A parallel repetition theorem for entangled two-player one-round games under product distributions

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Centre for Quantum Technologies

Outline

1. Introduction
   - The Model of Games
   - Parallel Repetition Theorems

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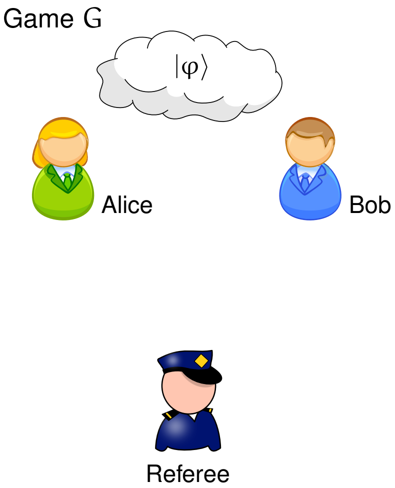
Two-Player One-Round Games



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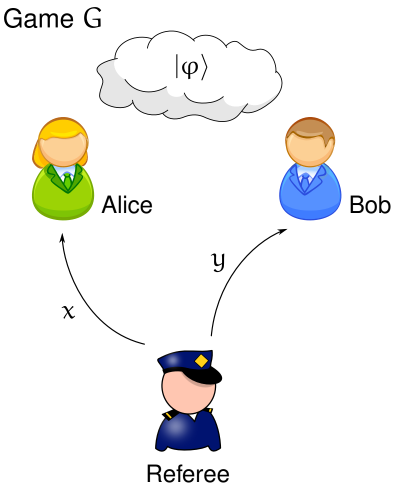

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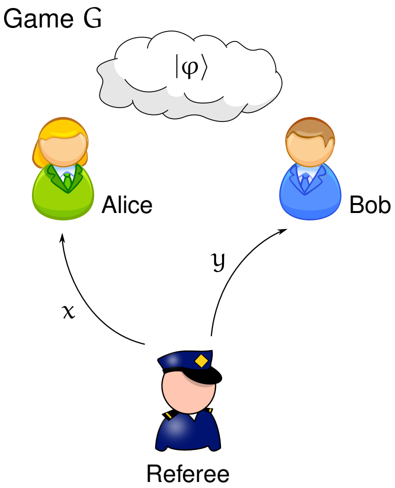

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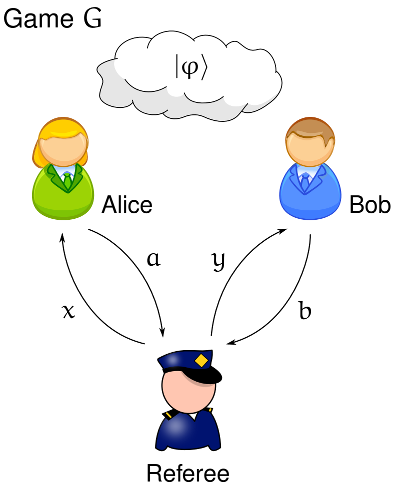

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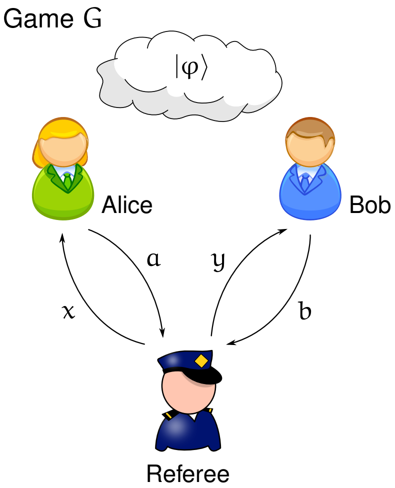

