Coloring Voronoi cells

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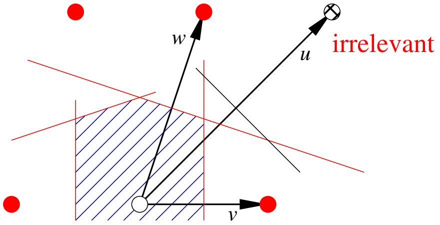
I. Introduction

Lattice and Voronoi polytope

- A subgroup $L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n \subset \mathbb{R}^n$ is a lattice if $\det(v_1, \ldots, v_n) \neq 0$.
- The Voronoi polytope is defined as

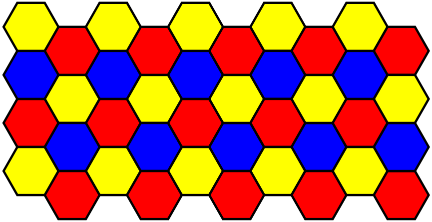
$$\mathcal{V} = \{x \in \mathbb{R}^n \text{ s.t. } \|x\| \leq \|x - v\| \text{ for } v \in L - \{0\}\}$$

- The translates $v + \mathcal{V}$ tile $\mathbb{R}^n$.
- The vectors v defining a facet of $\mathcal{V}$ are named relevant. The set of relevant vectors is named $Vor(L)$.
- Voronoi Theorem: A vector u is relevant if and only if it can not be written as $u = v + w$ with $\langle v, w \rangle \geq 0$.



Problem statement: Coloring lattices

- Given a lattice L, the problem that we consider is choosing a color on each element of L such that if $v - w$ is a relevant vector then v and w have different colors. The minimal number is named $\chi(L)$.
- For the root lattice A_2 just three colors suffice:



- Questions:
  - What is χ for remarkable lattices? (Leech, root lattices, ...)
  - How does χ vary when the lattice is perturbed?
  - What are the possible χ in a fixed dimension n?
  - How does the maximum value of χ depend on n?

II. Tools of the trade

Generalities on chromatic numbers

Finite graphs:

- Given $c > 2$ the problem of checking if graphs on n-vertices are colorable with c colors is NP-hard.
- For each $g > 0$ and $c > 0$ there exist graph with girth at least g and coloring number at least c.
- It is difficult to use graph symmetries since a graph can have many symmetries but colorings with no symmetries.

Lattice case:

- No general algorithm
- If we find a sublattice Λ such that Λ has no relevant vectors and L/Λ can be colored by c colors then $\chi(L) \leq c$.
- Lattice $2L$ does not contain relevant vectors and so $\chi(L) \leq 2^n$.
- If $L = L_1 \oplus_\perp L_2$ then $\chi(L) = \max(\chi(L_1), \chi(L_2))$.

Satisfiability for testing coloring

- Given a graph on n vertices, can it be colored with c colors?
- We defined a number of Boolean $B_{v,i}$ with v a vertex and $1 \leq i \leq c$ a color.
- We have following constraints:
  1. For vertex v adjacent to w we want for any i to have $\overline{B_{v,i}} \wedge \overline{B_{w,i}}$
  2. For any vertex v and colors $i < j$ we should have $\overline{B_{v,i}} \wedge \overline{B_{v,j}}$
  3. For any vertex v we want $B_{v,1} \wedge B_{v,2} \wedge \cdots \wedge B_{v,c}$
- This kind of satisfiability problem can be resolved for example with `minisat`.
- Computational situation:
  - It is `NP` problem, so cannot work for very large problems.
  - Proving `UNSAT` is much harder than `SAT`.
  - Sometimes fails with $n = 100$ and works with $n = 1.6e4$.
- Remark: Satisfaction Modulo Theories (SMT) is a foundation of modern computer technology (see `Z3`).

