Provable Sieving Algorithms for the Shortest Vector Problem and the Closest Vector Problem in the ℓ_p norm

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Overview

Lattice

$\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\} \in \mathbb{R}^d$: n linearly independent vectors

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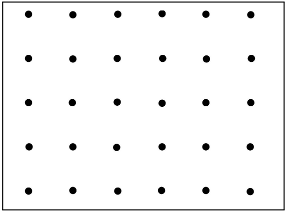

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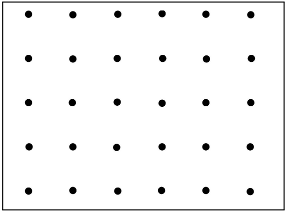

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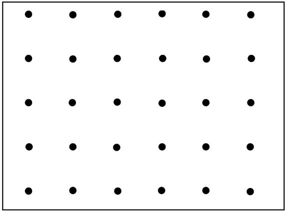

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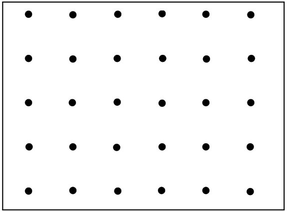

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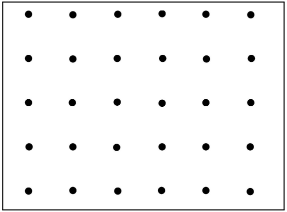

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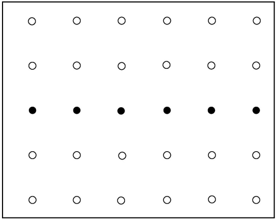

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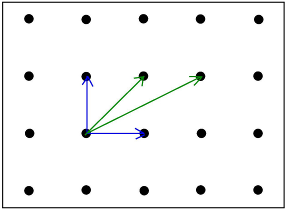

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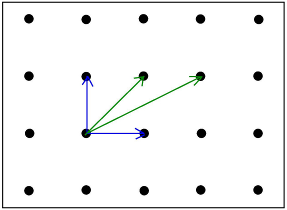

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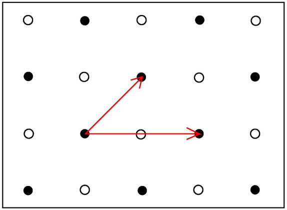

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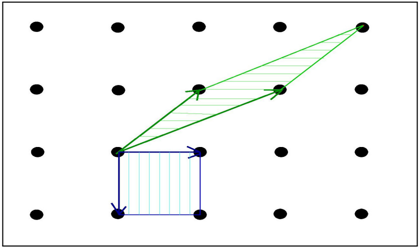

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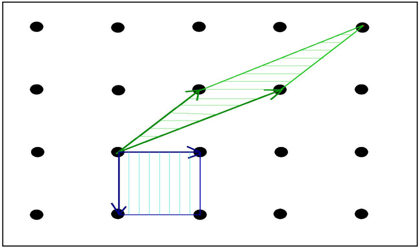

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- For any $\mathbf{z} \in \mathbb{R}^n$, there exists a unique $\mathbf{y} \in \mathscr{P}(\mathbf{B})$ such that $\mathbf{z} - \mathbf{y} \in \mathscr{L}(\mathbf{B})$.
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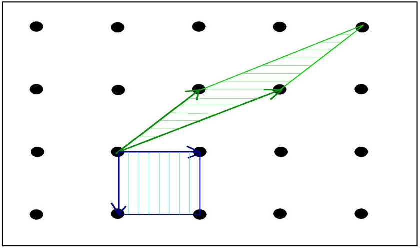

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- Translates $\mathscr{P}(\mathbf{B}) + \mathbf{v}$ where $\mathbf{v} \in \mathscr{L}$ form a partition of $\text{span}(\mathbf{B})$.

