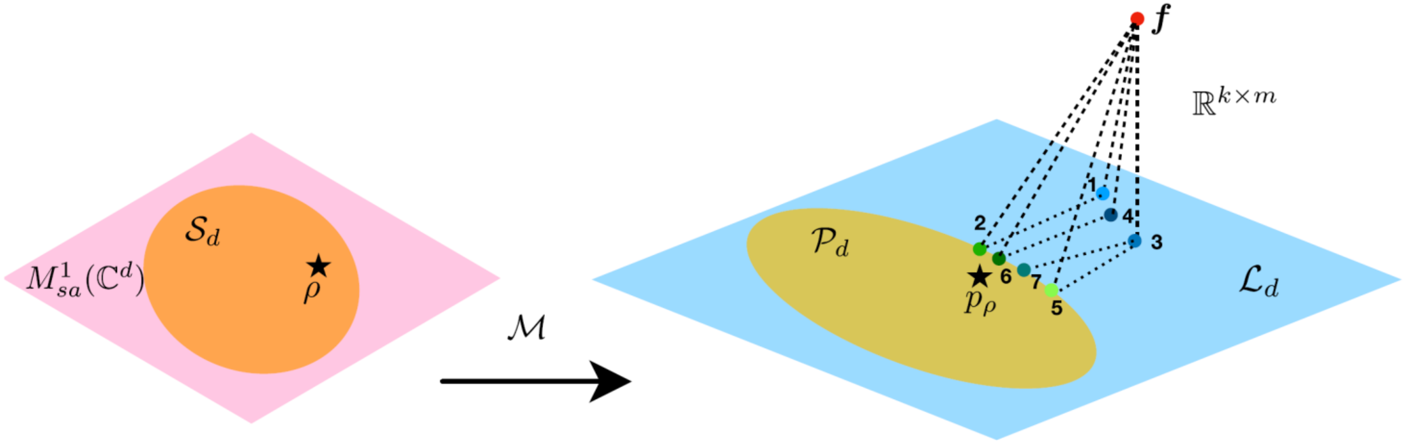
Dead simple:

1. Take data

$$f_i = \mathrm{Tr}(\rho\, A_i) + \epsilon_i$$

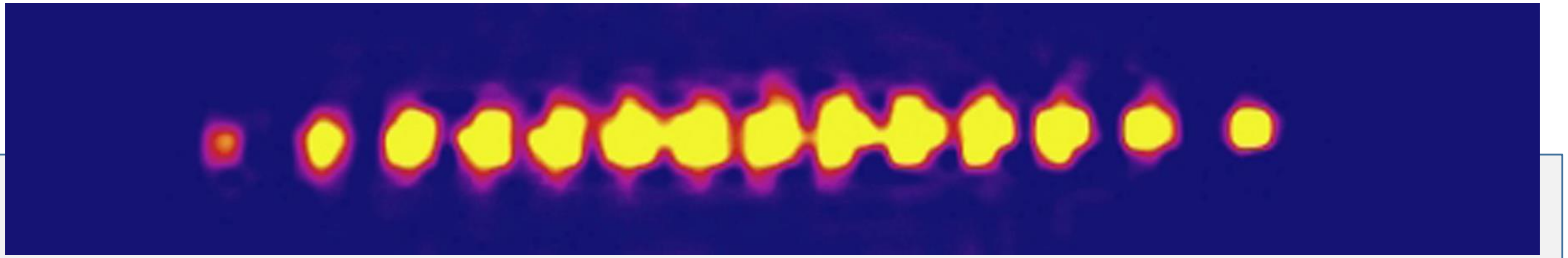
2. Find least-squares fit ρ_{LS}.

3. Modify eigenvalues to project onto state space.



4 / 4 Projected Least Squares

Dead simple:

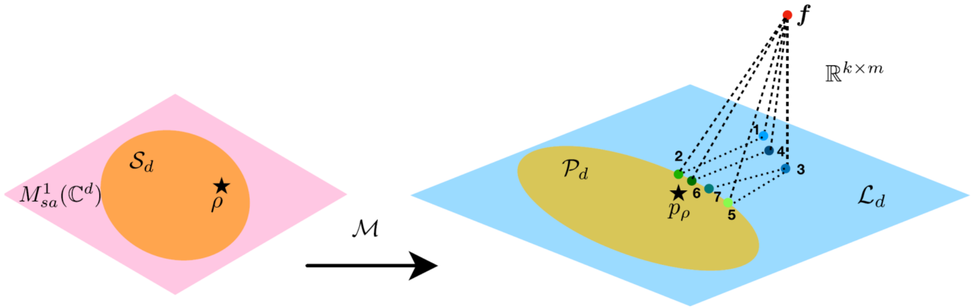


PLS:

- Simple numerics, simple theory
- Optimal scaling for local and some local-local models
- Treats product basis measurements: $n = O(\frac{r^2\, d^{1.6}}{\epsilon^2})$.
- **Relevant open problem: Is this optimal?**

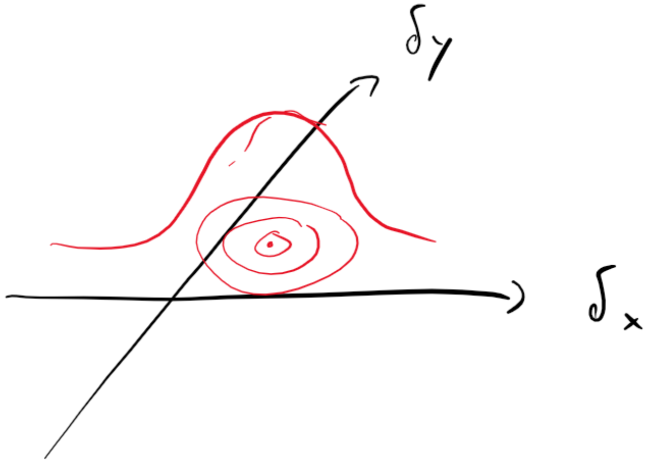
[Guta, Kahn, Kueng, Tropp, Acharya, Kypraios]

Summary

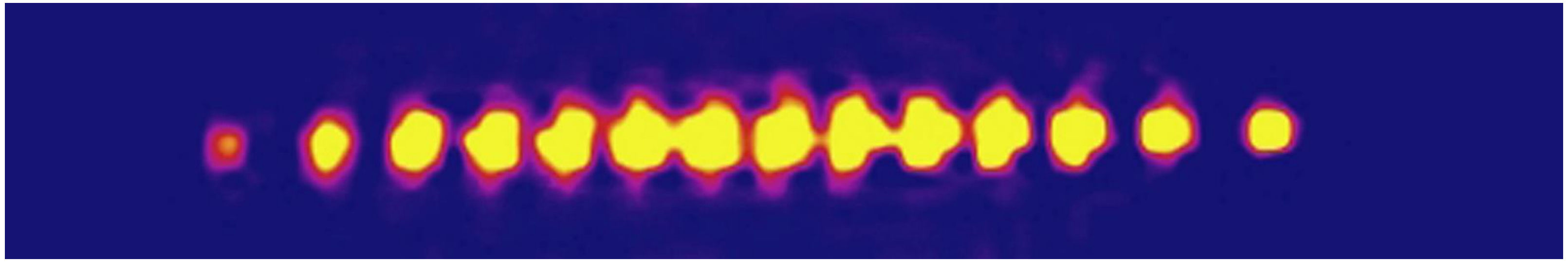


Quantum state tomography is:

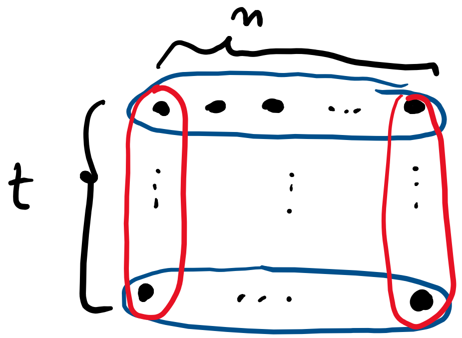
- …relevant in practice, rich in theory

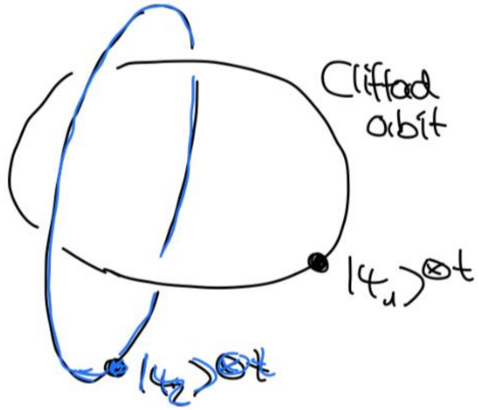


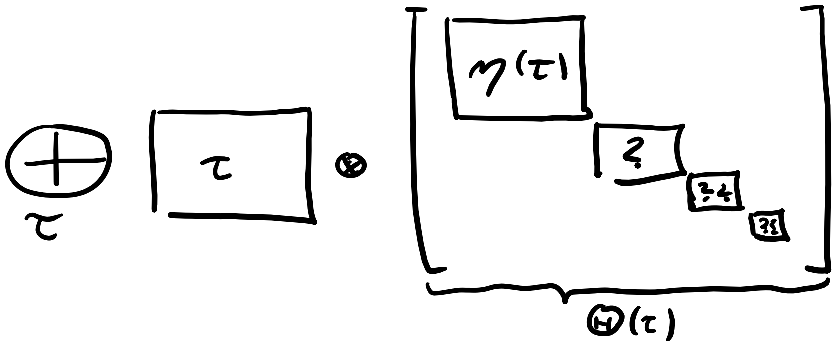
Well. That's it.



