The algebraic side of MIP* $=$ RE

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Connes embedding problem

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Theorem (Ji-Natarajan-Vidick-Wright-Yuen)

$MIP^* = RE$

Connes embedding problem

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MIP = RE*

Corollary

Connes embedding is false.

Connes embedding problem

Connes embedding problem: does every separable finite von Neumann algebra embeds in $R^{\mathcal{U}}$?

Theorem (Ji-Natarajan-Vidick-Wright-Yuen)

MIP = RE*

Corollary

Connes embedding is false.

Aim of this (expository) talk: try to bring the worlds of von Neumann algebras and nonlocal games together

Algebras

Defn. **Algebra**. Vector space $\mathcal{A}$ with a bilinear multiplication

$$\mathcal{A} \times \mathcal{A} \to \mathcal{A} : (a, b) \mapsto a \cdot b.$$

An algebra is

- *associative* if $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, and
- *unital* if there is an element $1 \in \mathcal{A}$ with $1 \cdot a = a \cdot 1 = a$.

Examples:

- $M_n(\mathbb{C})$: $n \times n$ matrices
- $B(H)$: bounded linear operators on a Hilbert space H

Algebras ct'd

Examples:

- $M_n(\mathbb{C})$: $n \times n$ matrices
- $B(H)$: bounded linear operators on a Hilbert space H

Both of these examples have another operation $x \mapsto x^*$ (where x^* is the adjoint)

An algebra with an antilinear multiplication-reversing operation

$$\mathcal{A} \to \mathcal{A} : x \mapsto x^*$$

is called a $*$-**algebra**.

In this talk, algebra = unital associative $*$-algebra

Homomorphisms

Defn. **Homomorphism**. A linear function $\phi : \mathcal{A} \to \mathcal{B}$ between two algebras $\mathcal{A}$ and $\mathcal{B}$ such that all the operations of the algebra are preserved:

- $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in \mathcal{A}$,
- $\phi(a^*) = \phi(a)^*$ for all $a \in \mathcal{A}$, and
- $\phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$.

Example

$$M_n(\mathbb{C}) \to M_n(\mathbb{C}) \otimes M_k(\mathbb{C}) : x \mapsto x \otimes 1$$

is an algebra homomorphism.

Quotients

Defn. **Ideal**. A subspace $\mathcal{I}$ of an algebra $\mathcal{A}$ such that

$$\text{if } x \in \mathcal{I}, a \in \mathcal{A} \text{ then } ax, xa, x^* \in \mathcal{I}.$$

If $\mathcal{I}$ is an ideal of $\mathcal{A}$, then quotient space $\mathcal{A}/\mathcal{I}$ is also an algebra.

The **ideal generated by** $x_1, \ldots, x_n \in \mathcal{A}$ is the smallest ideal in $\mathcal{A}$ containing $x_1, \ldots, x_n$. (This could be $\mathcal{A}$ itself).

Given $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathcal{A}$, can make new algebra where $x_i = y_i$ by taking quotient

$$\mathcal{A}/(x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)$$

Finitely presented algebras

Another familiar algebra: polynomials $\mathbb{C}[x_1, \ldots, x_n]$

Noncommutative polynomials $\mathbb{C}\langle x_1, \ldots, x_n \rangle$:

like polynomials, but $x_i x_j \neq x_j x_i$

There's also a $*$-algebra version:

$$\mathbb{C}^*\langle x_1, \ldots, x_n \rangle = \mathbb{C}\langle x_1, \ldots, x_n, x_1^*, \ldots, x_n^* \rangle$$

Can get lots of new examples of algebras by taking quotients:

if $f_1, \ldots, f_m, g_1, \ldots, g_m \in \mathbb{C}^*\langle x_1, \ldots, x_n \rangle$, then

$$\mathbb{C}^*\langle x_1, \ldots, x_n : f_1 = g_1, \ldots, f_m = g_n \rangle :=$$
$$\mathbb{C}^*\langle x_1, \ldots, x_n \rangle / (f_1 - g_1, \ldots, f_m - g_m)$$

An algebra of this form is called a **finitely presented algebra**.