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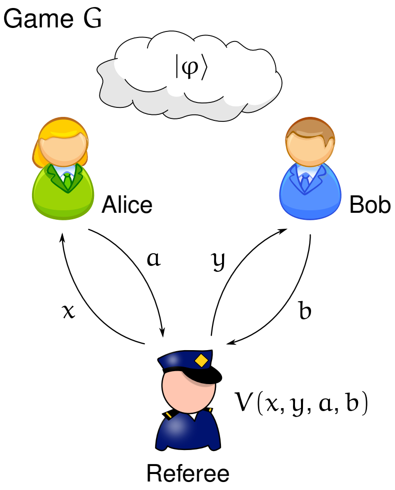

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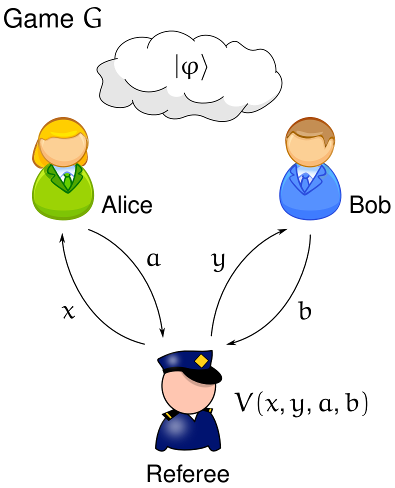

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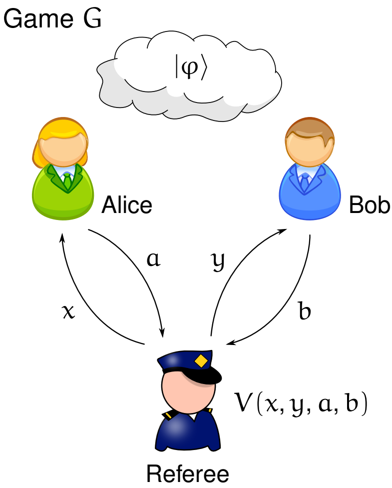

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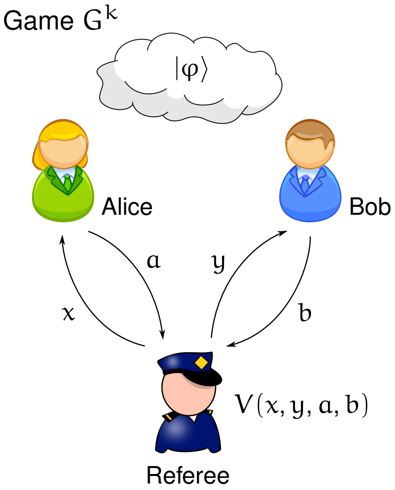
- They win if $V(x, y, a, b) = 1$.
- The value of G, denoted by $\omega^*(G)$, is the supremum of the achievable winning probability.

Parallel Repetition of Games



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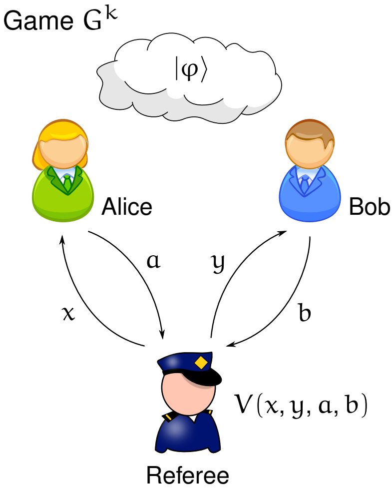

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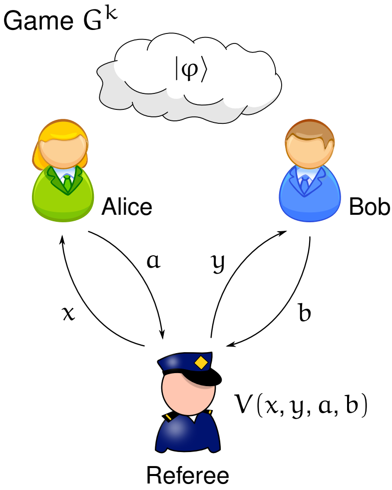