Schur-Weyl duality for the Clifford group with applications to Quantum Property Testing (among others)





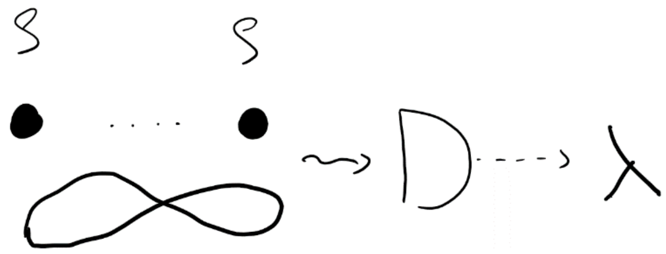


David Gross, University of Cologne

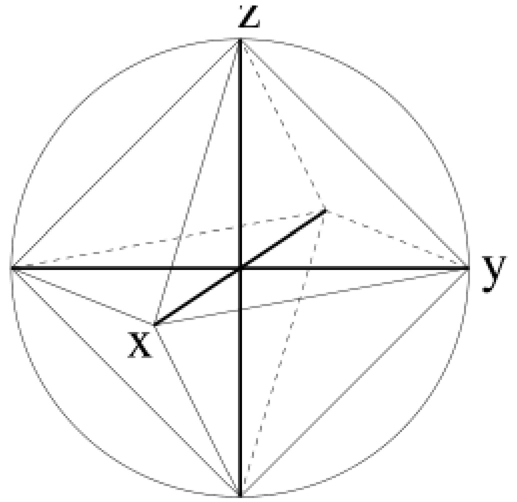
With: Sepehr Nezami, Michael Walter, Felipe Montealegre, Huangjun Zhu

Introduction

Testing under symmetry



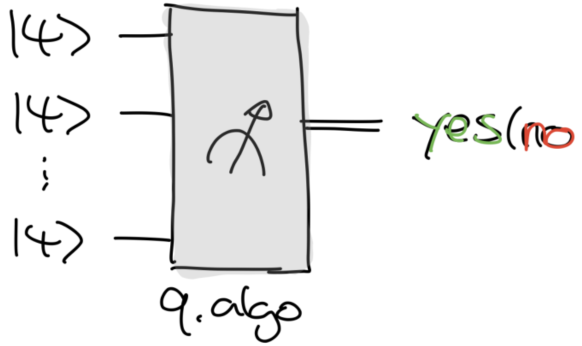
- Recall spectrum estimation problem…
- …solved by exploiting unitary and permutation symmetry.



Q: What if we replace unitary by *Clifford invariance*?

Problem [Montanaro, de Wolf]:

- Is there a dimension-independent t s.t. from t copies of a pure state $\psi^{\otimes t}$, can decide whether
  - ψ is a stabilizer state or
  - ψ is far away from the set of stabilizer states?



Schur-Weyl duality 1

- On t-th tensor power $\mathcal{H}^{\otimes t}$ of a Hilbert space $\mathcal{H}$, commuting actions:

$$U(\mathcal{H}) \ni U \mapsto U \otimes \cdots \otimes U,$$

$$S_t \ni \pi\colon |\psi_1\rangle \otimes \cdots \otimes |\psi_t\rangle \mapsto |\psi_{\pi_1}\rangle \otimes \cdots \otimes |\psi_{\pi_t}\rangle.$$

- Operator A commutes with $U^{\otimes t}$ iff

$$A = \sum_{\pi \in S_t} c_\pi \, \pi \qquad (1)$$

and vice versa.

[Nezami, Walter, DG 18]

- Under action of $S_t \times U(\mathcal{H})$:

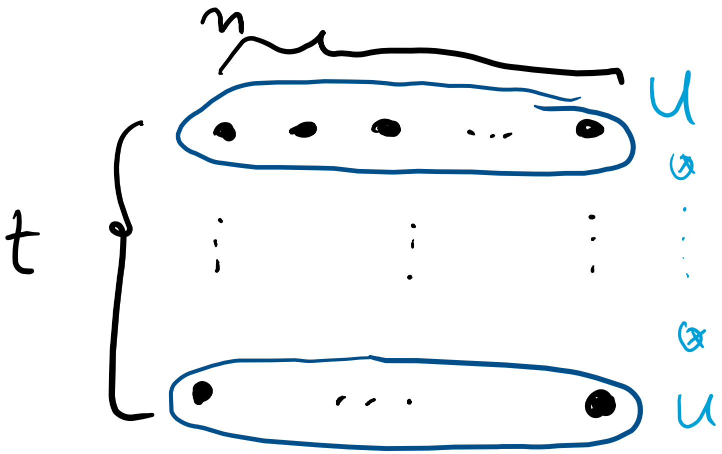
$$\mathcal{H}^{\otimes t} \simeq \bigoplus_\lambda S_\lambda \otimes U_\lambda \qquad (2)$$

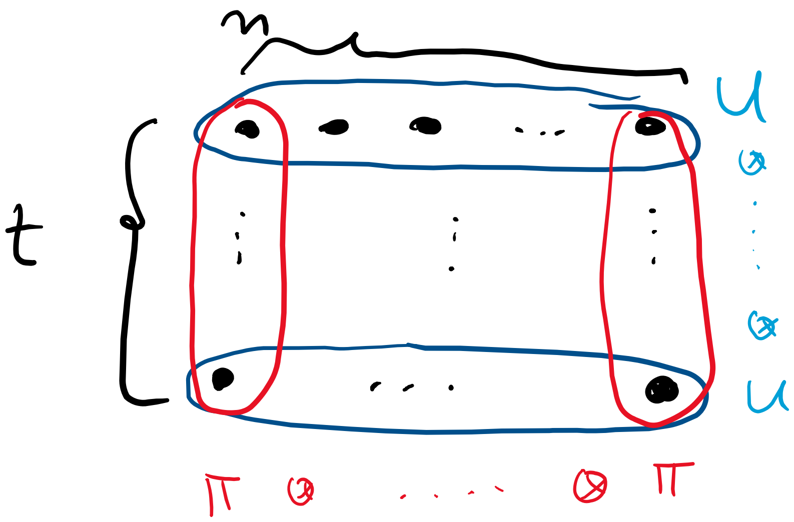
- S_λ irrep of S_t, U_λ irrep of $U(\mathcal{H})$.

[Montealegre-Mora, DG 19]

Schur-Weyl duality 2: Transversality

• = $\mathbb{C}^d$





Assume that each copy is already a tensor product

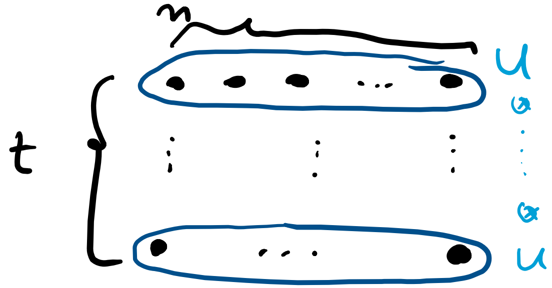
$$\mathcal{H} = (\mathbb{C}^d)^{\otimes n}$$

⇒ Permutations act *transversally*.

Both algebras:

- Have product basis
- Form groups!

Clifford group, prior results



Q: What if we replace unitary by *Clifford invariance*?