Finitely presented algebras ct'd

Finitely presented algebra. Algebra of the form

$$\mathbb{C}^*\langle x_1, \ldots, x_n : f_1 = g_1, \ldots, f_m = g_n \rangle := \mathbb{C}^*\langle x_1, \ldots, x_n \rangle / (f_1 - g_1, \ldots, f_m - g_m)$$

Examples:

- Algebra of a unitary: $\mathbb{C}^*\langle u : u^* u = uu^* = 1 \rangle$.
- Algebra of a self-adjoint projection: $\mathbb{C}^*\langle p : p^* = p = p^2 \rangle$.
- Algebra of a k-outcome projective measurement: $\mathbb{C}^*\langle p_1, \ldots, p_k : p_i^* = p_i = p_i^2, p_i p_j = 0 \text{ if } i \neq j, \sum_i p_i = 1 \rangle$.
- Clifford algebra of rank n: $\mathbb{C}^*\langle x_1, \ldots, x_n : x_i^* = x_i, x_i^2 = 1, x_i x_j = -x_j x_i \text{ if } i \neq j \rangle$.

Representations of finitely presented algebras

Homomorphisms $\mathcal{A} \to M_n(\mathbb{C})$ are called **representations** of $\mathcal{A}$

Let $\mathcal{A} = \mathbb{C}^*\langle x_1, \ldots, x_n : f_1 = g_1, \ldots, f_m = g_m \rangle$.

Nice fact about finitely presented algebras:

> Homomorphisms $\mathcal{A} \to \mathcal{B}$ correspond to tuples $(b_1, \ldots, b_k) \in \mathcal{B}$ such that $f_i(b_1, \ldots, b_k) = g_i(b_1, \ldots, b_k)$, $i = 1, \ldots, m$.

Example

$\mathcal{A} = \mathbb{C}^*\langle p : p^* = p = p^2 \rangle$.

$\mathcal{B} = M_n(\mathbb{C})$.

A homomorphism $\mathcal{A} \to \mathcal{B}$ is an element $P \in M_n(\mathbb{C})$ such that $P^* = P = P^2$, i.e. a projection

Finite-dimensional algebras

If $\phi : \mathcal{A} \to \mathcal{B}$ is injective, can think of $\mathcal{A}$ as a subset of $\mathcal{B}$

Say $\mathcal{A}$ **embeds** in $\mathcal{B}$

Matrices are nice algebras... what algebras $\mathcal{A}$ embed in $M_n(\mathbb{C})$?

These are called **matrix algebras**

$\mathcal{A}$ is a matrix algebra if and only if $\mathcal{A}$ is finite-dimensional

Is every algebra a matrix algebra?

No, easy to find non-finite-dimensional algebras

Can't use matrices to understand algebras in general.

RFD algebras

Can't use matrices to understand algebras in general... or can we?

Can only work with finitely many elements of $\mathcal{A}$ at a given time, so:

Definition

Algebra $\mathcal{A}$ *is* **residually finite-dimensional (RFD)** *if for every finite subset* F *of* $\mathcal{A}$, *there is a homomorphism*

$$\phi : \mathcal{A} \to M_n(\mathbb{C}),$$

for some $n \geq 1$, *such that* $\phi(x) \neq \phi(y)$ *for all* $x \neq y \in F$.

RFD algebras

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Unfortunately, there are non-RFD algebras as well

Approximate representations

Can't use matrices to understand algebras in general... or can we?

Maybe we are willing to tolerate some noise in our representation:

Definition

Let $\mathcal{A} = \mathbb{C}^*\langle x_1, \ldots, x_n : f_1 = g_1, \ldots, f_m = g_m \rangle$.