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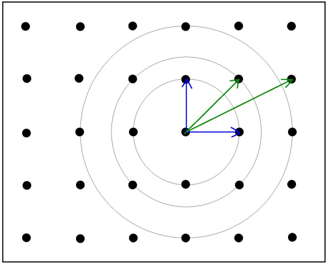

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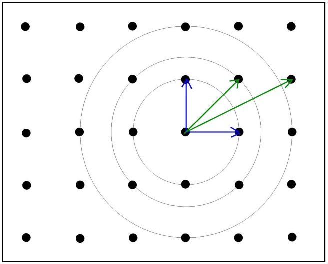

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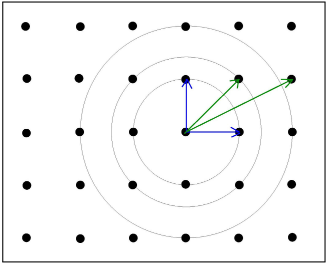

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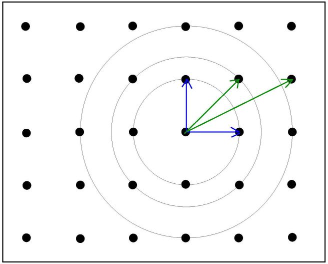

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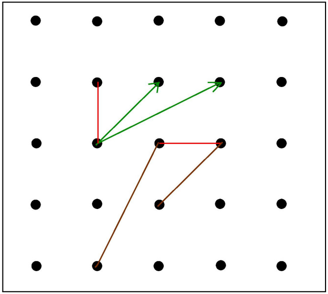

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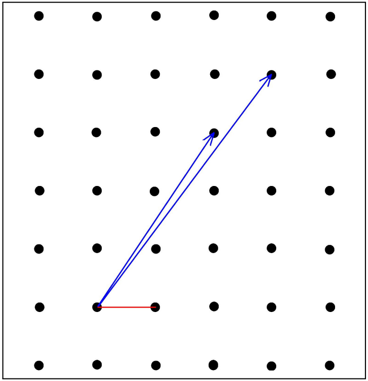
- Preliminary definitions
- Shortest Vector Problem and Closest Vector Problem
- Sieving algorithms for SVP and CVP

Shortest Vector Problem ($SVP_c^{(p)}$)

Input : A lattice specified by a basis $\mathbf{B}$

Output : Find a non-zero lattice vector of smallest norm upto some approximation factor c.
i.e. Find $\mathbf{v} \in \mathscr{L} \setminus \{\mathbf{0}\}$ such that $\|\mathbf{v}\| \leq c\|\mathbf{u}\|$ for any other $\mathbf{u} \in \mathscr{L} \setminus \{\mathbf{0}\}$.



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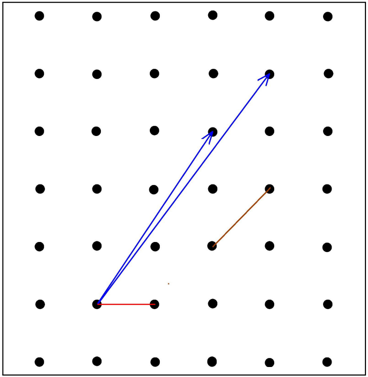

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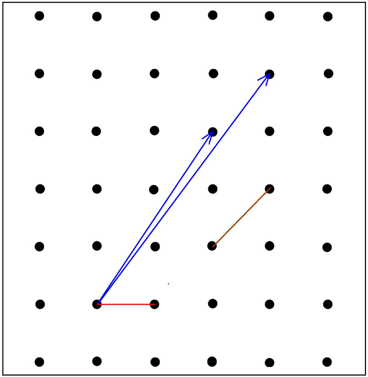

