Quantum Singular Value Transformation & Its Algorithmic Applications

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Quantum walks

Continuous-time quantum / random walks

Laplacian of a weighted graph

Let $G = (V, E)$ be a finite simple graph, with non-negative edge-weights $w \colon E \to \mathbb{R}_+$. The Laplacian is defined as

$$u \neq v\colon\ L_{uv} = w_{uv}, \text{ and } L_{uu} = -\sum_v w_{uv}.$$

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Evolution of the state:

$$\frac{d}{dt} p_u(t) = \sum_{v \in V} L_{uv} p_v(t) \qquad \Longrightarrow \qquad p(t) = e^{tL} p(0)$$

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Transition probability in one step (stochastic matrix)

$$P_{vu} = \Pr(\text{step to } v \mid \text{being at } u) = \frac{w_{vu}}{\sum_{v' \in U} w_{v'u}}$$

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How to erase history? The Szegedy quantum walk operator:

$$W' := U^\dagger \cdot \mathrm{SWAP} \cdot U$$

$$W := U^\dagger \cdot \mathrm{SWAP} \cdot U((2|0\rangle\langle 0| \otimes I) - I)$$

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Multiple steps of the quantum walk: $(\langle 0| \otimes I) W^k (|0\rangle \otimes I) = T_k(P)$

[$T_k(x) = \cos(k \arccos(x))$ Chebyshev polynomials: $T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x)$]

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Are we happy with Chebyshev polynomials?

Linear combination of (non-)unitary mat. [Childs & Wiebe '12, Berry et al. '15]

Suppose that $V = \sum_k |k\rangle\langle k| \otimes U^k$, and $Q : |0\rangle \mapsto \sum_i \sqrt{q_i}|i\rangle$ for $q_i \in [0, 1]$, then

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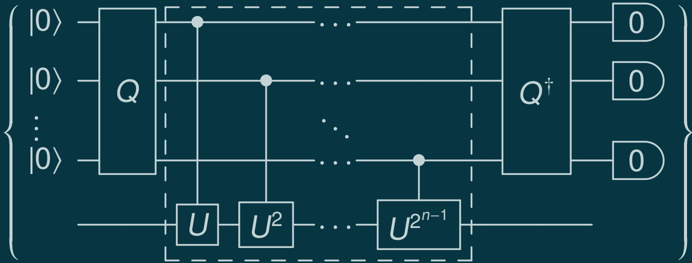

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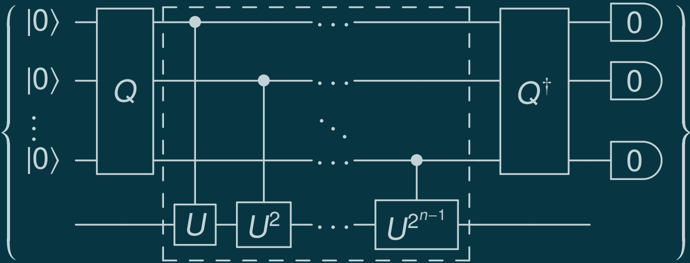

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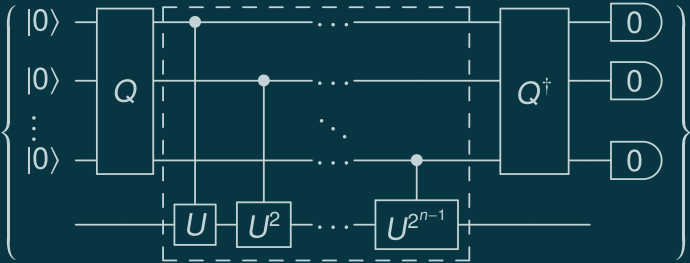

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Corollary: Quantum fast-forwarding (Apers & Sarlette 2018)

We can implement a unitary V such that

$$(\langle 0| \otimes I)V(|0\rangle \otimes I) \overset{\varepsilon}{\approx} P^t$$

with using only $\mathcal{O}\left(\sqrt{t \log(1/\varepsilon)}\right)$ quantum walk steps.

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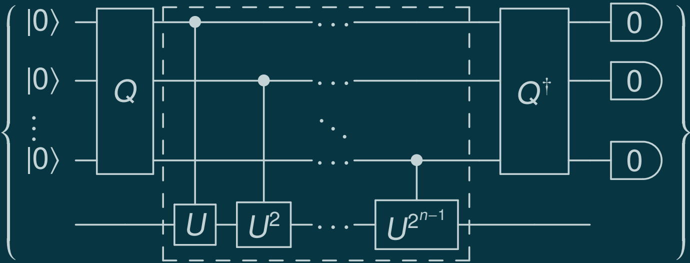

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Szegedy quantum walk based search

Suppose we have some unknown marked vertices $M \subset V$.