Commutant remains S_t for

- t=2
  [Dankert, Emerson 2005]

- t=3
  [Zhu; Webb; Gross and Kueng 2015; implicit in Nebe, Rains, Sloane 2006]

Must be augmented by one stabilizer code projection for

- t=4
  [Zhu, Grassl, Kueng, Gross; Helsen, Wallman, Flammia, Wehner 2016]

Applications of prior results

Representation theory of t-th tensor powers used in, e.g.:

- Randomized benchmarking
- Decoupling technique
- Non-malleable quantum one-time pads

$t = 2$

- Variance bounds for randomized benchmarking
- Stabilizer POVM optimal state-independent measurement for pure states

$t = 4$

Algebraic Theory of the Clifford commutant

Statement of main result

Theorem [Nezami, Walter, DG 18]

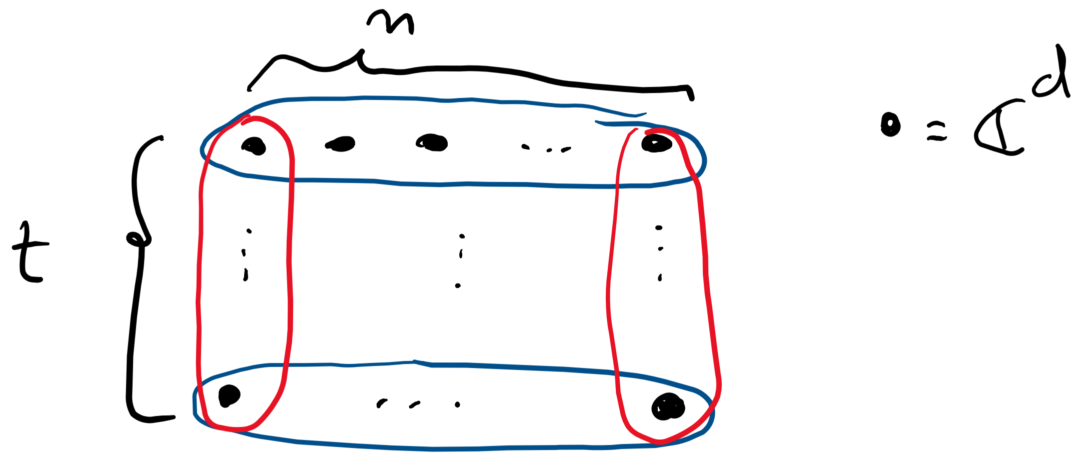
The commutant algebra of t-th tensor powers of the Clifford group over d^n is generated by t-th tensor powers of:

- Discrete orthogonal transformations
- Self-orthogonal CSS code projections

Stochastic orthogonal transformations

A $t \times t$ matrix O, entries in $\mathbb{Z}_d$, is orthogonal if

$$O^T O = \mathrm{Id} \mod \mathrm{d}$$



$\Downarrow$

$|F\rangle \quad F \in \mathbb{Z}_d^{t \times n}$

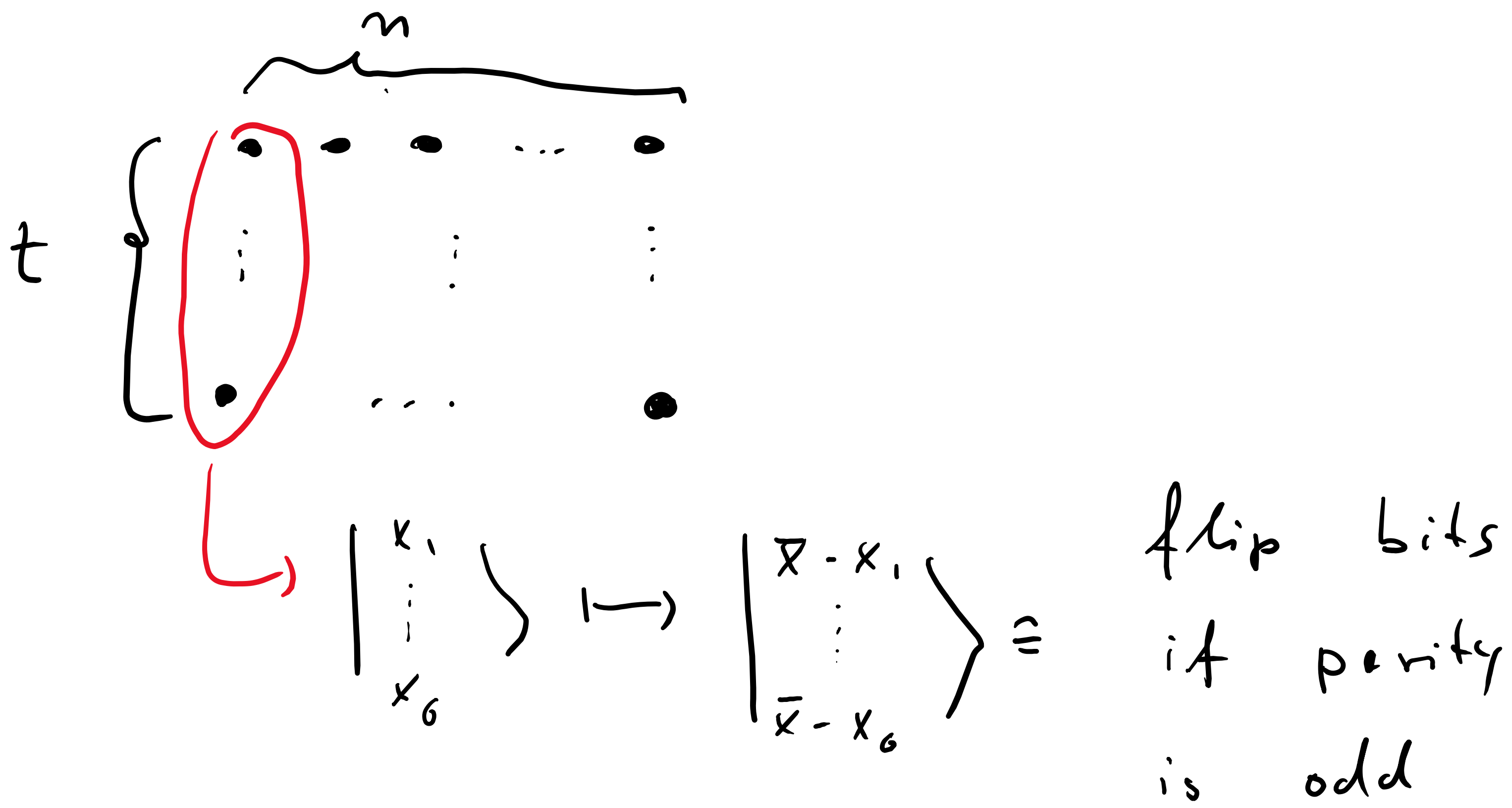
$O : |F\rangle \mapsto |OF\rangle$

Discrete orthogonal transformations

Example: **Anti-permutations**

- Binary complement of permutation matrices

$$\bar{\mathbb{1}}_{6\times 6} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$



Calderbank-Shor-Steane codes

Let $N \subset \mathbb{Z}_d^t$ be *self-orthogonal*:

$$\sum_i u_i v_i = 0 \mod d, \qquad u, v \in N.$$

$$X^u := X_1^{u_1} \otimes \cdots \otimes X_t^{u_t}$$

$$Z^v := X_1^{v_1} \otimes \cdots \otimes Z_t^{v_t}$$

$$\Rightarrow \quad X^u Z^v = Z^v X^u$$

A *self-orthogonal CSS code* is the common eigenspace of these commuting Paulis.

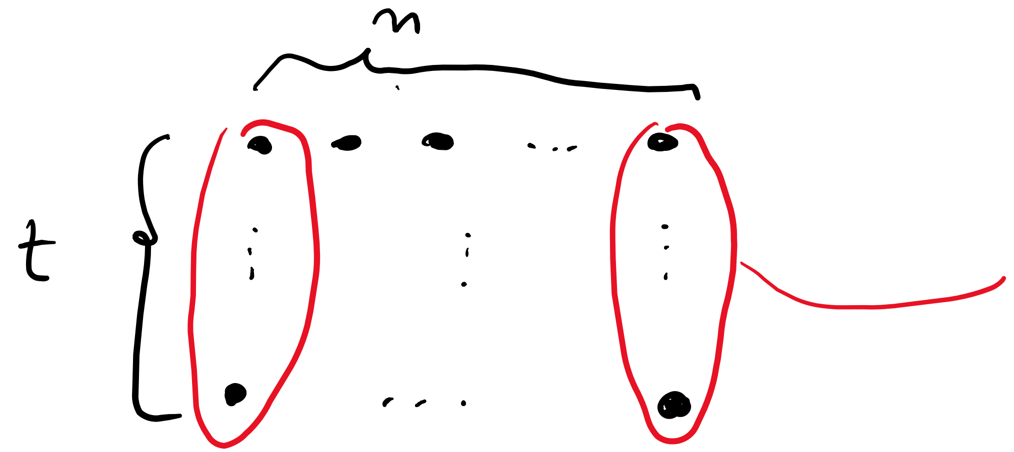
The commutant

Theorem [Nezami, Walter, DG 18]

Commutant generated by tensor powers of:

- Finite orthogonal transformations
- Self-orthogonal CSS code projections

- Transversal! ☺
- Not *quite* a group.
  rep-theoretic consequences → later.