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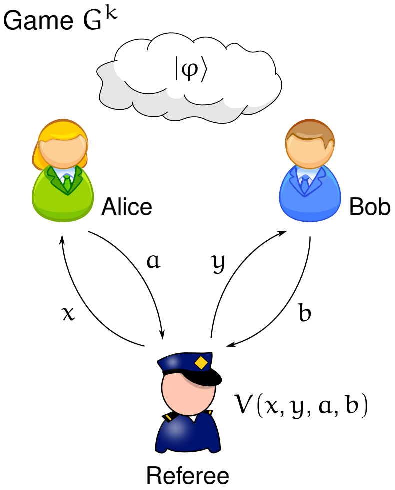

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- (x, y) is distributed according to $\mu^{\otimes k}$, where $\mu^{\otimes k}$ denotes k independent copies of μ.
- $V(x, y, a, b) = 1$ if the players win all the instances.

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   - Parallel Repetition Theorems
2. Proof of the Main Theorem
   - Idea Behind the Proof
   - Some Details

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Analogous result holds for the classical value (denoted by $\omega(G)$):

Theorem ([Raz '95] and [Holenstein '07])

$\exists$ *constant* C *s.t.*

$$\omega(G^k) \leqslant \left(1 - C(1 - \omega(G))^3\right)^{\frac{k}{\log(|\mathcal{A}| \cdot |\mathcal{B}|)}}$$

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- [Chailloux and Scarpa '13] showed it for games where the input distribution is uniform.

Our Contribution

Theorem (Main Theorem)

For any game G, where the input distribution μ is product on $\mathcal{X} \times \mathcal{Y}$, it holds that

$$\omega^*(G^k) = \left(1 - (1 - \omega^*(G))^3\right)^{\Omega\left(\frac{k}{\log(|\mathcal{A}| \cdot |\mathcal{B}|)}\right)} .$$

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  - If they could win the j-th instance with high probability then they would be able to win G with high probability.

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The global state, conditioned on success in $\mathcal{C}$, after the unitaries is of the form

$$\sigma = \sum_{x \in \mathcal{X}^{\times k}, y \in \mathcal{Y}^{\times k}} \mu^{\otimes k}(x, y) \, |xy\rangle\langle xy|^{\mathsf{XY}} \otimes |\phi_{xy}\rangle\langle\phi_{xy}|$$

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where (unnormalized) $|\phi_{xy}\rangle$ is shared between Alice and Bob.

How to generate the state in G^k?

Let us take a strategy for G where the players share

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- This way, we embedded a single instance of G into G^k, with being conditioned on success in $\mathcal{C}$.
  - If $\omega^*(G)$ is bounded away from 1 then success in the j-th instance is also bounded away from 1.

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Is there a way to generate the post-measurement state (approximately) without measurements?

Answer

Yes, if the input distribution μ is product.

How to do it without measurements?

Let $\left|\varphi_{x_j'}\right\rangle$ be the resulting state after Alice measures X_j in $|\varphi\rangle$ and gets x_j'. States $\left|\varphi_{y_j'}\right\rangle$ and $\left|\varphi_{x_j' y_j'}\right\rangle$ are defined similarly.

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- By the unitary equivalence of purifications, $\exists$ unitary $U_{x_j'}$ that Alice can apply to get $\left(U_{x_j'} \otimes \mathbb{1}_{\text{Bob}}\right)|\varphi\rangle \approx \left|\varphi_{x_j'}\right\rangle$.

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- Similarly, $\exists \mathbf{V}_{y_j'}$ s.t. $\left(\mathbb{1}_{\text{Alice}} \otimes \mathbf{V}_{y_j'}\right)|\varphi\rangle \approx \left|\varphi_{y_j'}\right\rangle$.

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Let $\left|\varphi_{x'_j}\right\rangle$ be the resulting state after Alice measures X_j in $|\varphi\rangle$ and gets x'_j. States $\left|\varphi_{y'_j}\right\rangle$ and $\left|\varphi_{x'_j y'_j}\right\rangle$ are defined similarly.

