Short simplex paths in lattice polytopes

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Lattices: Geometry, Algorithms and Hardness
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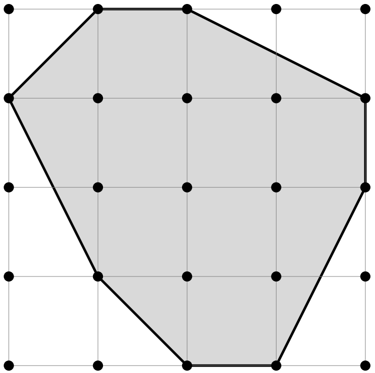
Lattice polytopes



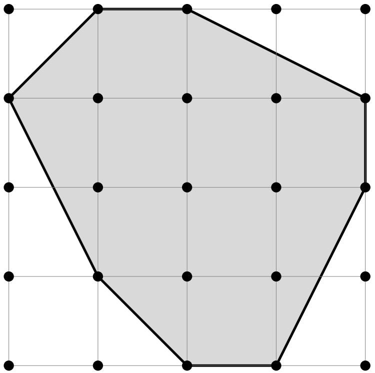
P is a **lattice polytope** in $[0, k]^n$ if

- all vertices are integral
- $P \subseteq [0, k]^n$

Appear in:

- polyhedral combinatorics
- integer programming
- fractional relaxations

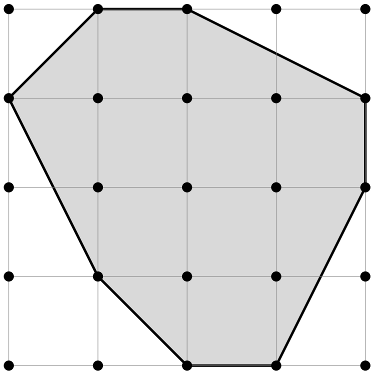
Diameter of lattice polytopes



Upper bounds:

- n if $k = 1$ [Naddef 89]
- kn [Kleinschmidt Onn 92]
- $\lfloor (k - \frac{1}{2})n \rfloor$ if $k \geq 2$
  [DP Michini 16]
- $kn - \lceil \frac{2}{3}n \rceil - (k - 3)$ if $k \geq 3$
  [Deza Pournin 18]

Diameter of lattice polytopes



Lower bounds:

- n if $k = 1$
- $\lfloor \frac{3}{2} n \rfloor$ if $k = 2$ [dP Michini 16]
- $\lfloor \frac{1}{2}(k+1)n \rfloor$ if $k < 2n$ [Deza Manoussakis Onn 18]
- $ck^{\frac{2}{3}}$ if $n = 2$, $k \to \infty$ [Balog Bárány 91]
- $c(n)k^{\frac{n}{n+1}}$ if n fixed, $k \to \infty$ [Deza Pournin Sukegawa 19]

LP on lattice polytopes

We study the LP problem:

$$\begin{aligned} \max \quad & c^\top x \\ \text{s.t.} \quad & x \in P \end{aligned}$$

- P is a lattice polytope in $[0, k]^n$
- $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, where $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$
- $c \in \mathbb{Z}^n$

GOAL: Simplex algorithm that traces "short" simplex paths on P from given vertex x^0 to optimal vertex x^*

Possibly, polynomially far from the worst case diameter

How "short" can a simplex path be?

Upper bound on simplex path length by [Kitahara Matsui Mizuno '12]

- $Q = \{x \in \mathbb{R}^n_+ \mid Dx = d\}$ lattice polytope in $[0, k]^n$ in **standard form** with $D \in \mathbb{Z}^{m \times n}$ and $d \in \mathbb{Z}^m$
- simplex path length
  $\leq (n - m) \cdot \min\{m, n - m\} \cdot k \cdot \log(k \min\{m, n - m\})$

- $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ lattice polytope in $[0, k]^n$
- $\bar{P} = \{(x, s) \in \mathbb{R}^{n+m}_+ \mid Ax + I_m s = b\}$
- $\bar{P}$ is a lattice polytope in $[0, \max\{k, S\}]^{n+m}$, where $S = \max_{x \in P}\{\|b - Ax\|_\infty\}$
- simplex path length $O(n^2 \max\{k, S\} \log(n \max\{k, S\}))$

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Can we eliminate dependence on S, i.e., on A, b?