An ϵ**-representation** *of* $\mathcal{A}$ *in* $M_n(\mathbb{C})$ *is a tuple* $(b_1, \ldots, b_n) \in M_n(\mathbb{C})$ *such that*

$$\|f_i(b_1, \ldots, b_n) - g_i(b_1, \ldots, b_n)\|_{hs} \leq \epsilon$$

for all $i = 1, \ldots, k$.

$$\|A\|_{hs} = \sqrt{\frac{\mathrm{tr}(A^* A)}{n}},$$ the little Hilbert-Schmidt norm

Interruption: approx rep'ns and nonlocal games

Non-local game given by:

input and output sets $\mathcal{I}_A = \mathcal{I}_B = I$, $\mathcal{O}_A = \mathcal{O}_B = O$, and

winning predicate $V : \mathcal{O}_A \times \mathcal{O}_B \times \mathcal{I}_A \times \mathcal{I}_B \to \{Win, Lose\}$.

A nonlocal game $\mathcal{G}$ is **synchronous** if whenever Alice and Bob receive the same inputs, they win if and only if their outputs are the same

Synchronous algebra $\mathcal{A}(\mathcal{G})$ of $\mathcal{G}$ is

$$\mathcal{A}(\mathcal{G}) := \mathbb{C}^*\langle m_a^x, x \in I, a \in O : \sum_{a \in O} m_a^x = 1 \text{ for all} x$$
$$(m_a^x)^* = m_a^x = (m_a^x)^2 \text{ for all } x, a$$
$$m_a^x m_b^y = 0 \text{ if } V(a, b|x, y) = Lose\rangle$$

Interruption: approx rep'ns and nonlocal games ct'd

$\mathcal{G}$ synchronous nonlocal game:

$$\mathcal{A}(\mathcal{G}) := \mathbb{C}^*\langle m_a^x, x \in I, a \in O : \sum_{a \in O} m_a^x = 1 \text{ for all} x$$
$$(m_a^x)^* = m_a^x = (m_a^x)^2 \text{ for all } x, a$$
$$m_a^x m_b^y = 0 \text{ if } V(a, b | x, y) = Lose\rangle$$

Theorem (Helton-Meyer-Paulsen-Satriano)

Perfect finite-dimensional strategies for $\mathcal{G}$ correspond to finite-dimensional representations $\mathcal{A}(\mathcal{G}) \to M_n(\mathbb{C})$

ϵ-rep'ns of $\mathcal{A}(\mathcal{G}) \equiv O(\text{poly}(\epsilon))$-perfect strategies for $\mathcal{G}$
(conjecture based on S.-Vidick)

Approximate representations

Can't use matrices to understand algebras in general... or can we?

Maybe we are willing to tolerate some noise in our representation:

Definition

Let $\mathcal{A} = \mathbb{C}^*\langle x_1, \ldots, x_n : f_1 = g_1, \ldots, f_m = g_m \rangle$.

An ϵ-**representation** *of* $\mathcal{A}$ *in* $M_n(\mathbb{C})$ *is a tuple* $(b_1, \ldots, b_n) \in M_n(\mathbb{C})$ *such that*

$$\|f_i(b_1, \ldots, b_n) - g_i(b_1, \ldots, b_n)\|_{hs} \leq \epsilon$$

for all $i = 1, \ldots, k$.

$$\|A\|_{hs} = \sqrt{\frac{\operatorname{tr}(A^* A)}{n}},$$ the little Hilbert-Schmidt norm

Approximate representations ct'd

Definition (Wrong!)

Let $\mathcal{A} = \mathbb{C}^*\langle x_1, \ldots, x_n : f_1 = g_1, \ldots, f_m = g_m \rangle$.