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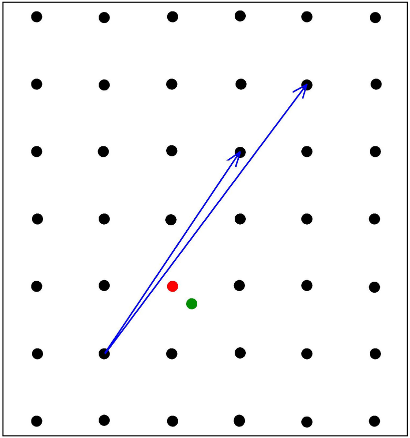

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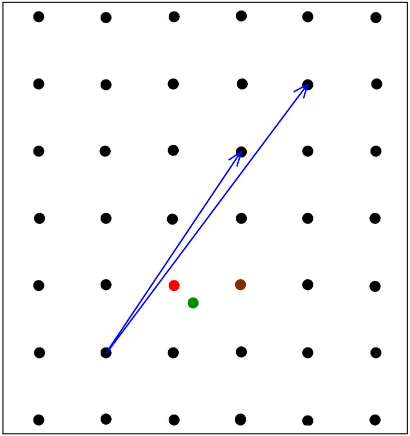

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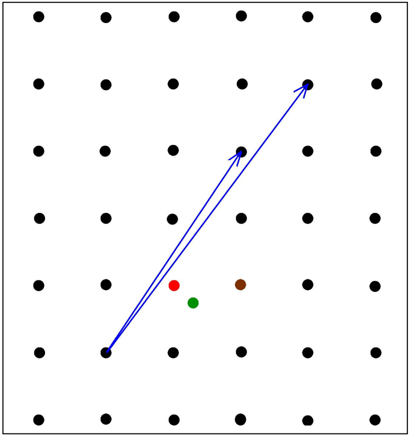

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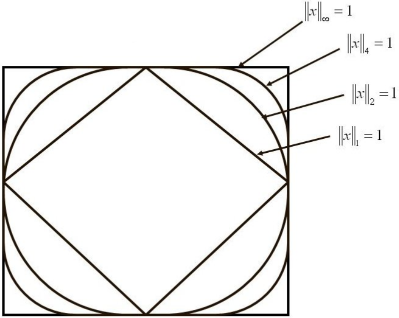

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- CVP in the ℓ_∞ norm is equivalent to integer programming.

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  - AKS sieving algorithm in the ℓ_p norm
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Prior Works : Sieving algorithms for SVP and CVP

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- Heuristic algorithms[5] for SVP run in time $(3/2)^{n/2}$ (Becker, Ducas, Gama, Laarhoven, 2016).

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[1] M.Ajtai,R.Kumar and D.Sivakumar, *A sieve algorithm for the shortest lattice vector problem*,STOC,2001.

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[3] M.Liu,X.Wang,G.Xu and X.Zheng, *Shortest lattice vectors in the presence of gaps*, IACR Cryptology ePrint Archive,2011.

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[5] A.Becker,L.Ducas,N.Gama and T.Laarhoven,*New directions in nearest neighbor searching with applications to lattice sieving*,ACM Symp. on Discrete Algo.,2016.

Prior Works : Sieving algorithms for SVP and CVP

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Sieving algorithm in the ℓ_p norm : AKS sieve

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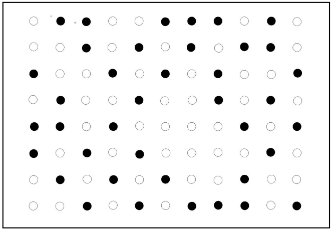

Figure: S

AKS sieving algorithm in the ℓ_p norm : AKS sieve

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- S : Set of N lattice vectors sampled in a ball of radius R.
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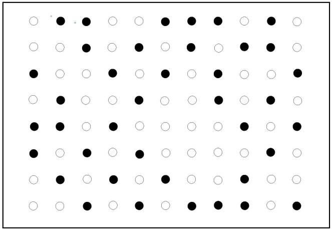