Short simplex paths in lattice polytopes

1st MAIN RESULT: Simplex algorithm for lattice polytopes in $[0, k]^n$ s.t. simplex path length $O(n^4 k \log(nk))$

Independent on:

- cost vector c
- description $Ax \leq b$ of P
- number of inequalities m

Short simplex paths in lattice polytopes

1st MAIN RESULT: Simplex algorithm for lattice polytopes in $[0, k]^n$ s.t. simplex path length $O(n^4 k \log(nk))$

The simplex path length is polynomially far from optimal:

- For fixed k, $\exists$ polytopes with diameter in $\Omega(n)$
- For fixed n and $k \to \infty$, $\exists$ polytopes with diameter in $\Omega(k^{\frac{n}{n+1}})$

Short simplex paths in lattice polytopes

1st MAIN RESULT: Simplex algorithm for lattice polytopes in $[0, k]^n$ s.t. simplex path length $O(n^4 k \log(nk))$

More questions:

- Most lattice polytopes in combinatorial optimization are defined via $0, \pm 1$ constraint matrices
- Can we exploit the largest absolute value α of the entries in the constraint matrix?

Short simplex paths in lattice polytopes

2nd MAIN RESULT: Simplex algorithm for lattice polytopes in $[0, k]^n$ s.t. simplex path length $O(n^2 k \log(nk\alpha))$

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Short simplex paths in lattice polytopes

2nd MAIN RESULT: Simplex algorithm for lattice polytopes in $[0, k]^n$ s.t. simplex path length $O(n^2 k \log(nk\alpha))$

- If $\alpha \leq \text{poly}(n, k)$, then simplex path length $O(n^2 k \log(nk))$
- If $\alpha \leq \text{poly}(n, k)$ and $k = 1$ then simplex path length $O(n^2 \log n)$

How does it work?

We move to an adjacent vertex by calling:

Oracle

Input: Polytope P, $c \in \mathbb{Z}^n$, vertex x^t of P

Output:

- Either a statement that x^t maximizes $c^\top x$ over P
- or a vertex x^{t+1} adjacent to x^t s.t. $c^\top x^{t+1} > c^\top x^t$

The input to the oracle is key to compute a short simplex path...

How does it work?

1. **Basic algorithm** length $\leq kn\,\|c\|_\infty$

2. **Scaling algorithm** length $O(kn\log\|c\|_\infty)$

3. **Preprocessing & scaling algorithm** length $O(n^4 k\log(nk))$

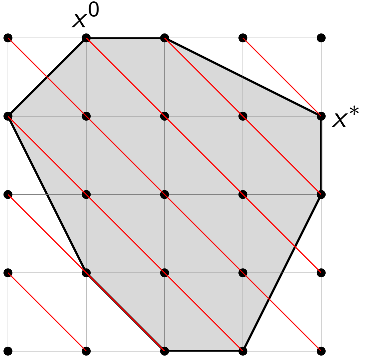
4. **Iterative algorithm** length $O(n^2 k\log(nk\alpha))$

Basic algorithm

Basic algorithm

Input: Lattice polytope P in $[0,k]^n$, $c \in \mathbb{Z}^n$, vertex x^0 of P
Output: A vertex x^* of P maximizing $c^\top x$.

for $t = 0, 1, 2, \ldots$ **do**
  Invoke **oracle**(P, c, x^t)
  If the **oracle** states that x^t is optimal, return x^t
  Otherwise, let x^{t+1} be the vertex returned by the **oracle**



Observation: The length of the simplex path generated is at most $c^\top x^* - c^\top x^0 \leq kn\|c\|_\infty$

Example: $c = (1,1)$

Basic algorithm

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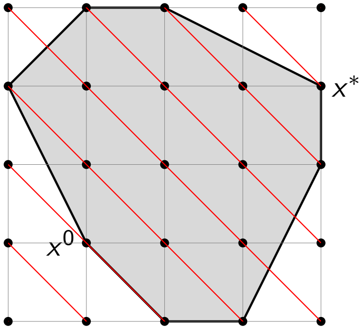

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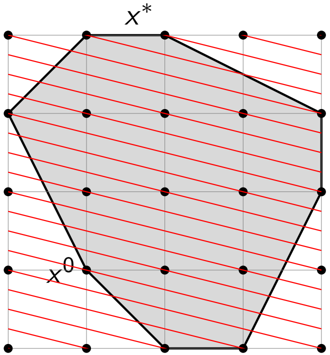


Observation: The length of the simplex path generated is at most $c^\top x^* - c^\top x^0 \leq kn\,\|c\|_\infty$

Example: $c = (1, 4)$

Scaling algorithm

Let $\ell := \lceil \log \|c\|_\infty \rceil$
For $t = 0, \ldots, \ell$, define the integral approximations of c:

$$c^t := \lceil \frac{c}{2^{\ell - t}} \rceil \qquad \text{(Note: } c^\ell = c\text{)}$$