$\mathcal{A}$ *is* **approximable by matrix algebras** *if*

for every finite subset F *of* $\mathcal{A}$,

there is an ϵ*-representation* $(b_1, \ldots, b_n)$ *of* $\mathcal{A}$ *such that*

$$f(b_1, \ldots, b_n) \neq g(b_1, \ldots, b_n)$$

for all $f \neq g \in F$

Problem: what should we set ϵ to?

Approximate representations ct'd

Definition (Wrong!)

Let $\mathcal{A} = \mathbb{C}^*\langle x_1, \ldots, x_n : f_1 = g_1, \ldots, f_m = g_m\rangle$.

$\mathcal{A}$ *is* **approximable by matrix algebras** *if*

for every $\epsilon > 0$ *and finite subset* F *of* $\mathcal{A}$*,*

there is an ϵ*-representation* $(b_1, \ldots, b_n)$ *of* $\mathcal{A}$ *such that*

$$f(b_1, \ldots, b_n) \neq g(b_1, \ldots, b_n)$$

for all $f \neq g \in F$

Problem: $f(b_1, \ldots, b_n)$ doesn't make sense for elements $f \in \mathcal{A}$ since, i.e., can have $f_1(b_1, \ldots, b_n) \neq g_1(b_1, \ldots, b_n)$

Approximate representations ct'd

Definition (Wrong!)

Let $\mathcal{A} = \mathbb{C}^* \langle x_1, \ldots, x_n : f_1 = g_1, \ldots, f_m = g_m \rangle$.

$\mathcal{A}$ *is* **approximable by matrix algebras** *if*

for every $\epsilon > 0$ *and finite subset* F *of* $\mathbb{C}^* \langle x_1, \ldots, x_n \rangle$,

there is an ϵ*-representation* $(b_1, \ldots, b_n)$ *of* $\mathcal{A}$ *such that*

$$f(b_1, \ldots, b_n) \neq g(b_1, \ldots, b_n)$$

for all $f, g \in F$ *with* $f \neq g$ *in* $\mathcal{A}$

Problem: If $\|f(b_1, \ldots, b_n) - g(b_1, \ldots, b_n)\| \leq O(\epsilon)$, this isn't very meaningful

Approximate representations ct'd

Definition (Correct)

Let $\mathcal{A} = \mathbb{C}^*\langle x_1, \ldots, x_n : f_1 = g_1, \ldots, f_m = g_m \rangle$.

$\mathcal{A}$ *is* **approximable by matrix algebras** *if*
there is a function $\delta : \mathcal{A} \to \mathbb{R}_{\geq 0}$ *such that*

(a) $\delta(a) = 0$ *if and only if* $a = 0$, *and*

(b) *for every* $\epsilon > 0$ *and finite subset* F *of* $\mathbb{C}^*\langle x_1, \ldots, x_n \rangle$, *there is an* ϵ*-representation* $(b_1, \ldots, b_n)$ *of* $\mathcal{A}$ *such that*

$$\|f(b_1, \ldots, b_n) - g(b_1, \ldots, b_n)\|_{hs} \geq \delta(f - g)$$

for all $f, g \in F$.

Technical note: this still may not be a very good definition if the b_i's are not bounded in ϵ-representations

Embedding problems

$\mathcal{A}$ is a matrix algebra if and only if $\mathcal{A}$ embeds in $M_n(\mathbb{C})$

$\implies$ snappy definition

Residual finite-dimensionality, approximable by matrix algebras

$\implies$ not so snappy

The product $\prod_{i \in I} \mathcal{A}_i$ of a collection of algebras $\{\mathcal{A}_i : i \in I\}$ is also an algebra with operations applied componentwise.

$\mathcal{A}$ is RFD if and only if $\mathcal{A}$ embeds in a product $\prod_{i \in I} M_{n_i}(\mathbb{C})$

$\mathcal{A}$ is approximable by matrix algebras iff $\mathcal{A}$ embeds in ???