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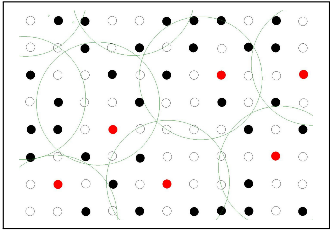

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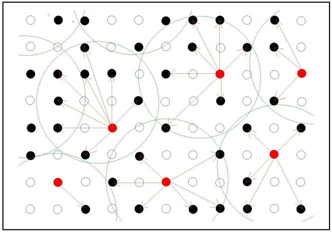

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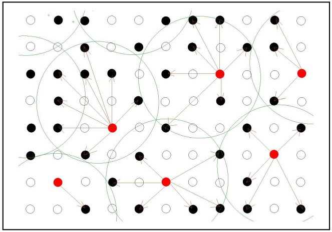

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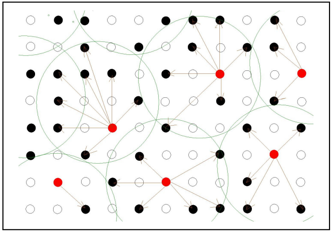

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Solution

- For each sampled vector, add a randomly chosen perturbation vector.

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- Take pair-wise difference of the vectors in the final set and **output the one with the smallest norm**.

Complexity of AKS sieve

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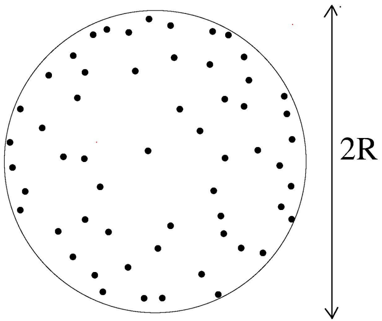

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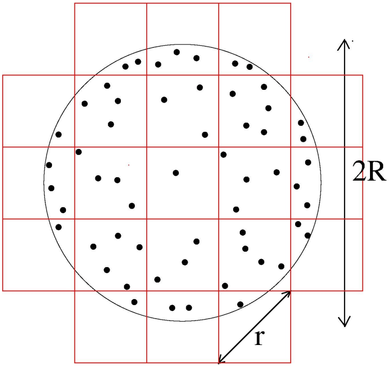

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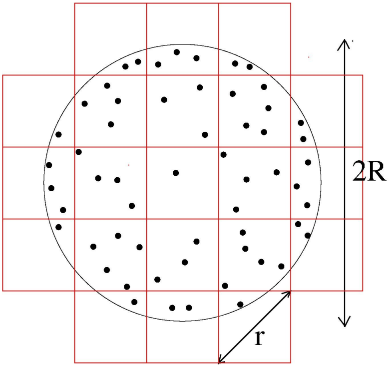

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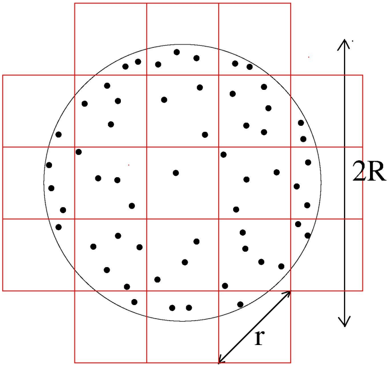

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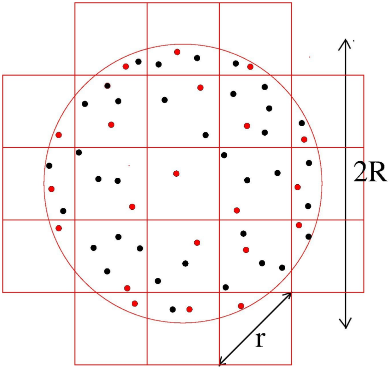

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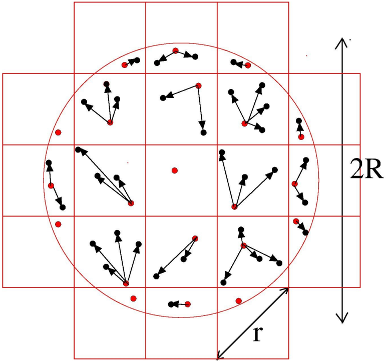
- At most one centre in each hypercube.
- Take difference.