Quadratically faster hitting

Hitting time: expected time to hit a marked vertex starting from the stationary distr. Starting from the quantum state $\sum_{v\in V} \sqrt{\pi_v}|v\rangle$ we can

- detect the presence of marked vertices ($M \neq 0$) in time $\mathcal{O}\big(\sqrt{HT}\big)$ (Szegedy 2004)
- find a marked vertex in time $\mathcal{O}\big(\frac{1}{\sqrt{\delta\varepsilon}}\big)$ (Magniez, Nayak, Roland, Sántha 2006)
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Starting from arbitrary distributions

Starting from distribution σ on some vertices we can

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- find a marked vertex in time $\widetilde{\mathcal{O}}\left(\sqrt{C_{\sigma,M}}\right)$ (Piddock; Apers, **G**, Jeffery 2019)

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Vertices: $\{S \subset N \colon |S| = K\}$; Edges: $\{(S, S') \colon |S \triangle S'| = 2\}$

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- Black box: Computes f on inputs corresponding to elements of $[n]$
- Question: Are there any $i \neq j \in [n] \times [n]$ such that $f(i) = f(j)$?
- Query complexity: $\mathcal{O}(n^{2/3})$ (Ambainis 2003) $\Omega(n^{2/3})$ (Aaronson & Shi 2001)

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Matrix Product Verification

- Black box: Tells any entry of the $n \times n$ matrices A, B or C.
- Question: Does $AB = C$ hold?
- Query complexity: $\mathcal{O}(n^{5/3})$ (Buhrman, Špalek 2004)

Block-encodings and

Quantum Singular Value Transformation

Block-encoding

A way to represent large matrices on a quantum computer efficiently

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Implementing arithmetic operations on block-encoded matrices

- Given block-encodings A_j we can implement convex combinations.
- Given block-encodings A, B we can implement block-encoding of AB.

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Quantum Singular Value Transformation (QSVT)

Main theorem about QSVT (G, Su, Low, Wiebe 2018)

Let $P\colon [-1,1] \to [-1,1]$ be a degree-d odd polynomial map.

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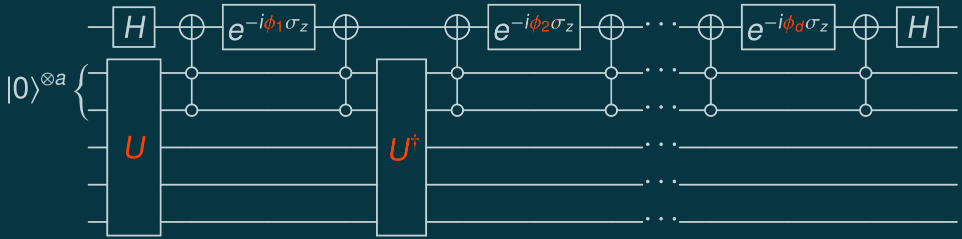

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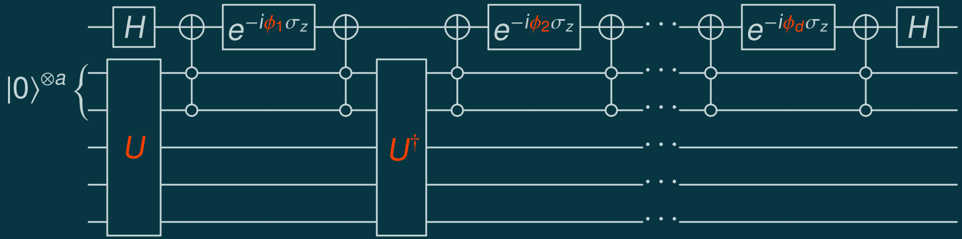

Simmilar result holds for even polynomials.

Singular vector transformation and projection

Fixed-point amplitude ampl. (Yoder, Low, Chuang 2014)

Amplitude amplification problem: Given U such that

$$U|\bar{0}\rangle = \sqrt{p}|0\rangle|\psi_{\text{good}}\rangle + \sqrt{1-p}|1\rangle|\psi_{\text{bad}}\rangle, \qquad \text{prepare } |\psi_{\text{good}}\rangle.$$

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$$|\psi\rangle = \sum_{i=i}^{k} \alpha_i |\psi_i\rangle \quad \text{to} \quad |\phi\rangle = \sum_{i=i}^{k} \alpha_i |\phi_i\rangle.$$

If $\varsigma_i \geq \delta$ for all $0 \neq \alpha_i$, we can ε-apx. using QSVT with compl. $\mathcal{O}\left(\frac{1}{\delta}\log\left(\frac{1}{\varepsilon}\right)\right)$.