Embedding problems ct'd

$\mathcal{A}$ is approximable by matrix algebras if and only if $\mathcal{A}$ embeds in

$$\prod_{i=1}^{\infty} M_{n_i}(\mathbb{C}) \; / \; \left((a_i)_{i=1}^{\infty} : \lim_{i \to +\infty} \|a_i\|_{hs} = 0 \right)$$

for some sequence $n_1, n_2, \ldots$

Idea: Choose some sequence of finite sets

$$F_1, F_2, \ldots \subset \mathbb{C}^*\langle x_1, \ldots, x_n \rangle$$

such that $\bigcup F_k$ is dense in $\mathbb{C}^*\langle x_1, \ldots, x_n \rangle$

For each k, choose $(1/k)$-representation $(b_{1k}, \ldots, b_{nk})$ for F_n

Send $x_i \mapsto (b_{ik})_{k=1}^{\infty}$

Direct limits of matrix algebras

$\mathcal{A}$ is approximable by matrix algebras if and only if $\mathcal{A}$ embeds in

$$\prod_{i=1}^{\infty} M_{n_i}(\mathbb{C}) \;/\; \left((a_i)_{i=1}^{\infty} : \lim_{i \to +\infty} \|a_i\|_{hs} = 0 \right)$$

for some sequence $n_1, n_2, \ldots$

Annoying that we have this unknown dimension sequence

What if we replace $M_{n_i}(\mathbb{C})$ with $M_\infty(\mathbb{C}) = \bigcup_{n \geq 1} M_n(\mathbb{C})$?

Problem: $\|\cdot\|_{hs}$ doesn't make sense on $M_\infty(\mathbb{C})$

Direct limits of matrix algebras ct'd

Need a version of $M_\infty(\mathbb{C})$ where $\|\cdot\|_{hs}$ makes sense

Note $\|x\|_{hs}^2 = \tau_n(x^* x)$, where

$\tau_n : M_n(\mathbb{C}) \to \mathbb{C} : x \mapsto \mathrm{tr}(x)/n$ is the normalized trace

$$\tau_{2n}(x \otimes 1) = \tau_{2n}\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = \tau_n(x)$$

Thus we get a chain of embeddings

$$M_2(\mathbb{C}) \hookrightarrow M_4(\mathbb{C}) \hookrightarrow \cdots \hookrightarrow M_{2^k}(\mathbb{C}) \hookrightarrow \cdots$$

where each map preserves the normalized trace (and hence preserves $\|\cdot\|_{hs}$)

Direct limits of matrix algebras ct'd

Get a chain of embeddings

$$M_2(\mathbb{C}) \hookrightarrow M_4(\mathbb{C}) \hookrightarrow \cdots \hookrightarrow M_{2^k}(\mathbb{C}) \hookrightarrow \cdots$$

where each map preserves the normalized trace (and hence preserves $\|\cdot\|_{hs}$)

Let $R_0 = \bigcup_{k \geq 1} M_{2^k}(\mathbb{C})$.

Can talk about trace τ and norm $\|\cdot\|_{hs}$ on R_0.

$\mathcal{A}$ is approximable by matrix algebras iff $\mathcal{A}$ embeds in

$$R_0^{\mathbb{N}} \;/\; \left((a_i)_{i=1}^{\infty} : \lim_{i \to +\infty} \|a_i\|_{hs} = 0 \right).$$

Embeddings in $B(H)$

We've looked at matrix algebras

What algebras embed in $B(H)$?