Example:

$$\begin{aligned} c &= (1,2,3,4,5,6,7) \\ c^0 &= (1,1,1,1,1,1,1) \\ c^1 &= (1,1,1,1,2,2,2) \\ c^2 &= (1,1,2,2,3,3,4) \\ c^3 &= (1,2,3,4,5,6,7) \end{aligned}$$

For $t = 0, \ldots, \ell$:

- $\|c^t\|_\infty \leq 2^t$

Scaling algorithm

For $t = 0, \ldots, \ell$: $\quad c^t := \lceil \frac{c}{2^{\ell-t}} \rceil$

Scaling algorithm

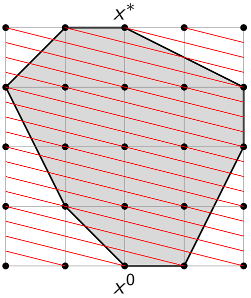
Input: Lattice polytope P in $[0, k]^n$, $c \in \mathbb{Z}^n$, vertex x^0 of P
Output: A vertex x^* of P maximizing $c^\top x$

for $t = 0, \ldots, \ell$ **do**
  Set $x^{t+1} :=$ **basic algorithm**(P, c^t, x^t)
Return the vertex $x^{\ell+1}$

Scaling algorithm

For $t = 0, \dots, \ell$: $\quad c^t := \lceil \frac{c}{2^{\ell - t}} \rceil$

Example: $c = (1, 4)$



Scaling algorithm

Input: Lattice polytope P in $[0, k]^n$, $c \in \mathbb{Z}^n$, vertex x^0 of P
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Set $x^{t+1} :=$ **basic algorithm**(P, c^t, x^t)
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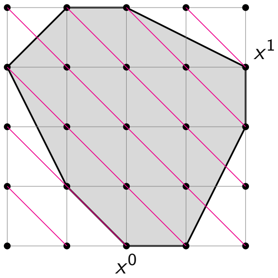
Scaling algorithm

For $t = 0, \ldots, \ell$: $\quad c^t := \lceil \frac{c}{2^{\ell - t}} \rceil$

Example: $c = (1, 4)$ $c^0 = (1, 1)$



Scaling algorithm

Input: Lattice polytope P in $[0, k]^n$, $c \in \mathbb{Z}^n$, vertex x^0 of P

Output: A vertex x^* of P maximizing $c^\top x$

for $t = 0, \ldots, \ell$ **do**

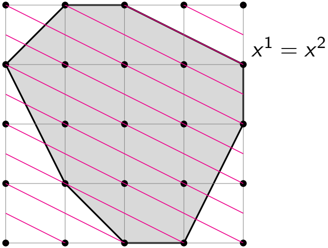
Set $x^{t+1} :=$ **basic algorithm**(P, c^t, x^t)

Return the vertex $x^{\ell+1}$

Scaling algorithm

For $t = 0, \ldots, \ell$: $c^t := \lceil \frac{c}{2^{\ell - t}} \rceil$

Example: $c = (1, 4)$ $c^1 = (1, 2)$



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for $t = 0, \ldots, \ell$ **do**
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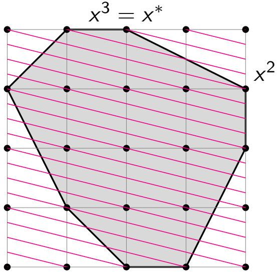
Return the vertex $x^{\ell+1}$

Scaling algorithm

For $t = 0, \ldots, \ell$: $\quad c^t := \lceil \frac{c}{2^{\ell - t}} \rceil$

Example: $c = (1,4)$ $c^2 = (1,4)$



Scaling algorithm

Input: Lattice polytope P in $[0,k]^n$, $c \in \mathbb{Z}^n$, vertex x^0 of P

Output: A vertex x^* of P maximizing $c^\top x$

for $t = 0, \ldots, \ell$ **do**

  Set $x^{t+1} :=$ **basic algorithm**(P, c^t, x^t)

Return the vertex $x^{\ell+1}$

Scaling algorithm

Proposition: Simplex path length $O(k\,n \log \|c\|_\infty)$

Scaling algorithm

Input: Lattice polytope P in $[0, k]^n$, $c \in \mathbb{Z}^n$, vertex x^0 of P
Output: A vertex x^* of P maximizing $c^\top x$

for $t = 0, \dots, \ell$ **do**
Set $x^{t+1} :=$ **basic algorithm**(P, c^t, x^t)
Return the vertex $x^{\ell+1}$