- We show that $I\left(X_j : \mathrm{Bob}\right)_{\varphi} \approx 0$ and $I\left(Y_j : \mathrm{Alice}\right)_{\varphi} \approx 0$.
- $I\left(X_j : \mathrm{Bob}\right)_{\varphi} \approx 0$ implies that Bob's part of $\left|\varphi_{x'_j}\right\rangle$ is mostly independent of x'_j.
- By the unitary equivalence of purifications, $\exists$ unitary $U_{x'_j}$ that Alice can apply to get $\left(U_{x'_j} \otimes \mathbb{1}_{\mathrm{Bob}}\right)|\varphi\rangle \approx \left|\varphi_{x'_j}\right\rangle$.
- Similarly, $\exists V_{y'_j}$ s.t. $\left(\mathbb{1}_{\mathrm{Alice}} \otimes V_{y'_j}\right)|\varphi\rangle \approx \left|\varphi_{y'_j}\right\rangle$.
- By [Jain, Radhakrishnan, Sen '08], if the distribution of $\left(x'_j, y'_j\right)$ is product then
$$\left(U_{x'_j} \otimes V_{y'_j}\right)|\varphi\rangle \approx \left|\varphi_{x'_j y'_j}\right\rangle.$$

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From Measurements to Unitaries (1 Sided)

Lemma

Let μ be a probability distribution on $\mathcal{X}$. Let

$$|\varphi\rangle \stackrel{\text{def}}{=} \sum_{x \in \mathcal{X}} \sqrt{\mu(x)}\, |xx\rangle^{X\tilde{X}} \otimes |\psi_x\rangle$$

be shared by Alice and Bob, where X, $\tilde{X}$, and some part of $|\psi_x\rangle$ are with Alice and the rest of $|\psi_x\rangle$ is with Bob.

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be shared by Alice and Bob, where X, $\tilde{X}$, and some part of $|\psi_x\rangle$ are with Alice and the rest of $|\psi_x\rangle$ is with Bob. Let $|\varphi_x\rangle \stackrel{\text{def}}{=} |xx\rangle \otimes |\psi_x\rangle$. If $I(X : \text{Bob})_\varphi \leqslant \varepsilon$ then there exist unitaries $\{U_x\}_{x \in \mathcal{X}}$ acting on Alice's space s.t.

$$\mathbb{E}_{x \leftarrow \mu}[\| |\varphi_x\rangle\langle\varphi_x| - (U_x \otimes \mathbb{1}_{\text{Bob}})\, |\varphi\rangle\langle\varphi|\, (U_x^* \otimes \mathbb{1}_{\text{Bob}}) \|_1] \leqslant 4\sqrt{\varepsilon}.$$

From Measurements to Unitaries (1 Sided)

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The proof easily follows from the unitary equivalence of purifications and Uhlmann's theorem.

From Measurements to Unitaries (2 Sided)

Lemma

Let μ be a prob. dist. on $\mathcal{X} \times \mathcal{Y}$ with marginals μ_X and μ_Y. Let

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$$\leqslant 8\sqrt{\varepsilon} + 2\,\|\mu - \mu_X \otimes \mu_Y\|_1 .$$

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$$\leqslant 8\sqrt{\varepsilon} + 2 \left\| \mu - \mu_X \otimes \mu_Y \right\|_1 .$$

Proved by [Jain, Radhakrishnan, Sen '08].

Key Lemma

Our main theorem follows from the following lemma. Show theorem

Lemma (Key Lemma)

Let $1/10 > \delta_1, \delta_2, \delta_3 > 0$ *s.t.* $\delta_3 = \delta_2 + \delta_1 \cdot \log(|\mathcal{A}| \cdot |\mathcal{B}|)$. *Let* $k' \stackrel{\text{def}}{=} \lfloor \delta_1 k \rfloor$.