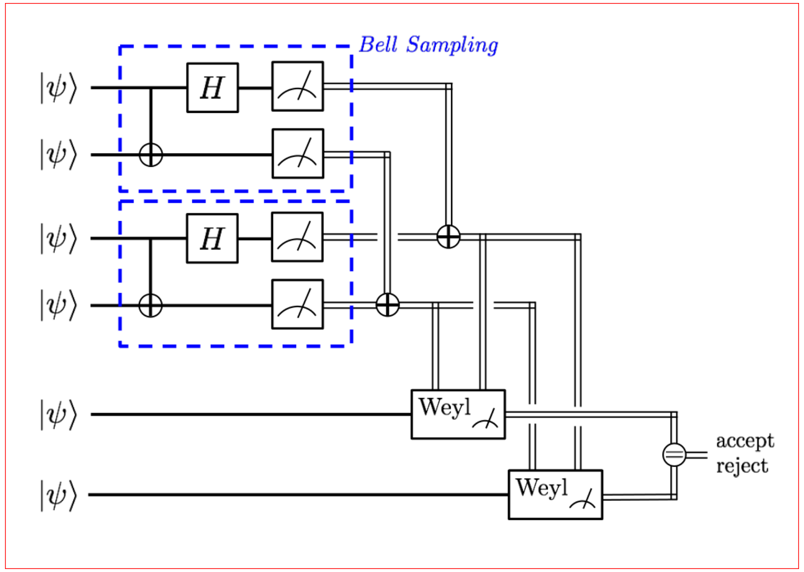


orthogonal $O^{\otimes n}$

or

CSS code projector $P^{\otimes n}$.

Applications

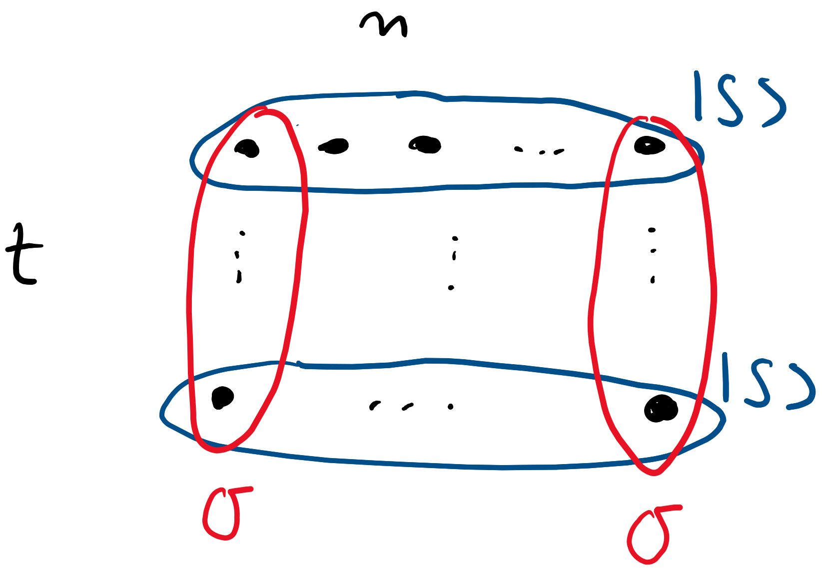


Stabilizer states have additional symmetry

Consider stabilizer state $|s\rangle$ on n qudits...

...and its t-th tensor power.

> Tensor powers of stabilizer states are invariant under the stochastic orthogonal group.



Proof: True for $|S\rangle = |0,\ldots,0\rangle$:

$$\left|\sigma \begin{bmatrix}0 & \cdots & 0\\ \vdots & & \vdots \\ 0 & \cdots & 0\end{bmatrix}\right\rangle = \left|\begin{bmatrix}0 & \cdots & 0\\ \vdots & & \vdots \\ 0 & \cdots & 0\end{bmatrix}\right\rangle$$

$$\Rightarrow \sigma^{\otimes n}|S\rangle^{\otimes t} = \sigma^{\otimes n} U^{\otimes t}|\underline{0}\rangle^{\otimes t}$$

$$= U^{\otimes t}\sigma^{\otimes n}|\underline{0}\rangle^{\otimes t} = |S\rangle^{\otimes t}$$

□

Application 1: Stabilizer testing

Thm. [Nezami, Walter, DG 18]

Let ψ be state on *n* qubits.

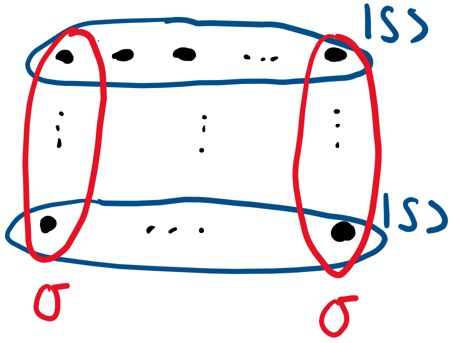
Measure projection on $(+1)$-eigenspace of anti-identity on $\psi^{\otimes 6}$.

If ψ is stabilizer, will accept with $p = 1$.

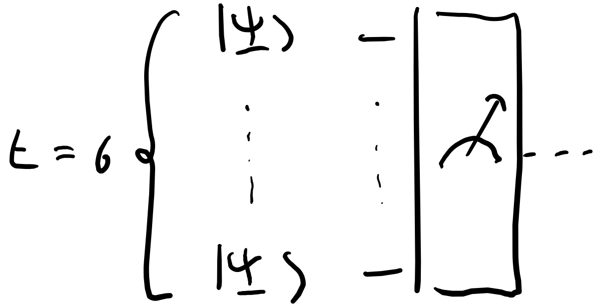
If

$$\max_S |\langle \psi | S \rangle|^2 \leq 1 - \epsilon,$$

accepts with $p \leq 1 - 4\epsilon$.



$$\bar{\mathbb{1}}_{6\times 6} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$



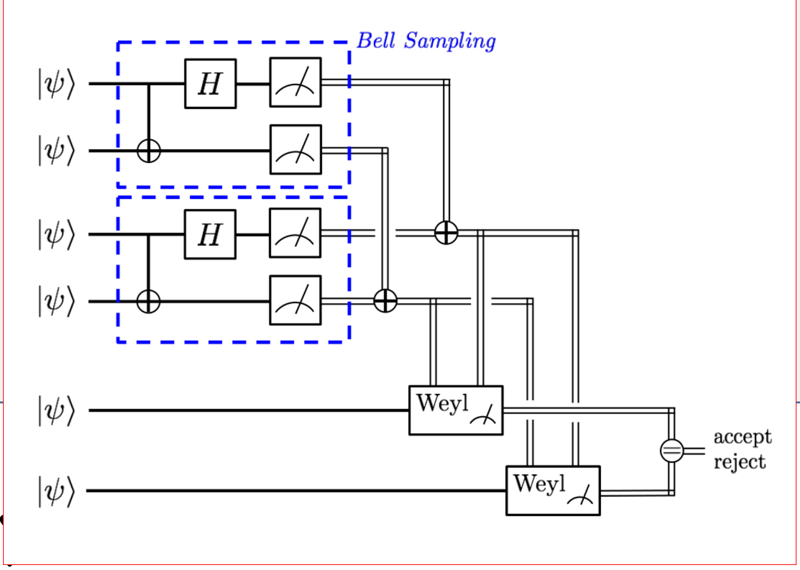
accept if find $+1$-eigenspace of $\bar{\mathbb{1}}^{\otimes n}$

Application 1: Stabilizer testing

Thm. [Nezami, Walter, DG 18]

Stabilizer Testing:

- Solves previous open problem
- Optimal in terms of degree $t = 6$, and in terms of error probability
- Works also for testing Cliffordness
- Has transversal circuit



accepts with $p \leq 1 - 4\epsilon$.

Application 2: Robust Hudson

Thm. [Nezami, Walter, DG 18]
Pure ψ on n qudits, d odd.