If $\mathcal{A} \subseteq B(H)$, then operator norm $\|\cdot\|_{op}$ on $B(H)$ restricts to $\mathcal{A}$

> $\mathcal{A} \subseteq B(H)$ iff $\mathcal{A}$ has a norm $\|\cdot\|$ satisfying certain conditions
>
> (pre-C^*-algebra)

This isn't a very practical criterion... a bit better:

Theorem (GNS theorem)

A **state** *f on $\mathcal{A}$ can be used to construct a homomorphism $\phi : \mathcal{A} \to B(H)$ called the* **GNS representation**

The GNS theorem

Theorem (GNS theorem)

A state f on $\mathcal{A}$ can be used to construct a homomorphism $\phi : \mathcal{A} \to B(H)$ called the **GNS representation**

A **state** f on an algebra $\mathcal{A}$ is a linear functional $\mathcal{A} \to \mathbb{C}$ satisfying certain conditions (positivity, $f(1) = 1$, boundedness)

Sometimes it's easier to construct a state on $\mathcal{A}$ than a norm

If GNS rep'n ϕ is an embedding, then f is **faithful**

A state f is **tracial** if $f(ab) = f(ba)$

C^*-algebras and von Neumann algebras

If working with $\mathcal{A} \subseteq B(H)$, might like $\mathcal{A}$ to be closed under limits

If $\mathcal{A} \subseteq B(H)$ is closed, then $\mathcal{A}$ is called a C^*-**algebra**

Given $\mathcal{A} \subseteq B(H)$, the closure $\overline{\mathcal{A}}$ is C^*-algebra containing $\mathcal{A}$

$\overline{\mathcal{A}}$ is called the C^*-**enveloping algebra** of $\mathcal{A}$

Sometimes $\overline{\mathcal{A}}$ is still too small:

> Suppose $W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq H$.
> Let P_i be orthogonal projection onto W_i,
> P be orthogonal projection onto $W := \bigcup W_i$.
> Then $P_i \not\to P$ as $i \to +\infty$,
> even though $\langle v, P_i w\rangle \to \langle v, Pw\rangle$ for all $v, w \in H$.
> Possible to have C^*-algebra $\mathcal{A}$ with $P_i \in \mathcal{A}$ for all i, $P \notin \mathcal{A}$.

von Neumann algebras ct'd

Suppose $W_1 \subsetneq W_2 \subseteq \cdots \subsetneq H$.
Let P_i be orthogonal projection onto W_i,
P be orthogonal projection onto $W := \bigcup W_i$.
Then $P_i \not\to P$ as $i \to +\infty$,
even though $\langle v, P_i w \rangle \to \langle v, Pw \rangle$ for all $v, w \in H$.
Possible to have C^*-algebra $\mathcal{A}$ with $P_i \in \mathcal{A}$ for all i, $P \notin \mathcal{A}$.

If $\mathcal{A} \subseteq B(H)$ contains these and other limits (technically, closed in weak operator topology) then $\mathcal{A}$ is a **von Neumann algebra**

Given $\mathcal{A} \subseteq B(H)$, **von Neumann enveloping algebra** is $\overline{\mathcal{A}}^{WOT}$, the closure in the weak operator topology

Universal property: if $\mathcal{B}$ is von Neumann algebra, any homomorphism $\mathcal{A} \to \mathcal{B}$ extends to homomorphism $\overline{\mathcal{A}}^{WOT} \to \mathcal{B}$.

Algebras we've seen so far

$R_0 = \bigcup_{k \geq 1} M_{2^k}(\mathbb{C})$:

Normalized trace τ on R_0 is a faithful state

$\implies$ GNS representation gives embedding $R_0 \subseteq B(H)$

$R := \overline{R_0}^{WOT}$ is the **(unique) hyperfinite II_1 factor**

What about

$$\mathcal{A} = R_0^{\mathbb{N}} \; / \; \left((a_i)_{i=1}^{\infty} : \lim_{i \to +\infty} \|a_i\|_{hs} = 0 \right).$$

Does $\mathcal{A}$ have a faithful tracial state?

Ultralimits and ultrapowers

Fun to think about the algebra

$$\mathcal{A} = R_0^{\mathbb{N}} \;/\; \left((a_i)_{i=1}^{\infty} : \lim_{i \to +\infty} \|a_i\|_{hs} = 0 \right).$$

Does $\mathcal{A}$ have a faithful tracial state?