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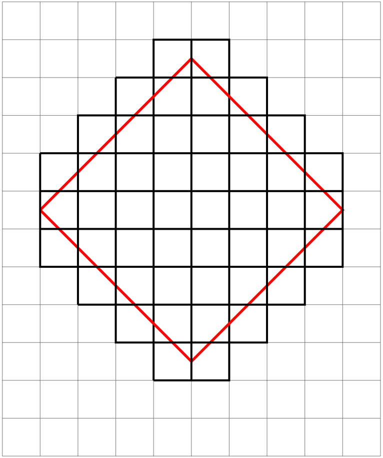
- $O\left(\left(2 + \frac{2}{\gamma}\right)^n\right)$ if $r = \gamma R$.



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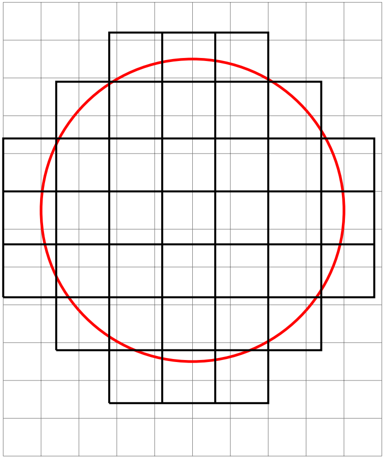

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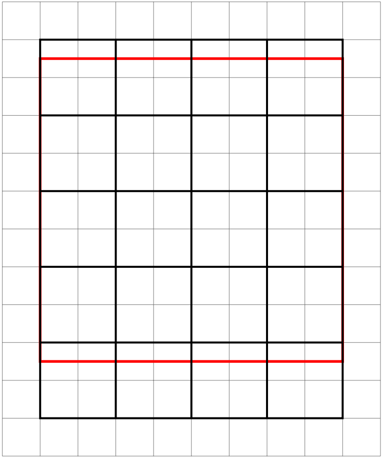

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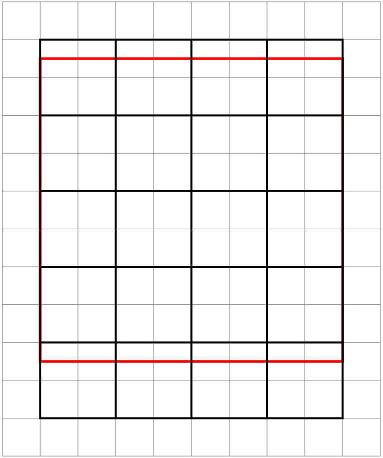

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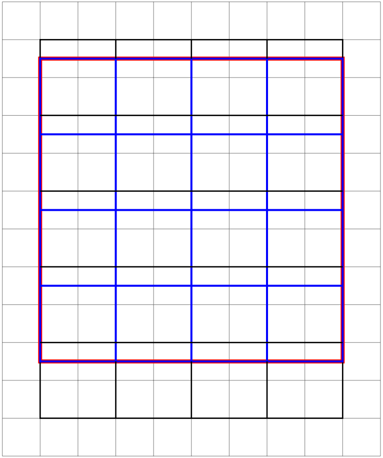

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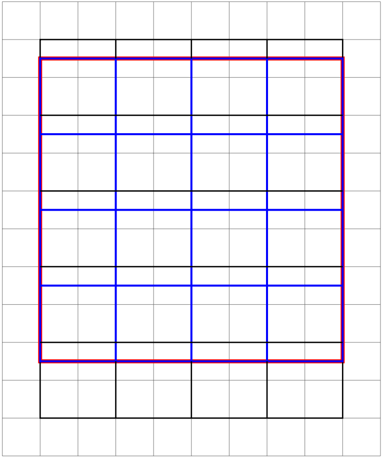

[1] D.Aggarwal and P.Mukhopadhyay, *Improved algorithms for the shortest vector problem and the closest vector problem in the infinity norm*, ISAAC, 2018.