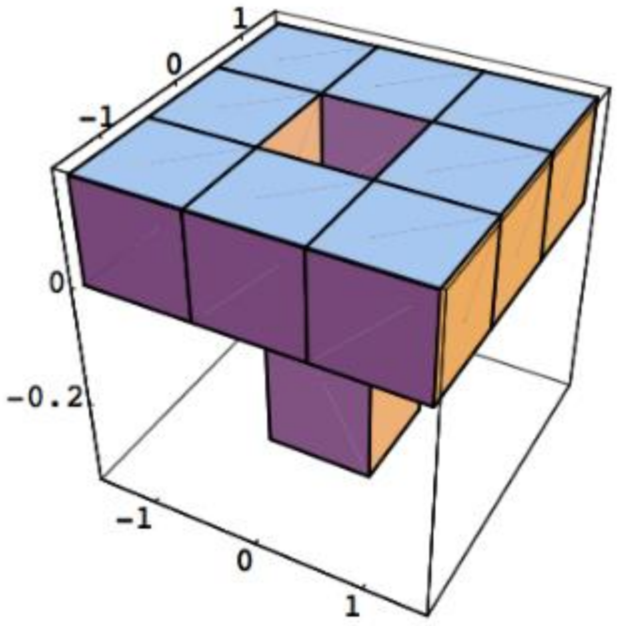
Wigner sum negativity for pure state:

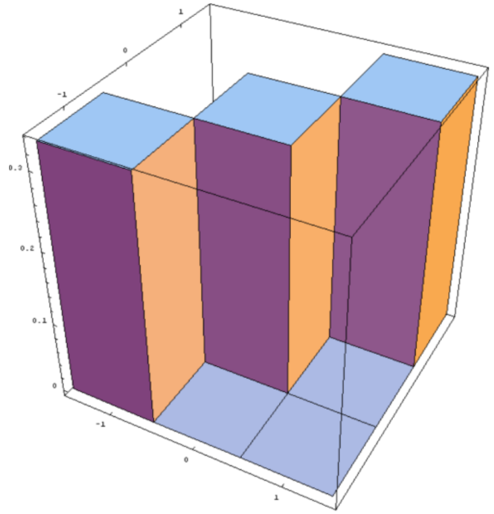
$$\mathrm{sn}(\psi) = \sum_{v, W_\psi(v) \leq 0} |W_\psi(v)|.$$

Then

$$\max_S |\langle \psi | S \rangle|^2 \leq 1 - d^2 \, \mathrm{sn}(\psi),$$

independent of n.





Application 3: exponential de Finetti

Thm.

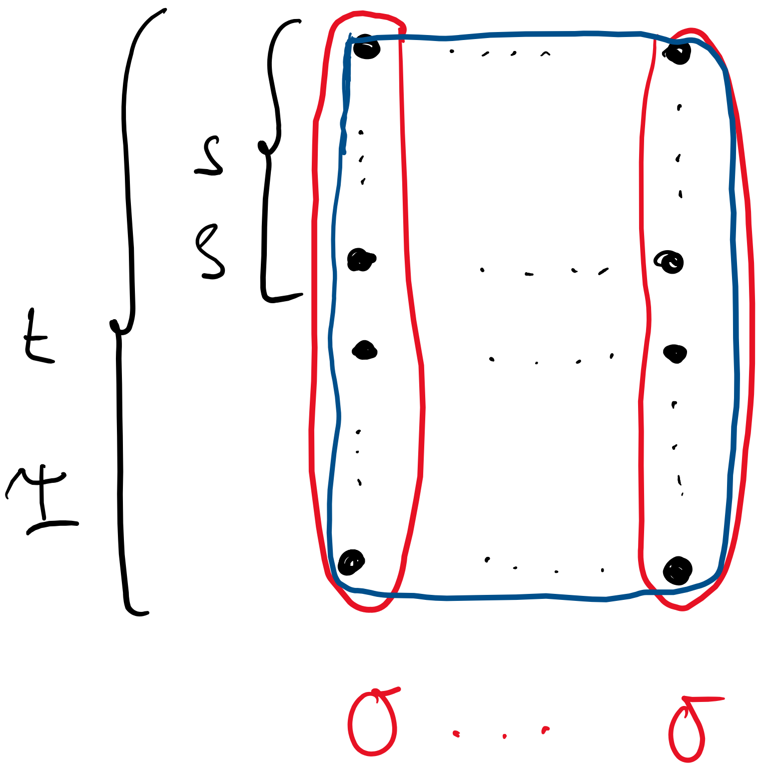
Let $\psi \in \left(\mathbb{C}^{2^n}\right)^{\otimes t}$ be invariant under stochastic orthogonal group.

Let ρ be the reduction to the first s copies.

There is a distribution over stabs s.t.:

$$\left\| \rho - \sum_S |S\rangle\langle S|^{\otimes s} p(S) \right\|_{tr}$$

$$\leq \exp\left(n^2 - (t-s) \right)$$



Finite analogue of [Leverrier 2017]

Application 4: Stabilizer rank

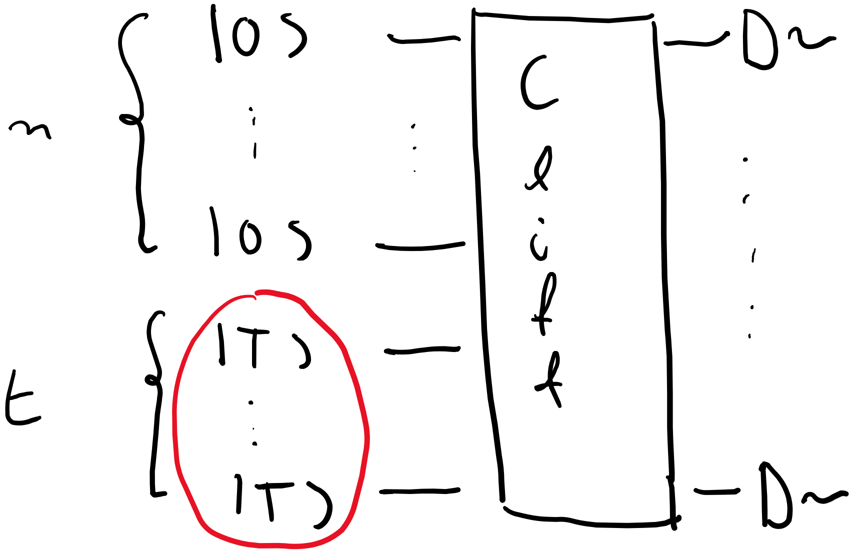
Theorem [Nezami, Walter, DG 18; Zhu, Grassl, Kueng, DG 16]

- For $t \leq 5$, the powers $|S\rangle^{\otimes t}$ of stabilizer states span symmetric space $\mathrm{Sym}^t(\mathbb{C}^{2^n})$.
- This fails for $t \geq 6$.

- For powers of single qubit states:

$$\mathrm{stabrank}(|\psi\rangle^{\otimes 5}) \leq \dim \mathrm{Sym}^5(\mathbb{C}^2) = 6 \ll 2^5 = 32.$$

- ⇒ Best-known general bound on stabilizer rank.



[Bravyi, Browne, Calpin, Campbell, Gosset, Howard]

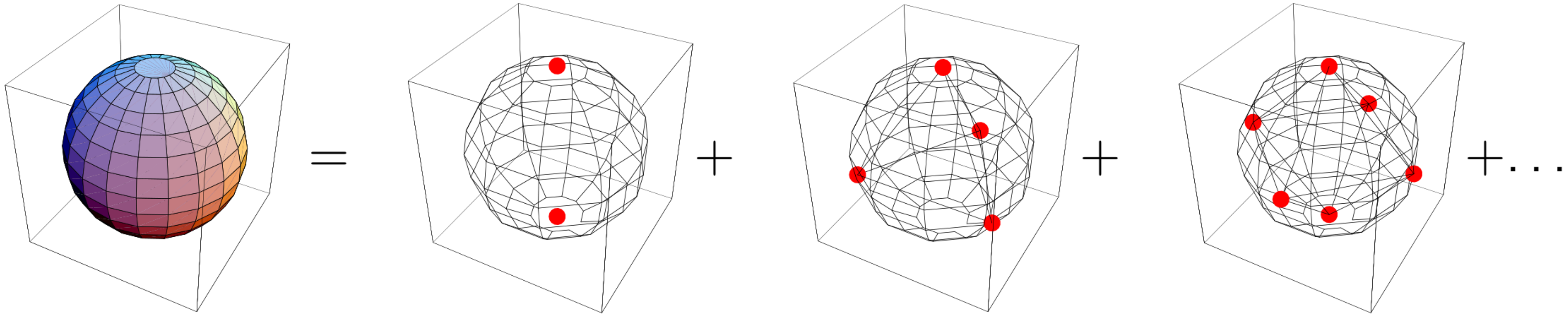
Application 5: Designs

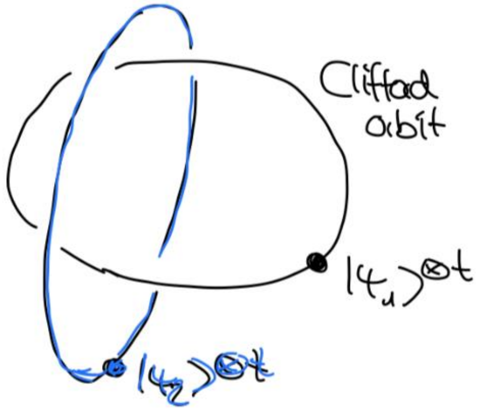
Def.: *t*-designs

finite set of points on sphere / unitaries that reproduce t-th moments.