Lower bounds on chromatic number

Fractional chromatic number

- Denote by $\mathcal{I}_G$ the set of all independent sets of G.
- The fractional chromatic number of G is the solution of the following linear program:

$$\min \left\{ \sum_{I \in \mathcal{I}_G} \lambda_I : \lambda_I \in \mathbb{R}_{\geq 0} \text{ for } I \in \mathcal{I}_G, \sum_{I \in \mathcal{I}_G \text{ with } v \in I} \lambda_I \geq 1 \text{ for } v \in V \right\}$$

- It is still NP-hard, but reasonably fast since we can use symmetries.

Subgraph

- For H an induced subgraph of G we have $\chi(G) \geq \chi(H)$.

Spectral lower bounds

- The advantage of spectral lower bounds is that they are computable in polynomial time.
- But they may not be very good lower bounds.

Spectral lower bounds

- **Hoffman lower bound**: If the eigenvalues of A are $\mu_1 \geq \cdots \geq \mu_n$ then $\chi(G) \geq 1 + \frac{\mu_1}{-\mu_n}$.
  For regular d-graphs, $\mu_1 = d$ and μ_n can be computed as an extremal problem. Numerically, it can be computed by the inflation method.
- **Inertia lower bound**: Denote n_+, n_- the number of positive, negative eigenvalues. We have $\chi(G) \geq 1 + \max\left(\frac{n_+}{n_-}, \frac{n_-}{n_+}\right)$.
- **Ando/Lin lower bound**: Denote $S_+ = \sum_{\mu>0} \mu^2$, $S_- = \sum_{\mu<0} \mu^2$ we have $\chi(G) \geq 1 + \frac{S_+}{S_-}$.
- **Elphick/Wocjan lower bound**: For all $1 \leq m \leq n$ we have

  $$\chi(G) \geq 1 + \frac{\sum_{i=1}^{m} \mu_i}{-\sum_{i=1}^{m} \mu_{n+1-i}}$$

  There are lower bounds that use the diagonal degree matrix.

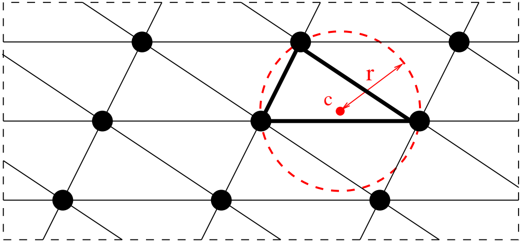
III. Special lattices

Case of the Leech and E_8 lattice

- Def: The Leech lattice Λ is the unique even unimodular lattice without roots in dimension 2.
- The length of vectors of the Leech lattice are 4, 6, 8, ...,
- The length of the relevant vectors of Λ are 4 and 6.
- There exist a copy of $\sqrt{2}\Lambda$ embedded into Λ. It does not contain relevant vectors. Thus $\chi(\Lambda) \leq (\sqrt{2})^{24} = 4096$.
- If there were a better coloring then one of the color class would have density greater than $\frac{1}{4096}$. This color class would give a better packing than Leech lattice. By *Cohn, Kumar, Miller, Radchenko, Viazovska, The sphere packing problem in dimension 24* this is impossible.
- Def: The root lattices are the lattices for which $Vor(L)$ is the set of shortest vectors.
- The irreducible root lattices are A_n, D_n, E_6, E_7, E_8.
- The same method works for the root lattice E_8.

Empty sphere and Delaunay polytopes

- Def: A sphere $S(c,r)$ of center c and radius r in an n-dimensional lattice L is said to be an empty sphere if:
  (i) $\|v-c\| \geq r$ for all $v \in L$,
  (ii) the set $S(c,r) \cap L$ contains $n+1$ affinely independent points.
- Def: A Delaunay polytope P in a lattice L is a polytope, whose vertex-set is $L \cap S(c,r)$.



- Delaunay polytopes define a tessellation of the Euclidean space $\mathbb{R}^n$
- Each Delaunay polytope define a natural induced subgraph.