Preprocessing algorithm

Preprocessing algorithm
Input: $c \in \mathbb{Q}^n$, positive integer N **Output:** $\check{c} \in \mathbb{Z}^n$ such that ▶ $\|\check{c}\|_\infty \leq 2^{4n^3} N^{n(n+2)}$ ▶ $\text{sign}(c^\top z) = \text{sign}(\check{c}^\top z)\ \forall z \in \mathbb{Z}^n$ with $\|z\|_1 \leq N - 1$

- Due to [Frank Tardos 87]
- Relies on the simultaneous approximation algorithm of [Lenstra Lenstra Lovász 82]

Setting $N := kn + 1$, x^* optimal for $\check{c} \Rightarrow$ optimal for c:

- $\forall x \in P \cap \mathbb{Z}^n$:
- $x^* - x \in \mathbb{Z}^n$ and $\|x^* - x\|_1 \leq kn$
- $\check{c}^\top (x^* - x) \geq 0 \quad \Rightarrow \quad c^\top (x^* - x) \geq 0$

Preprocessing & scaling algorithm

Preprocessing & scaling algorithm

Input: Lattice polytope P in $[0,k]^n$, $c \in \mathbb{Z}^n$, vertex x^0 of P

Output: A vertex x^* of P maximizing $c^\top x$

$\check{c} :=$ preprocessing algorithm$(c, N := kn)$

$x^* :=$ **scaling algorithm**$(P, \check{c}, x^0)$

Return x^*

Theorem 1: Simplex path length $O(n^4 k \log(nk))$

Iterative algorithm

GOAL: shorter simplex path length, dependent on α

$P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, where $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$
$\alpha :=$ largest absolute value of the entries of A

IDEA: Identify at each iteration one constraint of $Ax \leq b$ that is active at each optimal solution of $\max\{c^\top x \mid x \in P\}$

Inspired by [Tardos '86]

Iterative algorithm

Iterative algorithm

Input: Lattice polytope P in $[0,k]^n$, $c \in \mathbb{Z}^n$, vertex x^0 of P
Output: A vertex x^* of P maximizing $c^\top x$

0: Let $\mathcal{E} := \emptyset$ and $x^* := x^0$
1: Let $\bar{c}$ be the projection of c onto $\{x \in \mathbb{R}^n \mid a_i^\top x = 0\ \forall i \in \mathcal{E}\}$.
If $\bar{c} = 0$ return x^*
2: Let $\tilde{c} \in \mathbb{Z}^n$ be defined by $\tilde{c}_i := \left\lfloor \frac{n^3 k \alpha}{\|\bar{c}\|_\infty} \bar{c}_i \right\rfloor$ for $i = 1, \ldots, n$
3: Consider the following pair of primal and dual LP problems:

$$\begin{array}{lll} \max & \tilde{c}^\top x & \\ \text{s.t.} & a_i^\top x = b_i & i \in \mathcal{E} \\ & a_i^\top x \leq b_i & i \in [m] \setminus \mathcal{E} \\ & & (\tilde{P}) \end{array} \qquad \begin{array}{lll} \min & b^\top y & \\ \text{s.t.} & A^\top y = \tilde{c} & \\ & y_i \geq 0 & i \in [m] \setminus \mathcal{E} \\ & & (\tilde{D}) \end{array}$$

Compute optimal vertex $\tilde{x}$ of $(\tilde{P})$ with **scaling alg** from x^*
Compute an optimal solution $\tilde{y}$ to $(\tilde{D})$ [. . .]
Let $\mathcal{F} := \{i \mid \tilde{y}_i > nk\}$, and let $h \in \mathcal{F} \setminus \mathcal{E}$
$\mathcal{E} \leftarrow \mathcal{E} \cup \{h\}$, $x^* \leftarrow \tilde{x}$ and go back to step 1

Main results

(correctness)
Proposition: Vector x^* returned maximizes $c^\top x$ over P.

(short simplex paths)
Proposition: Simplex path length $O(n^2 k \log(nk\alpha))$

(polynomial runtime)
Proposition: The number of operations to construct the next vertex in the simplex path is bounded by $\text{poly}(n, m, \log \alpha, \log k)$.
If P is 'well-described' by $Ax \leq b$, then it is bounded by $\text{poly}(n, m, \log k)$.