Possible definition:

$$\tilde{\tau}((a_i)_{i=1}^{+\infty}) = \lim_{i \to +\infty} \tau(a_i)$$

Problem: limit might not exist (could even be $+\infty$)

Let's try to fix this:

$$R_0^{\mathbb{N}, bdd} = \{(a_i)_{i=1}^{\infty} \in R_0^{\mathbb{N}} : \sup_i \|a_i\|_{op} < +\infty\}.$$

Ultralimits and ultrapowers ct'd

$$R_0^{\mathbb{N},bdd} = \{(a_i)_{i=1}^{\infty} \in R_0^{\mathbb{N}} : \sup_i \|a_i\|_{op} < +\infty\}.$$

Now $(\tau(a_i))_{i=1}^{\infty}$ is bounded, has a convergent subsequence

Fix an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ (these exist by axiom of choice)

Ultralimit $\lim_U x_i$ exists for any bounded sequence $(x_i)_{i=1}^{\infty}$

The **ultrapower**

$$R_0^{\mathcal{U}} := R_0^{\mathbb{N},bdd} \ / \ \left((a_i)_{i=1}^{\infty} : \lim_{\mathcal{U}} \|a_i\|_{hs} = 0 \right).$$

has faithful tracial state

$$\tilde{\tau}\left((a_i)_{i=1}^{\infty}\right) = \lim_{\mathcal{U}} \tau(a_i)$$

and enveloping von Neumann algebra is $R^{\mathcal{U}}$.

Connes embedding problem

Suppose $\mathcal{A}$ is finitely presented algebra such that ϵ-representations are bounded for $\epsilon \leq 1$.

> $\mathcal{A}$ approximable by matrix algebras iff $\mathcal{A}$ embeds in $R^{\mathcal{U}}$

Which algebras embed in $R^{\mathcal{U}}$?

$R^{\mathcal{U}}$ has faithful tracial state

$\implies$ if $\mathcal{A}$ embeds in $R^{\mathcal{U}}$, $\mathcal{A}$ has faithful tracial state

If $\mathcal{A}$ embeds in $R^{\mathcal{U}}$, then $\overline{\mathcal{A}}^{WOT}$ embeds in $R^{\mathcal{U}}$

$\implies$ We might as well assume that $\mathcal{A}$ is von Neumann algebra

Defn. von Neumann algebra is **finite** if it has a faithful tracial state

> Connes embedding problem: Does every separable finite von Neumann algebra embed in $R^{\mathcal{U}}$?

Connes embedding ct'd

Theorem (Helton-Meyer-Paulsen-Satriano, Kim-Paulsen-Schafhauser)

$\mathcal{G}$ synchronous game

- *$\mathcal{G}$ has a perfect C_{qc}-strategy iff $\mathcal{A}(\mathcal{G})$ has a tracial state.*
- *$\mathcal{G}$ has a perfect C_{qa}-strategy iff there is a homomorphism $\mathcal{A}(\mathcal{G}) \to R^{\mathcal{U}}$.*

Theorem (Ji-Natarajan-Vidick-Wright-Yuen)

There is a synchronous nonlocal game $\mathcal{G}$ with a perfect C_{qc}-strategy but no perfect C_{qa}-strategy.

So $\mathcal{A}(\mathcal{G})$ has a tracial state, but no homomorphisms to $R^{\mathcal{U}}$

Connes embedding problem

Connes embedding problem: Does every separable finite von Neumann algebra embed in $R^{\mathcal{U}}$?