The root lattices A_n

- Def: The root lattice A_n is defined as

$$\mathsf{A}_n = \left\{ x \in \mathbb{Z}^{n+1} \text{ such that } \sum_{i=1}^{n+1} x_i = 0 \right\}$$

- Its relevant vectors are $e_i - e_j$ for $1 \leq i, j \leq n+1$.
- The preferred basis is $(v_i = e_{i+1} - e_i)_{1 \leq i \leq n}$.
- An index $n+1$ sublattice is defined by

$$v = \sum_{i=1}^{n} \alpha_i v_i \in \mathsf{A}_n \text{ such that } \sum_{i=1}^{n} \alpha_i \equiv 0 \pmod{n+1}$$

  and has no relevant vectors. So, $\chi(\mathsf{A}_n) \leq n+1$.
- The Delaunay polytopes of A_n are for $1 \leq k \leq n$:

$$J(n,k) = \left\{ x \in \{0,1\}^{n+1} \text{ s.t. } \sum_i x_i = k \right\} - ke_1$$

- $J(n,1)$ is a n-dimensional simplex. So $\chi(\mathsf{A}_n) = n+1$.

The root lattices D_n

- The Delaunay polytopes of the lattice $\mathbb{Z}^n$ are translations of $[0,1]^n$.
- Def: The root lattice D_n is

$$\mathsf{D}_n = \left\{ x \in \mathbb{Z}^n \text{ such that } \sum_{i=1}^{n} x_i \equiv 0 \pmod 2 \right\}$$

- The Delaunay polytopes are
  - The cross polytope $\beta_n = \text{conv}\left\{ e_1 \pm e_i \text{ for } 1 \leq i \leq n \right\}$
  - The half cube $\frac{1}{2}H_n = \text{conv}\left\{ x \in \{0,1\}^n \text{ s.t. } \sum_{i=1}^{n} x_i \equiv 0 \pmod 2 \right\}$
- Thm: For all n we have $\chi(\mathsf{D}_n) = \chi(\frac{1}{2}H_n)$ and for $n \leq 11$:

n	4	5	6	7	8	9	10	11
$\chi(\frac{1}{2}H_n)$	4	8	8	8	8	13*	$[13,15]$	$[15,18]$

*: J.I. Kokkala and P.R.J. Östergård, *The chromatic number of the square of the 8-cube*, Math. Comp. **87** (2018), 2551–2561.

The root lattice E_6

- The Delaunay polytopes of E_6 are the Schläfli polytope Sch and -Sch.
- It has 27 vertices, 51840 symmetries. The lattice E_6 is laminated on D_5 and Sch is formed of three layers:
  - One vertex
  - 16 vertices in the half cube $\frac{1}{2}H_5$
  - 10 vertices in the cross polytope β_5
- Maximum independent sets have size 3 and form just 1 orbit.
- Sch has chromatic number 9: There are two orbits of colorings, one orbit of size 160 and another of size 40 (use `libexact`). Thus $\chi(E_6) \geq 9$.
- Thm: $\chi(E_6) = 9$. Take a coloring in the orbit of size 40. For each triple take the difference of vectors in it. The spanned lattice is of index 9 and has no relevant vectors.
- Conj: All 9-colorings of E_6 are of this form.

The root lattice E_7

- The Delaunay polytopes of E_7 is the Gosset polytope Gos and an orbit of simplices.
- Gos has 56 vertices, 2903040 symmetries. Some laminations:

E_6	D_6	A_6
1 ○ point 27 —— Sch 27 —— $-Sch$ 1 ○ point	12 —— β_6 16 —— $\frac{1}{2}H_6$ 12 —— β_6	7 —— $J(6,1)$ 21 —— $J(6,2)$ 21 —— $-J(6,2)$ 7 —— $-J(6,1)$

- Maximum independent sets have size 2 or 4 and form 2 orbits.
- Gos has chromatic number 14: There are 40457 orbits of colorings (use `libexact` and symmetries). Thus $\chi(E_7) \geq 14$.
- The lattice E_7 is laminated on A_6 and we have $\chi(A_6) = 7$.
- Thm: $\chi(E_7) = 14$. We color the odd layers of A_6 by $\{1, \ldots, 7\}$ and the even layers by $\{8, \ldots, 14\}$.