Theorem [Nezami, Walter, DG 18]

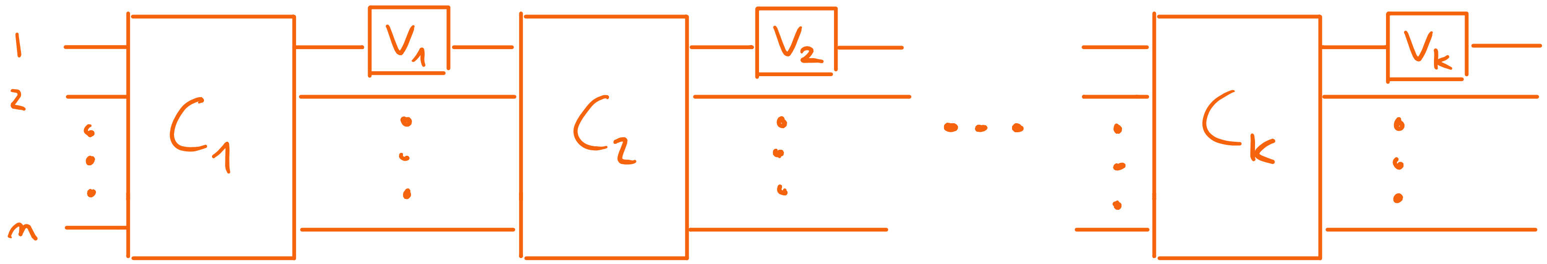
- Can construct exact t-designs from $\boldsymbol{n}$**-independent** number of Clifford orbits





Application 6: Quantum Homeopathy

Cliffords form unitary 3-design.   How many non-Cliffords have to be added to upgrade it to t-design?



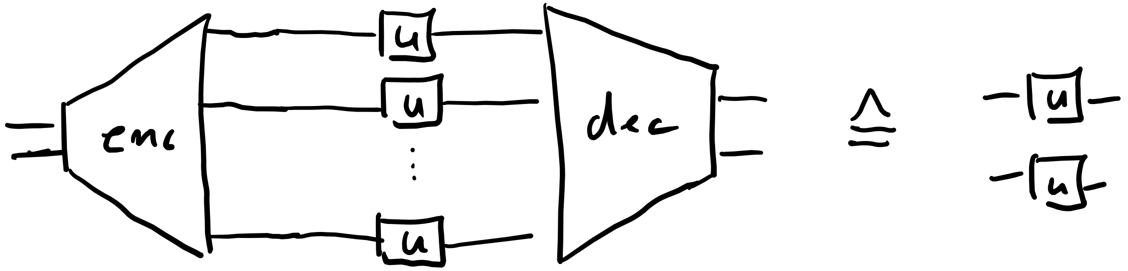
Theorem [HMHGER]: The family of circuits above is an ε-approximate design if

$$k = O(t^4 \log^2 t \log 1/\epsilon)$$

#non-Clifford gates *independent* of n!

Haferkamp, Montealegre-Mora, Heinrich, DG, Eisert, Roth. *Quantum homeopathy works: Efficient unitary designs with a system-size independent number of non-Clifford gates.* arXiv:2002:09524 (2020)

Representation-theoretic version



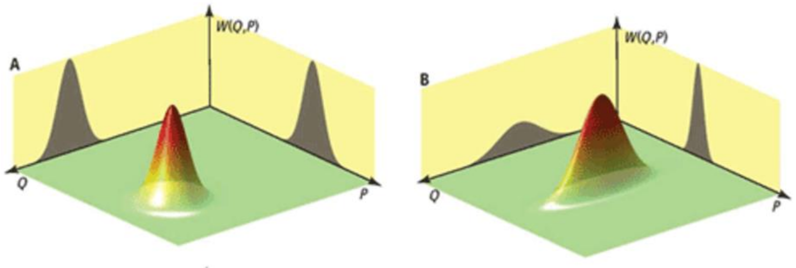
Use symplectic picture

Fact: In odd dimensions, Clifford group (up to Paulis) is *metaplectic representation*

$$\mu: \mathrm{Sp}\left(\mathbb{Z}_d^{2n}\right) \rightarrow U(\mathcal{H})$$

of a finite symplectic group.

Close analogue to canonical maps on phase space.



Howe-Kashiwara-Vergne Duality – CV

- Consider *metaplectic* representation

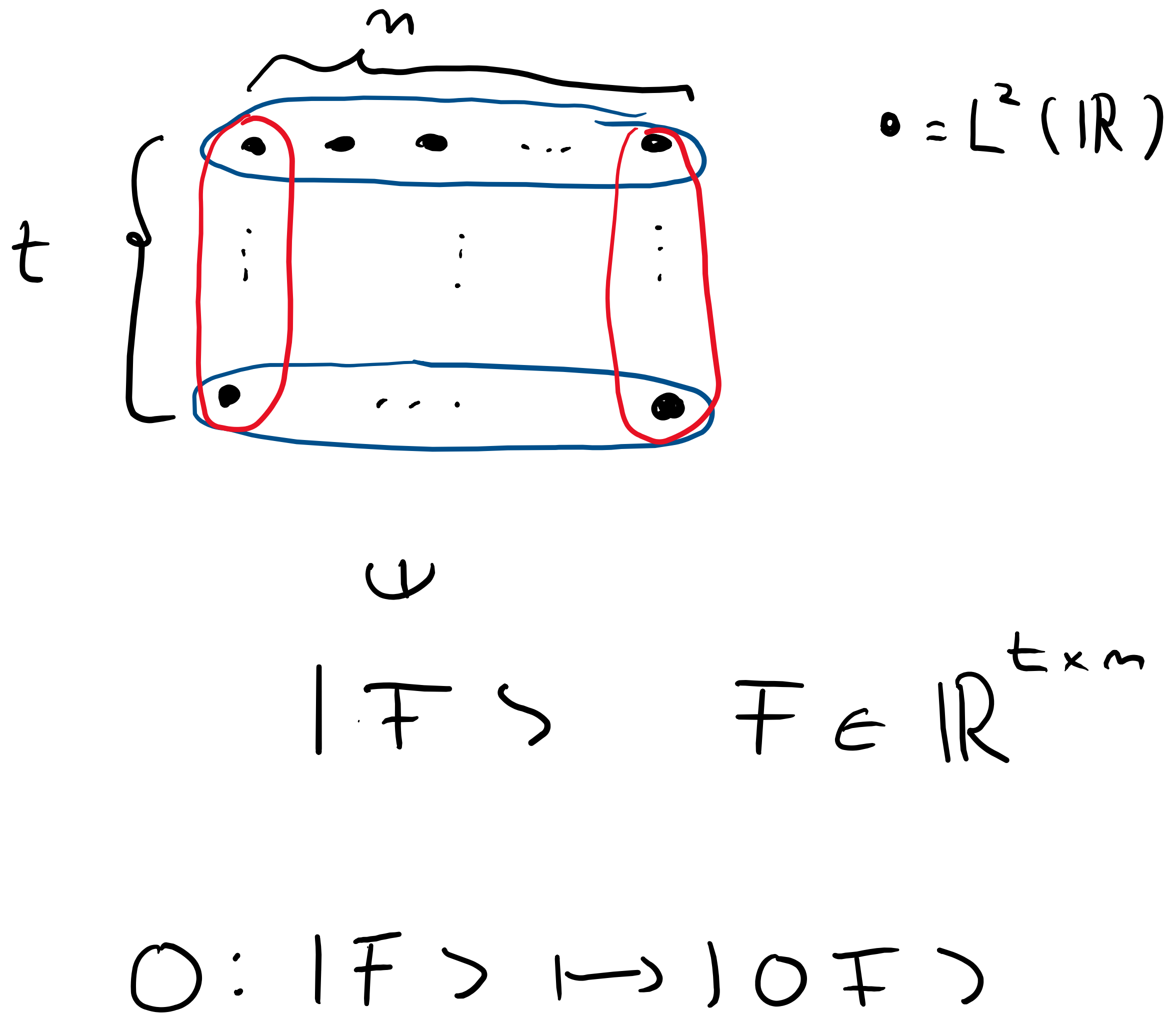
$$\mathcal{H} = L^2(\mathbb{R}^n), \quad \mu: \mathrm{Sp}(\mathbb{R}^{2n}) \to U(\mathcal{H})$$

- Tensor power $\mu^{\otimes t}$…
- …commutes with O(t) ⊃ S_t.

- Under $O(t) \times \mathrm{Sp}(\mathbb{R}^{2n})$:

$$\mathcal{H}^{\otimes t} \simeq \bigoplus_{\tau} \tau \otimes \Theta(\tau)$$

- τ irrep of $O(t)$, $\Theta(\tau)$ irrep of $\mathrm{Sp}(\mathbb{R}^{2n})$.