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$$\Pr\left[T^{(l)} = 1\right] \leqslant 2^{-\delta_2 k}$$

where $T_i \in \{0, 1\}$ *indicates success in the* i*-th repetition and* $T^{(l)} \stackrel{\text{def}}{=} \prod_{j=1}^{l} T_{i_j}$.

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$$\Pr\left[T^{(l)} = 1\right] \leqslant 2^{-\delta_2 k} \qquad \textit{or}$$

$$\Pr\left[T_{i_{l+1}} = 1 \middle| T^{(l)} = 1\right] \leqslant \omega^*(G) + 12\sqrt{10\delta_3}$$

where $T_i \in \{0, 1\}$ *indicates success in the* i*-th repetition and* $T^{(l)} \stackrel{\text{def}}{=} \prod_{j=1}^{l} T_{i_j}$.

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The purified state after the unitaries is

$$|\theta\rangle \stackrel{\text{def}}{=} \sum_{x \in \mathcal{X}^{\times k}, y \in \mathcal{Y}^{\times k}} \sqrt{\mu^{\otimes k}(x,y)}\, |xxyy\rangle^{X\tilde{X}Y\tilde{Y}} \otimes \sum_{a_{\mathcal{C}} \in \mathcal{A}^{\times l}, b_{\mathcal{C}} \in \mathcal{B}^{\times l}} |a_{\mathcal{C}} b_{\mathcal{C}}\rangle^{A_{\mathcal{C}} B_{\mathcal{C}}} \otimes \left|\gamma_{x,y,a_{\mathcal{C}},b_{\mathcal{C}}}\right\rangle^{E_A E_B}.$$

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Conditioning on success in $\mathcal{C}$ gives us the state

$$|\varphi\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{q}} \sum_{x,y} \sqrt{\mu^{\otimes k}(x,y)}\, |x x y y\rangle \sum_{\substack{a_\mathcal{C}, b_\mathcal{C} \text{ s.t.} \\ \prod_{i \in \mathcal{C}} T_i = 1}} |a_\mathcal{C} b_\mathcal{C}\rangle \otimes \left|\gamma_{x,y,a_\mathcal{C},b_\mathcal{C}}\right\rangle.$$

Lemma about the Relative Entropy

From $q > 2^{-\delta_2 k}$, we show the following simple lemma.

Lemma

$$\mathop{\mathbb{E}}_{x_{\mathcal{C}},y_{\mathcal{C}},a_{\mathcal{C}},b_{\mathcal{C}} \leftarrow \varphi^{X_{\mathcal{C}}Y_{\mathcal{C}}A_{\mathcal{C}}B_{\mathcal{C}}}} \left[S\left(\varphi_{x_{\mathcal{C}},y_{\mathcal{C}},a_{\mathcal{C}},b_{\mathcal{C}}}^{XY\tilde{X}_{\overline{\mathcal{C}}}\tilde{Y}_{\overline{\mathcal{C}}}E_A E_B} \middle\| \theta_{x_{\mathcal{C}},y_{\mathcal{C}}}^{XY\tilde{X}_{\overline{\mathcal{C}}}\tilde{Y}_{\overline{\mathcal{C}}}E_A E_B} \right) \right] \leqslant \delta_3 k$$

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Intuitively,

- going from θ to φ causes a difference of at most $-\log q < \delta_2 k$.
- further measuring $A_\mathcal{C}$ and $B_\mathcal{C}$ results in a difference of at most $|\mathcal{C}| \cdot \log(|\mathcal{A}| \cdot |\mathcal{B}|) \leqslant \delta_1 k \cdot \log(|\mathcal{A}| \cdot |\mathcal{B}|)$.

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Using the previous lemma, we can show the required upper bound for the mutual information.