Hoffman lower bounds for lattices

- The Hoffman lower bounds can be expressed on infinite graphs.
- Let us denote μ a measure on $Vor(L)$ with $\mu(v) = \mu(-v)$. We have

$$\chi(L) \geq 1 - \frac{\sup_{x\in\mathbb{R}^n} \sum_{u\in Vor(L)} \mu(u) e^{2\pi i u\cdot x}}{\inf_{x\in\mathbb{R}^n} \sum_{u\in Vor(L)} \mu(u) e^{2\pi i u\cdot x}}$$

- If we choose $\mu(u) = \frac{1}{|Vor(L)|}$ then we have

$$\chi(L) \geq 1 - |Vor(L)| \left(\inf_{x\in\mathbb{R}^n} \sum_{u\in Vor(L)} e^{2\pi i u\cdot x} \right)^{-1}$$

- The proof method uses Harmonic analysis and Fourier analysis. Expressing other spectral bounds on infinite graphs would be hard.

Hoffman lower bounds for root lattices

- The character of the adjoint representation corresponding to the group is expressed as

$$\mathrm{ch}_{ad}^{L}(g) = dim(L) + \sum_{u \in Vor(L)} e^{2\pi i u \cdot x}$$

- The critical values of the characters (see Serre, 2004) were computed by algebraic methods and this gives:

$$\begin{aligned}
Crit\,\mathrm{ch}_{ad}^{\mathsf{E}_6} &= \{-3, -2, 6, 14, 78\} \\
Crit\,\mathrm{ch}_{ad}^{\mathsf{E}_7} &= \left\{-7, -3, -2, 1, \tfrac{17}{5}, 5, 25, 133\right\} \\
Crit\,\mathrm{ch}_{ad}^{\mathsf{E}_8} &= \left\{-8, -4, -\tfrac{104}{27}, -\tfrac{57}{16}, -3, -2, 0, 5, 24, 248\right\}
\end{aligned}$$

- This gives $\chi(\mathsf{E}_6) \geq 9$, $\chi(\mathsf{E}_7) \geq 10$ and $\chi(\mathsf{E}_8) \geq 16$.
- We have $Crit\,\mathrm{ch}_{ad}^{\mathsf{A}_n} = \{-1, n(n+2)\}$.
- Also $Crit\,\mathrm{ch}_{ad}^{\mathsf{D}_n}$ is known.

The dual root lattices

- For a lattice $L \subset \mathbb{R}^n$ the dual lattice L^* is

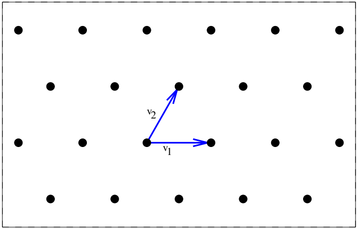
$$L^* = \{x \in \mathbb{R}^n \text{ s.t. } \langle x, y\rangle \in \mathbb{Z} \text{ for } y \in L\}$$

- $\chi(\mathsf{A}_n^*) = n + 1$ since the Delaunay polytopes are simplices and A_n is an index $n + 1$ sublattice without relevant vectors.
- $\chi(\mathsf{D}_n^*) = 4$ since $\mathsf{D}_n^* = \mathbb{Z}^n \cup ((1/2)^n + \mathbb{Z}^n)$
- Thm: We have $\chi(\mathsf{E}_n^*) = 16$ for $n = 6, 7, 8$.
- Lower bound is obtained by the fractional chromatic method applied to a sufficient set of vectors around the origin.
- Explicit coloring for E_6^* is obtained by finding an adequate sublattice.
- For E_7^* we could not find an index 16 sublattice that works.
- Instead we consider $\mathsf{E}_7^*/4\mathsf{E}_7^*$ and color the 16384 points with 16 colors with `minisat` in 2 minutes. Note that computing spectral lower bounds for this graph did not finish in 2 hours.