Direct implementation of the pseudoinverse (HHL)

Singular value decomposition and pseudoinverse

Suppose $A = \sum_i \varsigma_i |w_i\rangle\langle v_i|$ is a singular value decomposition.
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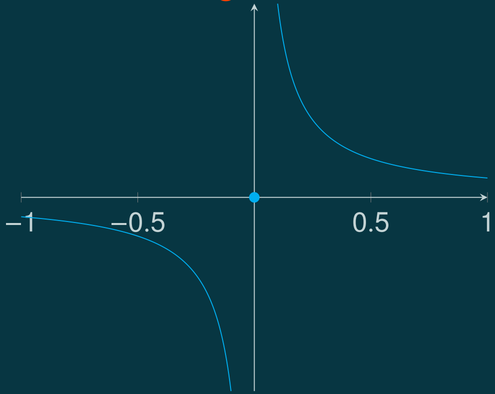

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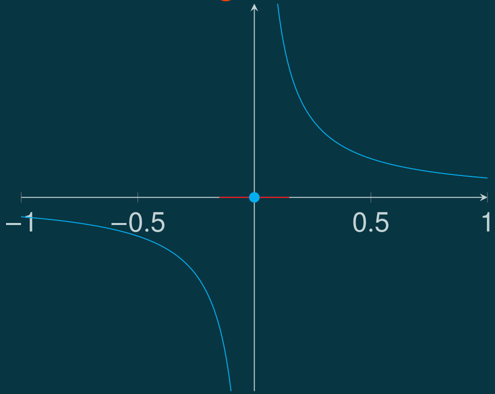

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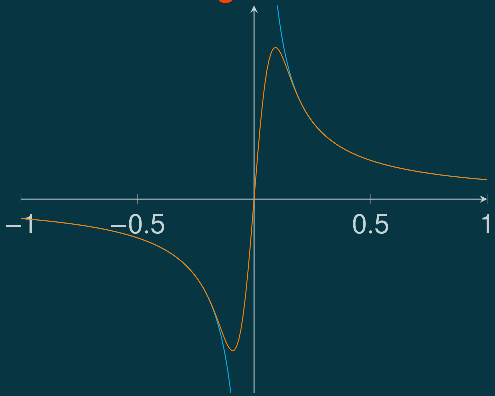

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- Complexity can be improved to $\widetilde{\mathcal{O}}(\kappa)$ using variable-time amplitude-amplification.
- Other variants are possible, such as weighted and generalized least-squares.

The special case of Hermitian matrices

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Removing parity constraint for Hermitian matrices

Let $P\colon [-1,1] \to [-\frac{1}{2}, \frac{1}{2}]$ be a degree-d polynomial map. Suppose that U is an a-qubit block-encoding of a Hermitian matrix H.

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$$U' = \begin{bmatrix} P(H) & . \\ . & . \end{bmatrix},$$

using d times U and $U^\dagger$, 1 controlled U, and $\mathcal{O}(ad)$ extra two-qubit gates.

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Proof sketch

Approximate to ε-precision $\sin(tx)$ and $\cos(tx)$ with polynomials of degree as above. Then use QSVT and combine even/odd parts.

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- Use the previous procedure but with the map $e^{-\frac{\beta}{2}(H - H_0 I)}$
- The final complexity is $\widetilde{O}\left(\beta\sqrt{N}\right)$

Summarizing the various speed-ups by QSVT

Speed-up	Source of speed-up	Examples of algorithms
Exponential	Dimensionality of the Hilbert space	Hamiltonian simulation
	Precise polynomial approximations	Improved HHL algorithm
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Some other applications

- Fast QMA amplification, fast quantum OR lemma
- Quantum Machine learning: PCA, principal component regression
- "Non-commutative measurements" (for ground state preparation)
- Fractional queries
- ⋮

Quantum algorithms for optimization

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In general we want to find the best solution $\min_{x \in X} f(x)$

- Unstructured: can be solved with $\mathcal{O}\left(\sqrt{|X|}\right)$ queries (Dürr & Høyer 1996)