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&= I(X : \mathrm{Bob} | X_{\mathcal{C}} Y_{\mathcal{C}} A_{\mathcal{C}} B_{\mathcal{C}})_{\varphi}
\end{aligned}$$

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&= I(X : \mathrm{Bob} | X_{\mathcal{C}}Y_{\mathcal{C}}A_{\mathcal{C}}B_{\mathcal{C}})_{\varphi} \\
&= \sum_{i \in \overline{\mathcal{C}}} I\left(X_i : \mathrm{Bob} \middle| X_{\mathcal{C}\cup[i-1]}Y_{\mathcal{C}}A_{\mathcal{C}}B_{\mathcal{C}}\right)_{\varphi}
\end{aligned}
$$

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Distribution of Questions

Using the same lemma, we show that for most of the coordinates in $\overline{\mathcal{C}}$ the distribution of questions is close to μ in $|\varphi\rangle$.

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&\geqslant \mathop{\mathbb{E}}_{x_{\mathcal{C}},y_{\mathcal{C}},a_{\mathcal{C}},b_{\mathcal{C}} \leftarrow \varphi^{X_{\mathcal{C}}Y_{\mathcal{C}}A_{\mathcal{C}}B_{\mathcal{C}}}} \left[S\left(\varphi^{XY}_{x_{\mathcal{C}},y_{\mathcal{C}},a_{\mathcal{C}},b_{\mathcal{C}}} \middle\| \theta^{XY}_{x_{\mathcal{C}},y_{\mathcal{C}}} \right) \right]
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$$
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&\geqslant \mathop{\mathbb{E}}_{x_{\mathcal{C}},y_{\mathcal{C}},a_{\mathcal{C}},b_{\mathcal{C}} \leftarrow \varphi^{X_{\mathcal{C}}Y_{\mathcal{C}}A_{\mathcal{C}}B_{\mathcal{C}}}} \left[S\left(\varphi^{XY}_{x_{\mathcal{C}},y_{\mathcal{C}},a_{\mathcal{C}},b_{\mathcal{C}}} \middle\| \theta^{XY}_{x_{\mathcal{C}},y_{\mathcal{C}}} \right) \right] \\
&= \sum_{i \in \overline{\mathcal{C}}} \mathop{\mathbb{E}}_{r_i \leftarrow \varphi^{R_i}} \left[S\left(\varphi^{X_i Y_i}_{r_i} \middle\| \theta^{X_i Y_i}_{x_{\mathcal{C}\cup[i-1]},y_{\mathcal{C}\cup[i-1]}} \right) \right]
\end{aligned}
$$

where $R_i \stackrel{\text{def}}{=} X_{\mathcal{C}\cup[i-1]} Y_{\mathcal{C}\cup[i-1]} A_{\mathcal{C}} B_{\mathcal{C}}$.

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Using the same lemma, we show that for most of the coordinates in $\overline{\mathcal{C}}$ the distribution of questions is close to μ in $|\varphi\rangle$.

$$\begin{aligned}
\delta_3 k &\geqslant \mathop{\mathbb{E}}_{x_{\mathcal{C}},y_{\mathcal{C}},a_{\mathcal{C}},b_{\mathcal{C}} \leftarrow \varphi^{X_{\mathcal{C}}Y_{\mathcal{C}}A_{\mathcal{C}}B_{\mathcal{C}}}} \left[S\left(\varphi^{XY\tilde{X}_{\overline{\mathcal{C}}}\tilde{Y}_{\overline{\mathcal{C}}}E_A E_B}_{x_{\mathcal{C}},y_{\mathcal{C}},a_{\mathcal{C}},b_{\mathcal{C}}} \middle\| \theta^{XY\tilde{X}_{\overline{\mathcal{C}}}\tilde{Y}_{\overline{\mathcal{C}}}E_A E_B}_{x_{\mathcal{C}},y_{\mathcal{C}}} \right) \right] \\
&\geqslant \mathop{\mathbb{E}}_{x_{\mathcal{C}},y_{\mathcal{C}},a_{\mathcal{C}},b_{\mathcal{C}} \leftarrow \varphi^{X_{\mathcal{C}}Y_{\mathcal{C}}A_{\mathcal{C}}B_{\mathcal{C}}}} \left[S\left(\varphi^{XY}_{x_{\mathcal{C}},y_{\mathcal{C}},a_{\mathcal{C}},b_{\mathcal{C}}} \middle\| \theta^{XY}_{x_{\mathcal{C}},y_{\mathcal{C}}} \right) \right] \\
&= \sum_{i\in\overline{\mathcal{C}}} \mathop{\mathbb{E}}_{r_i \leftarrow \varphi^{R_i}} \left[S\left(\varphi^{X_i Y_i}_{r_i} \middle\| \theta^{X_i Y_i}_{x_{\mathcal{C}\cup[i-1]},y_{\mathcal{C}\cup[i-1]}} \right) \right] \\
&\geqslant \sum_{i\in\overline{\mathcal{C}}} \mathop{\mathbb{E}}_{r_i \leftarrow \varphi^{R_i}} \left[\left\| \varphi^{X_i Y_i}_{r_i} - \mu \right\|_1^2 \right]
\end{aligned}$$