IV. Gram matrix iso-Delaunay and iso-edge domains

Gram matrix and lattices

- Denote by S^n the vector space of real symmetric $n \times n$ matrices and $S^n_{>0}$ the convex cone of real symmetric positive definite $n \times n$ matrices.
- Take a basis $(v_1, \dots, v_n)$ of a lattice L and associate to it the Gram matrix $G_{\mathbf{v}} = (\langle v_i, v_j \rangle)_{1 \le i,j \le n} \in S^n_{>0}$.
- Example: take the hexagonal lattice generated by $v_1 = (1, 0)$ and $v_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

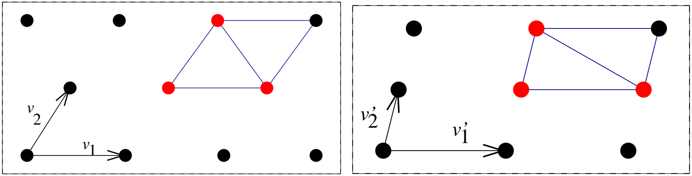


$$G_{\mathbf{v}} = \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

- This gives a parameter space of dimension $n(n+1)/2$.
- All programs for lattices (SVP, CVP, Arithmetic equivalence, etc.) use Gram matrices as input.

Iso-Delaunay domains

- Take a lattice L and select a basis $v_1, \ldots, v_n$.
- We want to assign the Delaunay polytopes of a lattice. Geometrically, this means that



  are part of the same iso-Delaunay domain.
- Formally, an iso-Delaunay domain is a set of positive definite matrices corresponding to the same iso-Delaunay domains.
- A primitive iso-Delaunay domain is a domain of maximum dimension and this means that all Delaunay are simplices.

Equalities and inequalities for iso-Delaunay domains

- Take $M = G_v$ with $v = (v_1, \dots, v_n)$ a basis of lattice L.
- If $V = (w_1, \dots, w_N)$ with $w_i \in \mathbb{Z}^n$ are the vertices of a Delaunay polytope of empty sphere $S(c, r)$ then:

$$\|w_i - c\|_M = r \text{ i.e. } w_i^T M w_i - 2 w_i^T M c + c^T M c = r^2$$

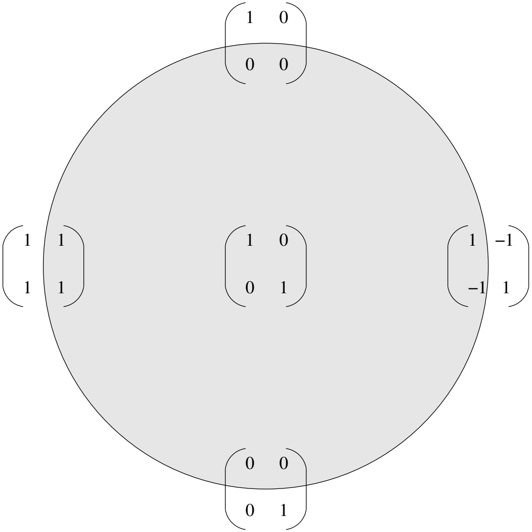
- Substracting one obtains

$$\left\{ w_i^T M w_i - w_j^T M w_j \right\} - 2 \left\{ w_i^T - w_j^T \right\} M c = 0$$

- Inverting matrices, one obtains $Mc = \psi(M)$ with ψ linear and so one gets linear equalities on M.
- Similarly $||w - c|| \geq r$ translates into a linear inequality on M: Take $V = (v_0, \dots, v_n)$ a simplex ($v_i \in \mathbb{Z}^n$), $w \in \mathbb{Z}^n$. If one writes $w = \sum_{i=0}^n \lambda_i v_i$ with $1 = \sum_{i=0}^n \lambda_i$, then one has