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- NP–hard problems:
  Quadratic speed-ups for Schöning's algorithm for 3-SAT (Ampl. ampl.)
  Quadratic speed-ups for backtracking (Montanaro 2015)
  Polynomial speed-ups for dynamical programming, e.g., TSP $2^n \to 1.73^n$
  (Ambainis, Balodis, Iraids, Kokainis, Prūsis, Vihrovs 2018)

Continuous optimization

Convex optimization

- Linear programs, semidefinite programs
  SDPs: $\widetilde{O}\big((\sqrt{n}+\sqrt{m})s\gamma^5\big)$ (Brandão et al., van Apeldoorn et al. 2016-18)
  Zero-sum games: $\widetilde{O}\big((\sqrt{n}+\sqrt{m})/\varepsilon^3\big)$, $\widetilde{O}\big(s/\varepsilon^{3.5}\big)$ (Apeldoorn & **G** 2019)
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Application of the SDP solver: Shadow tomography (Aaronson 2017)

Given n-dimensional quantum state ρ, estimate the probability of acceptance of each the two-outcome measurements $E_1, \ldots, E_m$, to within additive error ε.

- $$\widetilde{O}\left(\frac{\log^4(m)\log(n)}{\varepsilon^4}\right) \text{ copies suffice}$$

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The main taks is Gibbs sampling from a linear-combination of vectors.

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- $y^{(t+1)} = y^{(t)} + \frac{\varepsilon}{4} e_a$ and $x^{(t+1)} = x^{(t)} + \frac{\varepsilon}{4} e_b$

The main taks is Gibbs sampling from a linear-combination of vectors.
For SDPs: we need to Gibbs sample from a linear-combination of matrices.

Statistics, estimation and stochastic algorithms

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  Generalizes approximate counting (Brassard, Høyer, Mosca, Tapp 1998)

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- Testing equality of a distribution on $[n]$ (with query complexity)
  To an unknown distribution $\widetilde{\mathcal{O}}\left(n^{1/2}\right)$ (Bravyi, Hassidim, Harrow 2009; **G**, Li 2019)
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- Estimating the (Shannon / von Neumann) entropy of a distribution on $[n]$
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  density operator: query complexity $\widetilde{\mathcal{O}}(n)$ (**G** & Li 2019)

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Recommendation systems – Netflix challange

	Inside Out	Good Will Hunting	Mean Girls	Terminator	Titanic	Warrior
Tina Fey	3	1	5	1	?	1
Helen Mirren	2	?	?	2	5	1
Sylvester Stallone	1	3	1	4	2	5
Tom Hanks	?	3	1	?	4	3
George Clooney	2	2	1	3	1	4

Image source: https://towardsdatascience.com

The assumed structure of preference matrix:

Movies: a linear combination of a small number of features
User taste: a linear weighing of the features

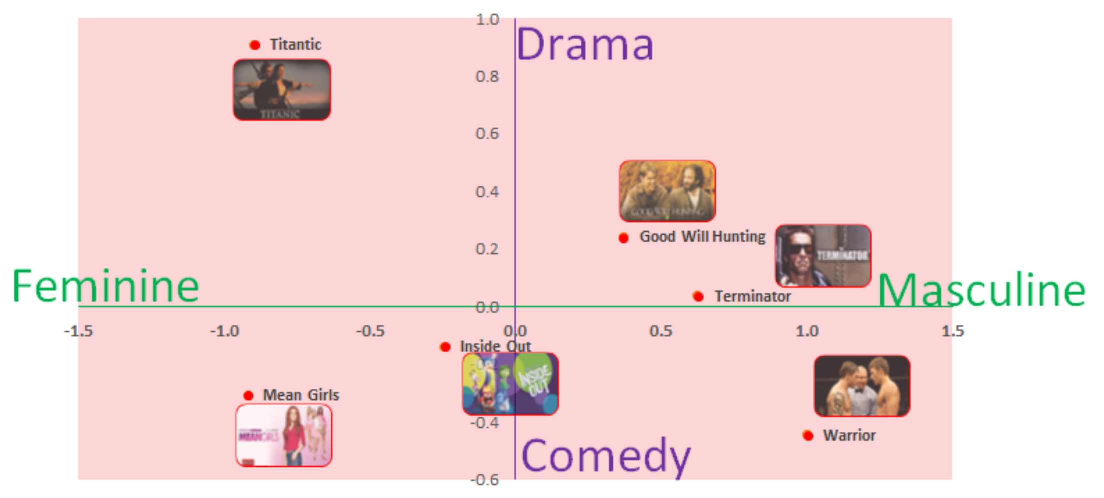

Image source: https://towardsdatascience.com

Major difficulty: how to input the data?