Correctness - idea

$$\begin{array}{lll} \max & \tilde{c}^\top x & \\ \text{s.t.} & a_i^\top x = b_i & i \in \mathcal{E} \\ & a_i^\top x \le b_i & i \in [m] \setminus \mathcal{E} \end{array} \qquad (\tilde{P}) \qquad \begin{array}{lll} \min & b^\top y & \\ \text{s.t.} & A^\top y = \tilde{c} & \\ & y_i \ge 0 & i \in [m] \setminus \mathcal{E} \end{array} \qquad (\tilde{D})$$

At each iteration, we restrict to a face F of P defined as

$$F := \{x \in \mathbb{R}^n \mid a_i^\top x \le b_i \text{ for } i \in [m] \setminus \mathcal{E},\ a_i^\top x = b_i \text{ for } i \in \mathcal{E}\}$$

We prove that each optimal solution of $\max\{c^\top x \mid x \in P\}$ lies in F

Correctness - idea

$$\begin{array}{lll} \max & \hat{c}^\top x & \\ \text{s.t.} & a_i^\top x = b_i & i \in \mathcal{E} \\ & a_i^\top x \le b_i & i \in [m] \setminus \mathcal{E} \end{array} \qquad (\hat{P}) \qquad \begin{array}{lll} \min & b^\top y & \\ \text{s.t.} & A^\top y = \tilde{c} & \\ & y_i \ge 0 & i \in [m] \setminus \mathcal{E} \end{array} \qquad (\tilde{D})$$

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Let $\hat{c}$ be defined by $\hat{c}_i := \frac{n^3 k \alpha}{\|\bar{c}\|_\infty} \bar{c}_i$ for $i = 1, \ldots, n \quad \Rightarrow \quad \tilde{c} = \lfloor \hat{c} \rfloor$

Correctness - idea

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We prove that each optimal solution of $\max\{c^\top x \mid x \in P\}$ lies in F

Complementary slackness conditions for $(\hat{P})/(\tilde{D})$:
If $\tilde{y}$ optimal for $(\tilde{D})$ then $\forall\, \hat{x}$ optimal for $(\hat{P})$:

$$\tilde{y}_i > \textit{some const} \quad \Rightarrow \quad a_i^\top \hat{x} = b_i \qquad i \in [m] \setminus \mathcal{E} \quad (*)$$

$\Rightarrow$ to solve $(\hat{P})$ set primal constraints in $(*)$ to equality

Correctness - idea

$$\begin{array}{llll} \max & \hat{c}^\top x & & \\ \text{s.t.} & a_i^\top x = b_i & i \in \mathcal{E} & \\ & a_i^\top x \leq b_i & i \in [m] \setminus \mathcal{E} & \end{array} \quad (\hat{P}) \qquad \begin{array}{lll} \min & b^\top y & \\ \text{s.t.} & A^\top y = \tilde{c} & \\ & y_i \geq 0 & i \in [m] \setminus \mathcal{E} \end{array} \quad (\tilde{D})$$

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Correctness - key lemma

$$\begin{array}{lll} \max & \hat{c}^\top x & \\ \text{s.t.} & a_i^\top x = b_i & i \in \mathcal{E} \\ & a_i^\top x \leq b_i & i \in [m] \setminus \mathcal{E} \end{array} \qquad (\hat{P}) \qquad \begin{array}{lll} \min & b^\top y & \\ \text{s.t.} & A^\top y = \tilde{c} & \\ & y_i \geq 0 & i \in [m] \setminus \mathcal{E} \end{array} \qquad (\tilde{D})$$

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Complementary slackness conditions for $(\hat{P})/(\tilde{D})$:
If $\tilde{y}$ optimal for $(\tilde{D})$ then $\forall\ \hat{x}$ optimal for $(\hat{P})$:

$$\tilde{y}_i > nk \quad \Rightarrow \quad a_i^\top \hat{x} = b_i \qquad\qquad i \in [m] \setminus \mathcal{E} \quad (*)$$

$\Rightarrow$ to solve $(\hat{P})$ set primal constraints in $(*)$ to equality

Short simplex paths - idea

At each iteration, we restrict to a face F of P defined as

$$F := \{x \in \mathbb{R}^n \mid a_i^\top x \leq b_i \text{ for } i \in [m] \setminus \mathcal{E},\ a_i^\top x = b_i \text{ for } i \in \mathcal{E}\}$$

We prove that at each iteration the dimension of F decreases by 1
$\Rightarrow$ at most n iterations

At each iteration, we run the **scaling algorithm** to solve $(\tilde{P})$
Obs: F is a lattice polytope in $[0, k]^n$ and $\|\tilde{c}\|_\infty \leq n^3 k \alpha$.

At each iteration the **scaling algorithm** constructs a simplex path of length at most $nk \log \|\tilde{c}\|_\infty \in O(nk \log(nk\alpha))$

Theorem 3: Simplex path length $O(n^2 k \log(nk\alpha))$

Thank you!