Comparison of space and time complexity

(Linear sieve)

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$$1 \leq p \leq \infty$$

ALGORITHM	SPACE	TIME
Blömer and Naewe,2009	$2^{2.023n+o(n)}$	$2^{3.849n+o(n)}$
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Mixed sieving

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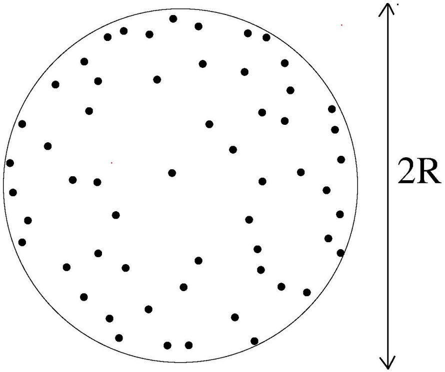

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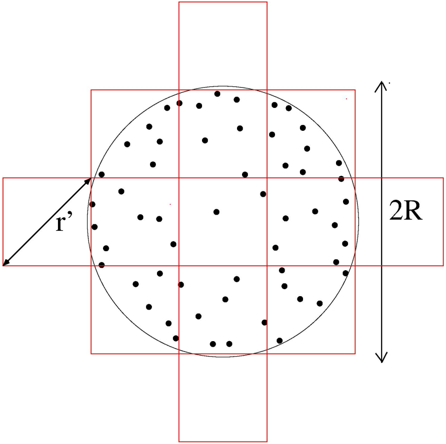

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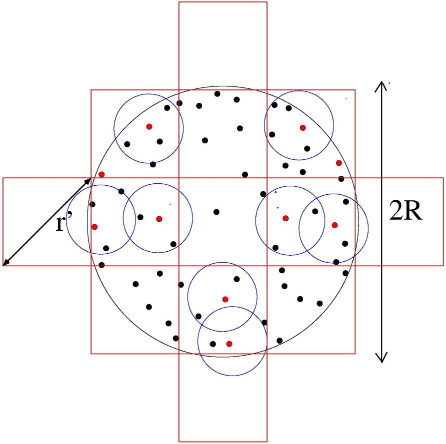

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Comparison of space and time complexity

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ALGORITHM	SPACE	TIME
List sieve,2011[2]	$2^{1.233n+o(n)}$	$2^{2.465n+o(n)}$
Mukhopadhyay,2019	$2^{2.25n+o(n)}$	$2^{2.25n+o(n)}$
Aggarwal et al.,2015[3]	2^n	2^n

[1] D.Micciancio, P.Voulgaris, *Faster exponential time algorithms for the shortest vector problem.*, SODA, 2010.

[2] G.Hanrot,X.Pujol,D.Stehle,*Algorithms for the shortest and closest lattice vector problems.*,International Conference on Coding and Cryptology,2011.

[3] D.Aggarwal, D.Dadush, O.Regev and N.Stephens-Davidowitz,*Solving the shortest vector problem in 2^n time using Discrete Gaussian sampling*, STOC, 2015.

Approximation algorithms for large constant approximation factor

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  - Reduction from approximate CVP to approximate SVP (Blömer and Naewe,2009).

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Liu,Wang,Xu and Zheng,2011[1]	$2^{0.401n+o(n)}$	$2^{0.802n+o(n)}$
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[1] M.Liu,X.Wang,G.Xu and X.Zheng, *Shortest lattice vectors in the presence of gaps*, IACR Cryptology ePrint Archive,2011.

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Thank You