Data conversion: classical to quantum

- Given $b \in \mathbb{R}^m$ prepare

$$|b\rangle = \sum_{i=1}^{m} \frac{b_i}{||b||}$$

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- Given $b \in \mathbb{R}^m$ prepare

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- Given $A \in \mathbb{R}^{m\times n}$ construct quantum circuit (block-encoding)

$$U = \begin{pmatrix} A/||A||_F & \cdot \\ \cdot & \cdot \end{pmatrix}.$$

How to preserve the exponential advantage?

Solution: assume QRAM (readable in superposition)

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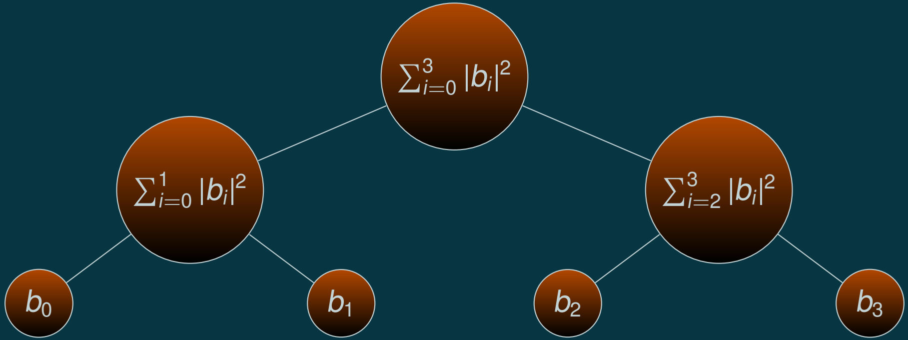
Data structure



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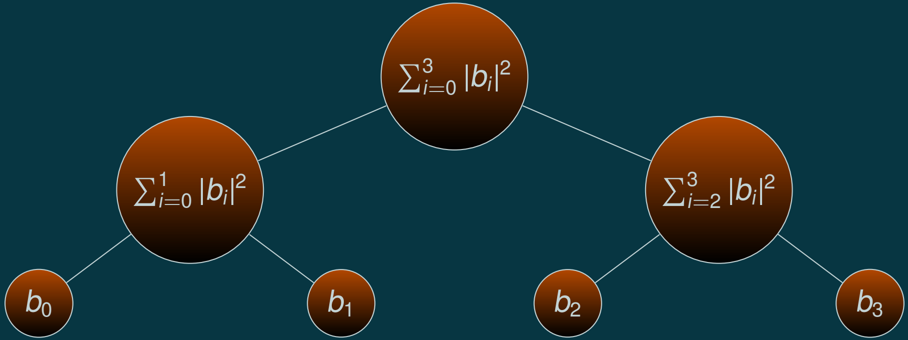
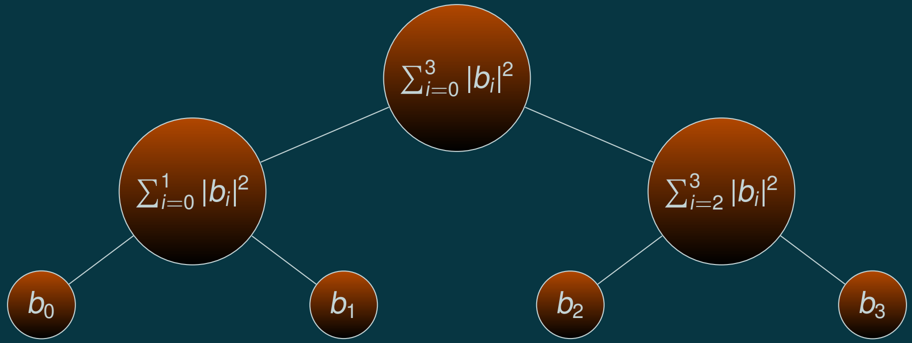


Firs prepare: $\sqrt{\sum_{i=0}^{1} |b_i|^2}|0\rangle + \sqrt{\sum_{i=2}^{3} |b_i|^2}|1\rangle$

Solution: assume QRAM (readable in superposition)