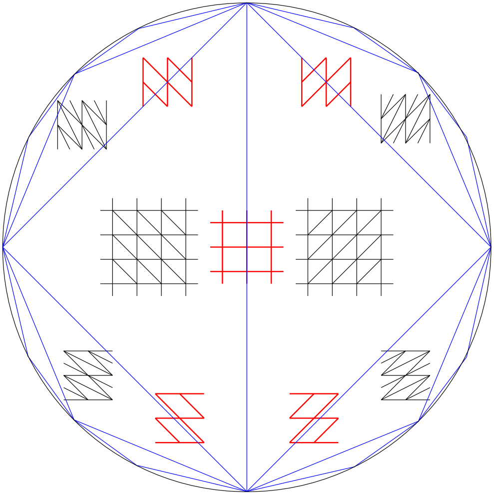
$$\|w - c\| \geq r \Leftrightarrow w^T M w - \sum_{i=0}^{n} \lambda_i v_i^T M v_i \geq 0$$

Plane representation of $S^2_{\geq 0}$



Iso-Delaunay domains in $S^2_{>0}$

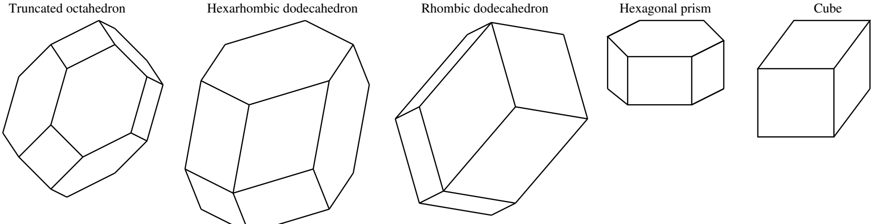
Primitive and non-primitive iso-Delaunay domains in $S^2_{>0}$:



Enumeration results on iso-Delaunay domains

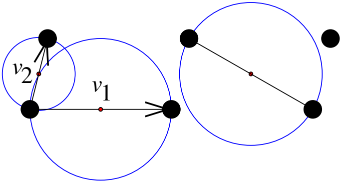
The equivalence used is the arithmetic equivalence $A \mapsto PAP^T$ for $P \in \mathrm{GL}_n(\mathbb{Z})$ which corresponds to changing basis in the lattice.

Dimension	Nr. iso-Delaunay domain	Nr. prim. iso-Delaunay domains
1	1	1
2	2	1
3	5 Fedorov, 1885	1 Fedorov, 1885
4	52 Delaunay & Shtogrin 1973	3 Voronoi, 1905
5	110244 MDS, AG, AS & CW, 2016	222 Engel & Gr. 2002



Iso-edge domain

- A parity class is a vector $v \in L - \frac{1}{2}L$. There are $2^n - 1$ up to translation by L. The middle of each edge of a Delaunay polytope is a parity class.
- A iso-edge domain is the assignation of an edge to each translation class.



- For the edge $[0, v_1]$ of center $v_1/2$ we have the set of inequalities:

$$\|v - v_1/2\| \geq \|v_1/2\| \text{ for } v \in L$$

- If we express in term of the basis (v_1, v_2) back into Gram matrices we obtain:

$$A[x - (1/2, 0)] \geq A[(1/2, 0)] \text{ for } x \in \mathbb{Z}^2$$

Implications on chromatic numbers

- A iso-Delaunay domain will be contained in an iso-edge domain. An iso-edge domain will contain a finite number of iso-Delaunay domains.
- Enumeration results:

n	$\|prim.iso - edge\|$	n	$\|prim.iso - edge\|$
2	1	4	3 Baranovski & Ryshkov 1973
3	1	5	76 Baranovski & Ryshkov 1973

- What matters for chromatic numbers is facets of Voronoi and so edges of Delaunay polytopes and so iso-Edge domains.
- The maximum chromatic number is attained in the generic case. When one goes to the boundary of an iso-Edge domain, some edges disappear and so the chromatic number decrease.
- For dimension $n \geq 6$ we do not know the full list.