JNWVY $\implies$ synchronous game $\mathcal{G}$ such that
$\mathcal{A}(\mathcal{G})$ has a tracial state, but no homomorphisms to $R^{\mathcal{U}}$

Is $\mathcal{A}(G)$ a von Neumann algebra?
No, but let τ be tracial state on $\mathcal{A}(\mathcal{G})$
let $\phi : \mathcal{A}(\mathcal{G}) \to B(H)$ be GNS representation
let $\mathcal{A}_1 = \overline{\phi(\mathcal{A}(\mathcal{G}))}^{WOT}$

$\mathcal{A}_1$ is a separable finite von Neumann algebra with no homomorphisms to $R^{\mathcal{U}}$ $\implies$ strong counterexample to CEP

Computability of embeddings

Word problem for groups undecidable:

$\implies$ "fin.-pres. algebra $\mathcal{A}$ has tracial state?" is coRE-hard

Kharlampovich: "$\mathcal{A}$ has homomorphism to $R^{\mathcal{U}}$" is coRE-hard

S.: coRE-hard to tell if $\mathcal{A}(\mathcal{G})$ has tracial state or morphism to $R^{\mathcal{U}}$

JNWVY: RE-hard to tell if $\mathcal{A}(\mathcal{G})$ has homomorphism to $R^{\mathcal{U}}$

Theorem

Mousavi-Nezhadi-Yuen Π_2^0-complete to tell if three-player nonlocal game has perfect C_{qa}-strategy

Conjecture: it is Π_2^0-complete to tell if $\mathcal{A}(\mathcal{G})$ has morphism to $R^{\mathcal{U}}$

Hyperlinear groups

Let G be a group (set with multiplication, inverses)

Group algebra $\mathbb{C}G$ has faithful tracial state

Definition

A group is **hyperlinear** *if* $\mathbb{C}G$ *embeds in* $R^{\mathcal{U}}$

Hyperlinearity can also be defined concretely in terms of ϵ-rep'ns

If $G = \langle S : R \rangle$ is finitely presented, then $\mathbb{C}G$ is finitely presented

Connes embedding for groups: is every group hyperlinear?

Still seems open.

Why should we care?

Binary constraint system games

Binary constraint system (BCS):

Given by variables $x_1, \ldots, x_n$

and constraints (V_i, f_i), $i = 1, \ldots, m$

where $V_i \subset \{x_1, \ldots, x_n\}$,

$f_i : \{-1,1\}^{V_i} \to \{0,1\}$ are multilinear polynomials

Given BCS, can write down synchronous nonlocal game $\mathcal{G}$

and algebra

$$\begin{aligned}\mathcal{A}(\mathcal{G}) = \mathbb{C}^*\langle x_1, \ldots, x_n : & x_i^* = x_i, x_i^2 = 1 \\ & x_j x_k = x_k x_j \text{ if } j,k \in V_i \text{ for some } i \\ & p_i = 0\rangle.\end{aligned}$$

Mermin-Peres magic square game is example of BCS game where constraint system is linear

BCS-MIP*: one-round proof systems with perfect completeness where protocol is based on BCS game

Surmise: JNWVY $\implies$ BCS-MIP* $=$ RE

We often talk about different classes of BCSs:

linear systems, 3SAT, 2SAT, HORN-SAT

It also makes sense to talk about different subclasses of BCS-MIP*

Classically, 2SAT-MIP, HORN-MIP, LIN-MIP, etc. are all easy

Atserias-Kolaitis-Severini: 2SAT-MIP*, HORN-MIP*, etc. are easy, except perhaps LIN-MIP*

If LIN-MIP* $=$ RE, then it would imply analogue of Schaefer's dichotomy theorem for subclasses of BCS-MIP*

Hyperlinear groups ct'd

If BCS is linear, $\mathcal{A}(\mathcal{G})$ is (central quotient of a) group algebra

S.: if G finitely presented, $\mathbb{C}G$ embeds in $\mathcal{A}(\mathcal{G})$ for some linear system game $\mathcal{G}$

There is a non-hyperlinear group iff

there is a linear system game $\mathcal{G}$ such that $\mathcal{A}(\mathcal{G})$ has tracial state, but no homomorphism to $R^{\mathcal{U}}$

If LIN-MIP* $=$ RE, then there is a non-hyperlinear group

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The end!