where $R_i \stackrel{\text{def}}{=} X_{\mathcal{C}\cup[i-1]} Y_{\mathcal{C}\cup[i-1]} A_{\mathcal{C}} B_{\mathcal{C}}$.

Distribution of Questions

Using the same lemma, we show that for most of the coordinates in $\overline{\mathcal{C}}$ the distribution of questions is close to μ in $|\varphi\rangle$.

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\begin{aligned}
\delta_3 k &\geqslant \mathop{\mathbb{E}}_{x_{\mathcal{C}},y_{\mathcal{C}},a_{\mathcal{C}},b_{\mathcal{C}} \leftarrow \varphi^{X_{\mathcal{C}}Y_{\mathcal{C}}A_{\mathcal{C}}B_{\mathcal{C}}}} \left[S\left(\varphi^{XY\tilde{X}_{\overline{\mathcal{C}}}\tilde{Y}_{\overline{\mathcal{C}}}E_A E_B}_{x_{\mathcal{C}},y_{\mathcal{C}},a_{\mathcal{C}},b_{\mathcal{C}}} \middle\| \theta^{XY\tilde{X}_{\overline{\mathcal{C}}}\tilde{Y}_{\overline{\mathcal{C}}}E_A E_B}_{x_{\mathcal{C}},y_{\mathcal{C}}} \right) \right] \\
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&\geqslant \sum_{i \in \overline{\mathcal{C}}} \left(\mathop{\mathbb{E}}_{r_i \leftarrow \varphi^{R_i}} \left[\left\| \varphi^{X_i Y_i}_{r_i} - \mu \right\|_1 \right] \right)^2
\end{aligned}
$$

where $R_i \stackrel{\text{def}}{=} X_{\mathcal{C}\cup[i-1]} Y_{\mathcal{C}\cup[i-1]} A_{\mathcal{C}} B_{\mathcal{C}}$.

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$$\mathop{\mathbb{E}}_{r_j \leftarrow \varphi^{R_j}} \left[\left\|\varphi_{r_j}^{X_j Y_j} - \varphi_{r_j}^{X_j} \otimes \varphi_{r_j}^{Y_j}\right\|_1\right] \leqslant \sqrt{10\delta_3}$$

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With these, and by treating R_j as public coins, it's easy to show that we can embed G into G^k.

Summary

We proved the following parallel repetition theorem.

Theorem (Main Theorem)

For any game G, where the input distribution μ is product on $\mathcal{X} \times \mathcal{Y}$, it holds that

$$\omega^*(G^k) = \left(1 - (1 - \omega^*(G))^3\right)^{\Omega\left(\frac{k}{\log(|\mathcal{A}|\cdot|\mathcal{B}|)}\right)}.$$

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A parallel repetition theorem for arbitrary games where the exponent only depends on k and $|\mathcal{A}| \cdot |\mathcal{B}|$ is still unknown.

Thank you for your attention!

The manuscript is available at arXiv:1311.6309.