V. Lattices in principal domain

The principal domain

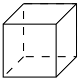
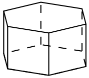
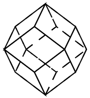
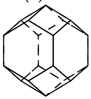
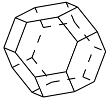
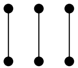
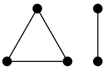
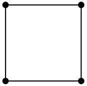
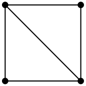
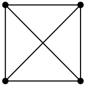
- A lattice Λ is in the principal domain if there are vectors $\{v_0, v_1, \ldots, v_n\}$ such that
  - $\{v_1, \ldots, v_n\}$ is a basis of Λ.
  - $v_0 + v_1 + \cdots + v_n = 0$
  - For $i < j$ we have $v_i \cdot v_j \leq 0$.
- The Delauney graph $D(\Lambda)$ of Λ is the graph with two vertices on (v_i) with two vertices adjacent if $v_i \cdot v_j < 0$.
- For such a lattice Λ, we have $\chi(\Lambda) \leq n + 1$. More generally if (G_i) is the decomposition into biconnected components of $D(\Lambda)$ then

$$\chi(\Lambda) \leq 1 + \max_i |V(G_i)|$$

- The chromatic number of Λ is at least the maximal length of a cycle in $D(\Lambda)$.

Three dimensional case

In dimension 3 all lattices are in the principal domain. Their lattice, Voronoi cell, Delaunay graph, and chromatic numbers:

$\mathbb{Z}^3$	$\mathsf{A}_2 \perp \mathbb{Z}$	A_3	$\mathbb{Z}\begin{pmatrix}2\\0\\0\end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix}0\\2\\0\end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix}-1\\-1\\2\end{pmatrix}$	A_3^*
				
cube	hexagonal prism	rhombic dodecahedron	elongated dodecahedron	truncated octahedron
				
2	3	4	4	4

Beyond dimension three, we have to consider lattices which are not in the principal domain.

VI. Exponential growth

Bounds

- We are interested in maximum chromatic number of lattices. Does it grow exponentially?
- We already know $\chi(L) \leq 2^n$.
- For $S \subset \{1, \dots, n\}$ we define the cut metric

$$\begin{array}{rcl} \delta_S : \{1, \dots, n\}^2 & \to & \mathbb{R} \\ (x, y) & \mapsto & \left\{ \begin{array}{ll} 1 & |S \cap \{x, y\}| = 1 \\ 0 & \text{otherwise} \end{array} \right. \end{array}$$

- The polytope $\mathrm{CUT}_n = \mathrm{conv}(\delta_S, S \subset \{1, \dots, n\})$ has 2^{n-1} vertices (since $\delta_S = \delta_{\{1,\dots,n\}-S}$) and they form a clique.
- The lattice $Latt_n$ spanned by the δ_S is the cut lattice and CUT_n is a Delaunay polytope of it. So $\chi(Latt_n) \geq 2^{n-1}$ with $\dim Latt_n = \frac{n(n-1)}{2}$.

Exponential growth

With high probability, the chromatic number of a random n-dimensional lattice grows exponentially in n. Moreover, there are n-dimensional lattices Λ_n with

$$\chi(\Lambda_n) \geq 2^{(0.0990\cdots - o(1))n}$$

- The proof is existential, it does not give those lattices.
- The probability refers to the density of the quotient $\mathrm{SL}(n, \mathbb{R}) / \mathrm{SL}(n, \mathbb{Z})$ which is of finite covolume (but not cocompact).
- A quasi-linear factor can be added in front.
- The proof depends on some elementary argument and the Minkowski-Hlawka lower bounds on lattice packings.