H-K-V Duality – finite, and odd, dimensions

- Consider *metaplectic* representation

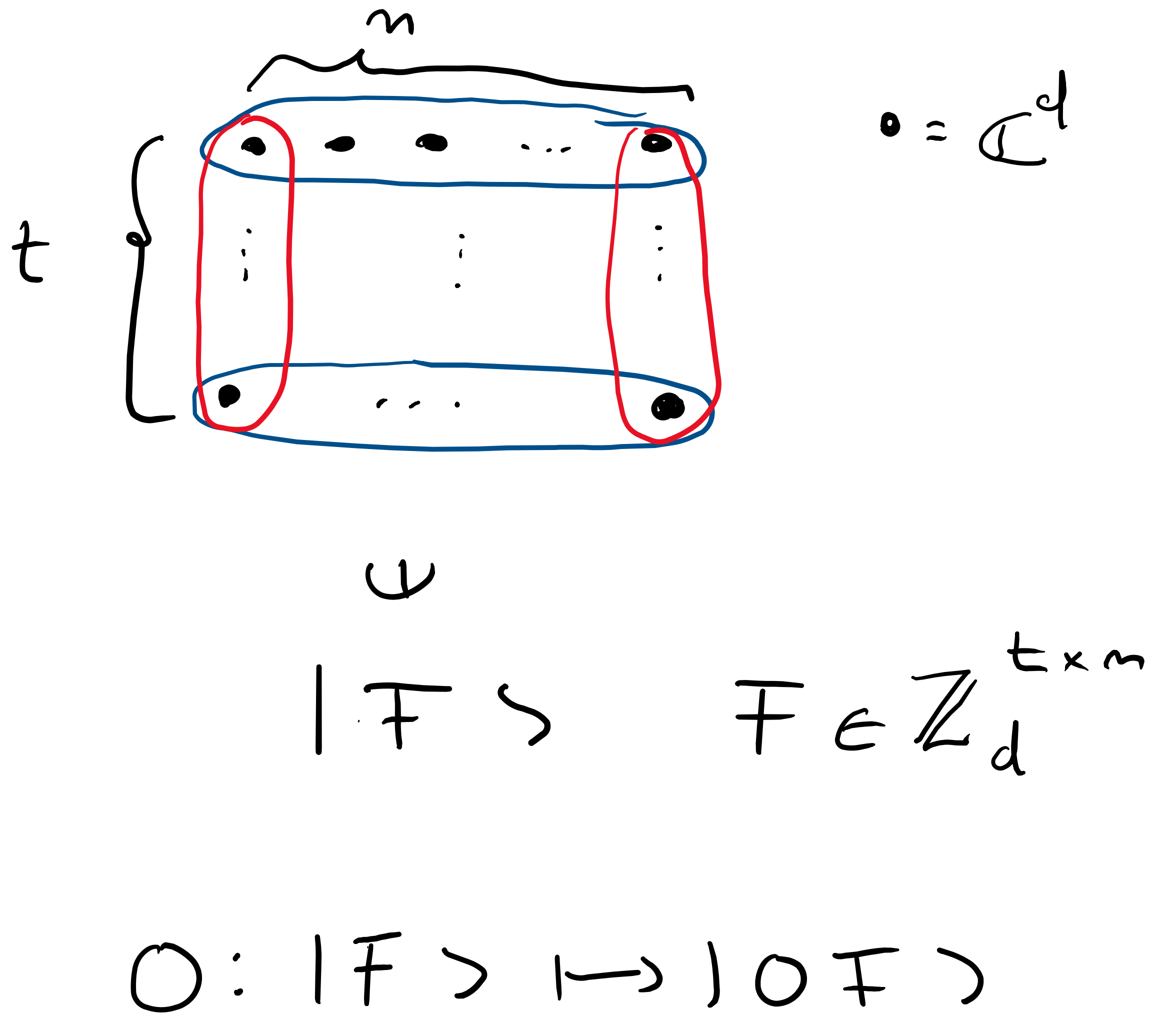
$$\mathcal{H} = (\mathbb{C}^d)^{\otimes n}, \quad \mu: \mathrm{Sp}\left(\mathbb{Z}_d^{2n}\right) \to U(\mathcal{H})$$

- Tensor power $\mu^{\otimes t}$…
- …commutes with O(t) $\supset S_t$.

- Under $O(t) \times \mathrm{Sp}\left(\mathbb{Z}_d^{2n}\right)$:

$$\mathcal{H}^{\otimes t} \simeq \bigoplus_{\tau} \tau \otimes \Theta(\tau)$$

- τ irrep of $O(t)$, $\Theta(\tau)$ reducible.



H-K-V Duality – finite, and odd, dimensions

$$\mathcal{H}^{\otimes t} \simeq \bigoplus_{\tau} \tau \otimes \Theta(\tau)$$

- τ irrep of $O(t)$, $\Theta(\tau)$ reducible.
- Failure of Howe duality over finite fields known since 70s…
- …building on Nezami-Walter-DG, Gurevich-Howe 2016…
- we can reduce out this space ☺

Theorem [Montealegre, DG 2019]

$$\mathcal{H}^{\otimes t} \simeq \bigoplus_{r} \bigoplus_{\tau} \eta(\tau) \otimes \mathrm{Ind}(\eta(\tau))$$

Rank of Sp(V)-representations

- Sp(V) contains a large Abelian subgroup

$$\begin{bmatrix} \mathbb{1} & A_1 \\ 0 & \mathbb{1} \end{bmatrix} \begin{bmatrix} \mathbb{1} & A_2 \\ 0 & \mathbb{1} \end{bmatrix} = \begin{bmatrix} \mathbb{1} & A_1 + A_2 \\ 0 & \mathbb{1} \end{bmatrix}$$

- ⇒ Restriction of any rep π to Abelian group decomposes Hilbert space into 1D irreps:

$$\pi \begin{pmatrix} \mathbb{1} & A \\ 0 & \mathbb{1} \end{pmatrix} |\Phi_B\rangle = \exp(i \operatorname{tr} AB)\, |\Phi_B\rangle$$

Def.: $\operatorname{rank} \pi = \max_B \operatorname{rank} B$

[Gurevich-Howe 2017]

The rank of Sp(V)-representations

Def.: $\operatorname{rank} \pi = \max_B \operatorname{rank} B$

Fact.: The rank of $\mu^{\otimes t}$ is t.

$\underline{t=1:}$

$$\mu \begin{pmatrix} \mathbb{1} & A \\ 0 & \mathbb{1} \end{pmatrix} |\underline{x}\rangle = \omega^{(\underline{x}, A\underline{x})} |\underline{x}\rangle = \omega^{\operatorname{tr} A \underbrace{\underline{x}\underline{x}^T}_{B}} |\underline{x}\rangle.$$

($\underline{x} \in \mathbb{Z}_d^n$)