Data structure



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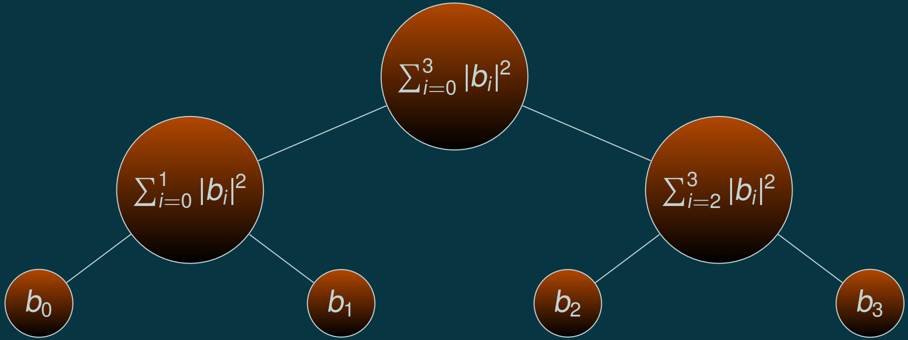

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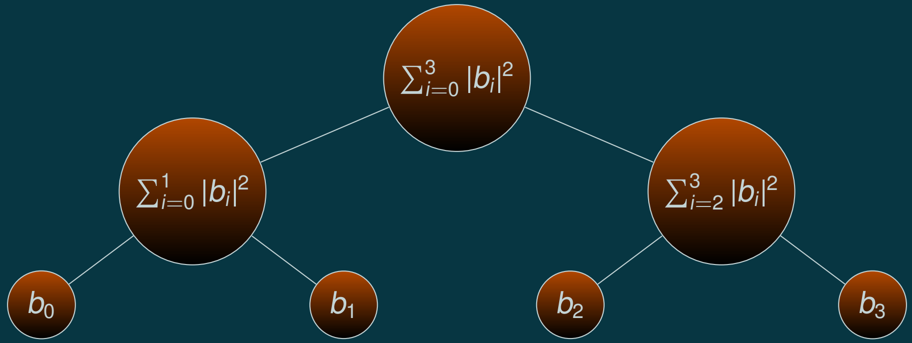
Solution: assume QRAM (readable in superposition)

Data structure



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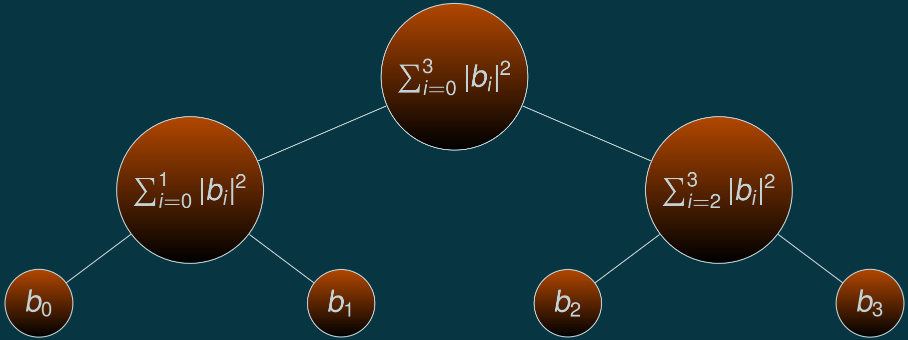
Add phases to get $b_0|00\rangle + b_1|01\rangle + b_2|10\rangle + b_3|11\rangle$

On-line updates to the data structure

Data structure



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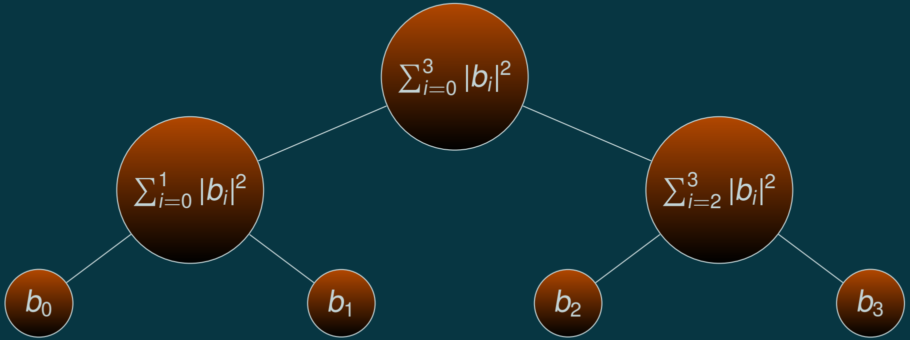

Cost is about the depth: $\log(\text{dimension})$

More about this in Ewin Tang's afternoon talk