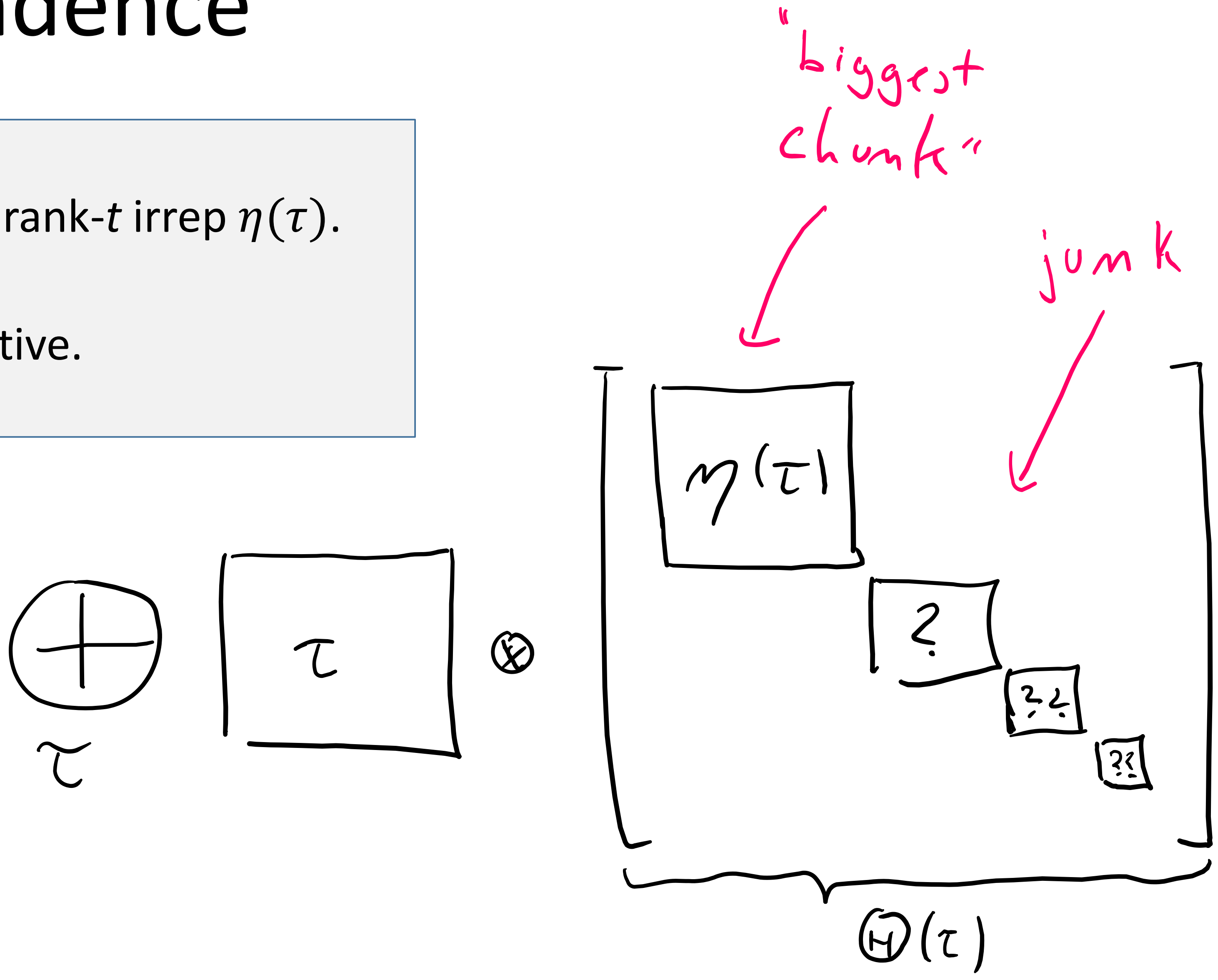
[Gurevich-Howe 2017]

The η-correspondence

Thm [Gurevich-Howe 2017]

- $\Theta(\tau)$ contains exactly one rank-t irrep $\eta(\tau)$.
- The map $\tau \mapsto \eta(\tau)$ is injective.

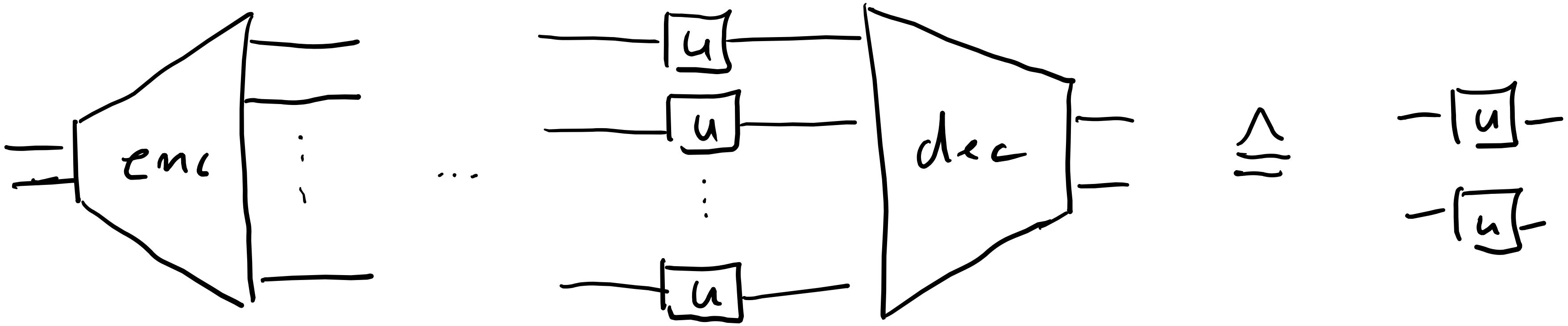
$$\mathcal{H}^{\otimes t} \simeq \bigoplus_{\tau} \tau \otimes \Theta(\tau)$$



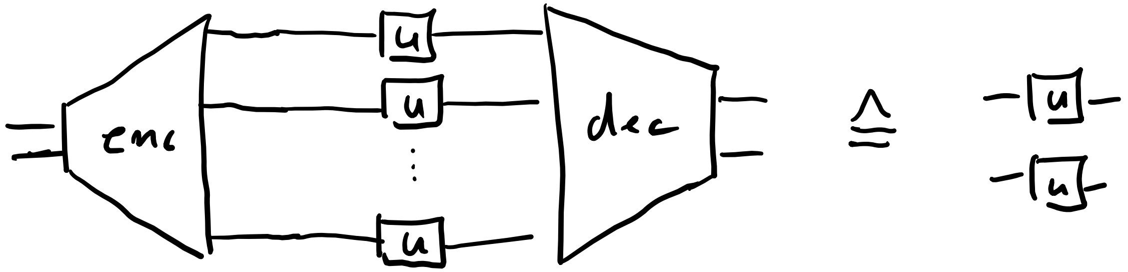
Where do the rank-deficient reps come from?

Idea: Can one "imbed lower tensor powers into *t*-th tensor power"?

…that's what transversal gates on quantum codes do!



…from CSS codes!



Thm [Montealegre-Mora, DG]

Let $N \subset \mathbb{Z}_d^t$ be isotropic, let C_N be the associated CSS code.

- Then $C_N^{\otimes t}$ is isomorphic to $\mu^{\otimes s}$, $s = t - 2\dim N$.
- All rank-deficient subreps arise this way! ;-)

$$\Sigma^{\otimes t} = \left[\bigoplus_{\tau \in \mathrm{Irr}\, O_t} \tau \otimes \eta(\tau) \right] \oplus \left[\bigoplus_{\tau \in \mathrm{Irr}\, O_{t-2}} \tau \otimes \eta(\tau) \right] \otimes \mathbb{C}^{\# N\text{'s}} \oplus \dots$$

irred. and inequ.

Outlook

For the future:

- Treat the representation spaces also for qubits (not just in odd dimensions)
- Results assume $n \geq t$ (the *stable range*). Work on that.
- Quantum info applications of duality?

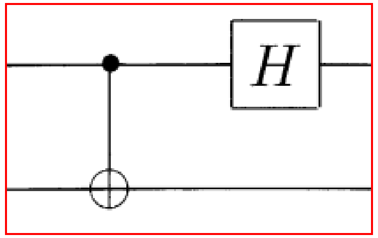
$$\mathcal{H}^{\otimes t} = \left[\bigoplus_{\tau \in \mathrm{Irr}\, O_t} \tau \otimes \eta(\tau) \right] \oplus \left[\bigoplus_{\tau \in \mathrm{Irr}\, O_{t-2}} \tau \otimes \eta(\tau) \right] \otimes \mathbb{C}^{\# N's} \oplus \cdots$$

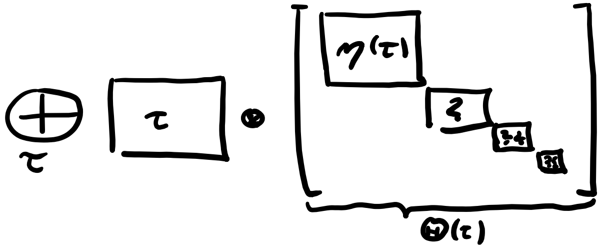
Summary

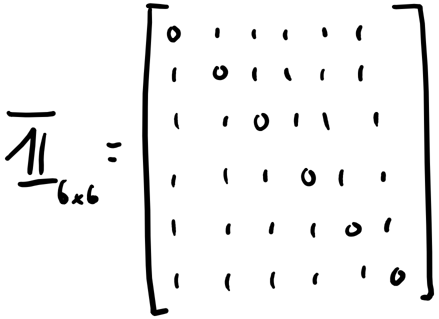
We have

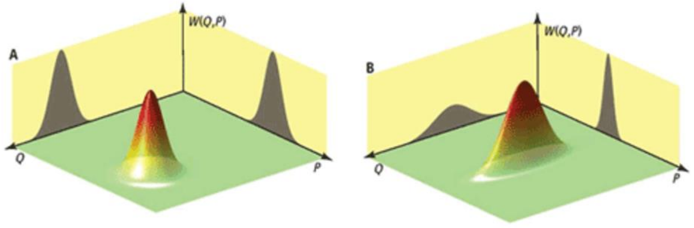
- …worked out the commutant algebra of powers of the Clifford group
- …have found, and continue to find, many applications
- …made progress on the failure of Howe-Kashiwara-Vergne Duality for finite dimensions

Thank you!









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With: Sepehr Nezami, Michael Walter, Felipe Montealegre, Huangjun Zhu