Limitations for Quantum PCPs

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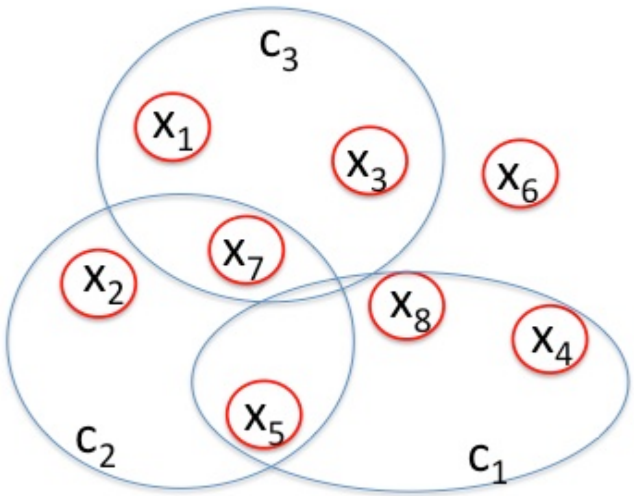
Based on joint work arXiv:1310.0017 with

Aram Harrow
MIT

Simons Institute, Berkeley, February 2014

Constraint Satisfaction Problems



(k, Σ, n, m)-CSP :

k: arity

Σ: alphabet

n: number of variables

m: number of constraints

Constraints: $C_j : \Sigma^k \to \{0, 1\}$

Assignment: $\sigma : [n] \to \Sigma$

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Quantum Constraint Satisfaction Problems

(k, d, n, m)-*q*CSP *H*

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Constraints: P_j *k*-local projection
Assignment: $|\psi>$ quantum state

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$$\mathrm{unsat}(H) := \min_{|\psi\rangle} \frac{1}{m} \sum_{j=1}^{m} \langle\psi|P_j|\psi\rangle = \frac{1}{m} \lambda_{\min}\left(\sum_j P_j\right)$$

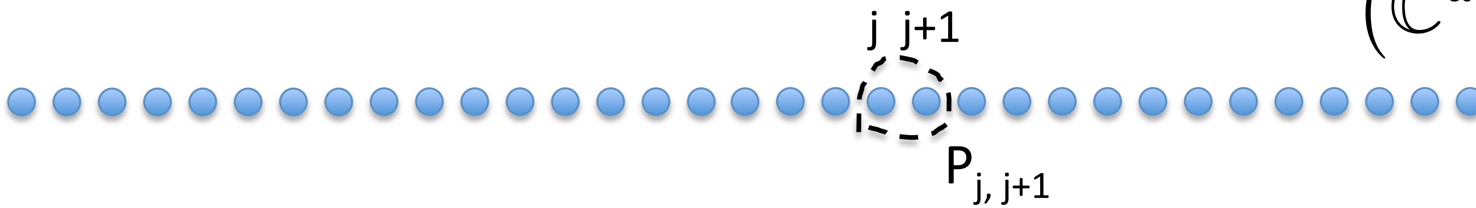
min eigenvalue

Hamiltonian

Quantum Constraint Satisfaction Problems

Ex 1: (2, 2, n, n-1)-qCSP on a line

$$\left(\mathbb{C}^d\right)^{\otimes n}$$



$$H = \sum_j P_j = \sum_j \mathrm{id}_{1,\ldots,j-1} \otimes P_{j,j+1} \otimes \mathrm{id}_{j+2,\ldots,n}$$

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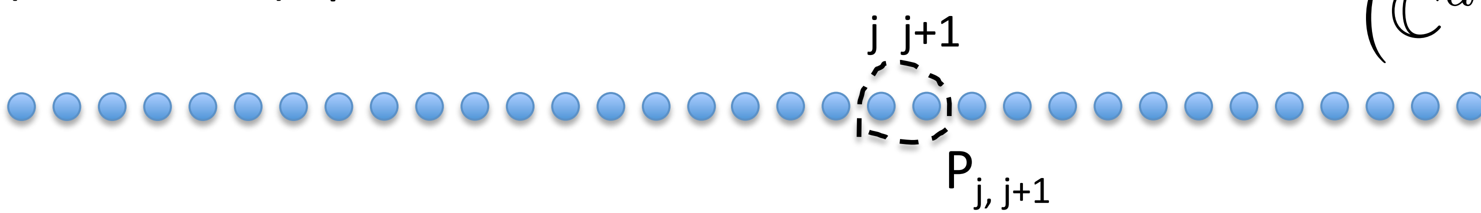

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Ex 2: (2, 2, n, m)-qCSP with diagonal projectors:

$$P_j = \sum_{x_{j_1}, x_{j_2}} C_j(x_{j_1}, x_{j_2}) |x_{j_1}, x_{j_2}\rangle\langle x_{j_1}, x_{j_2}|$$

$$\begin{aligned}
\mathrm{unsat}\left(\sum_j P_j\right) &= \min_{\{x_1,\ldots,x_n\}\in\{0,1\}^n} \sum_j \langle x_1,\ldots,x_n|P_j|x_1,\ldots,x_n\rangle / m \\
&= \min_{\{x_1,\ldots,x_n\}\in\{0,1\}^n} C_j(x_{j_1}, x_{j_2}) / m \\
&= \mathrm{unsat}(C)
\end{aligned}$$

PCP Theorem

PCP Theorem (Arora, Safra; Arora-Lund-Motwani-Sudan-Szegedy '98)
There is a $\varepsilon > 0$ s.t. it's NP-hard to determine whether for a CSP, unsat $= 0$ or unsat $> \varepsilon$

- Compare with Cook-Levin thm:
  It's NP-hard to determine whether unsat $= 0$ or unsat $> 1/m$.
- Equivalent to the existence of **P**robabilistically **C**heckable **P**roofs for NP.
- (Dinur '07) Combinatorial proof.
- Central tool in the theory of hardness of approximation.

Quantum Cook-Levin Thm

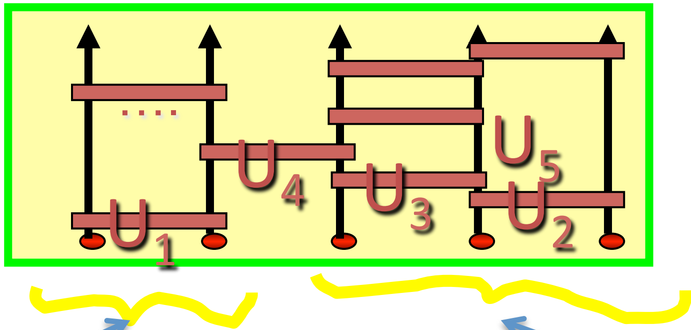
Local Hamiltonian Problem

locality local dim

Given a (k, d, n, m)-qcsp H with constant *k, d* and m = poly(n), decide if unsat(H)=0 or unsat(H)>Δ

Thm (Kitaev '99) The local Hamiltonian problem is QMA-complete for Δ = 1/poly(n)

QMA is the quantum analogue of NP, where the proof and the computation are quantum.



input

proof

Quantum PCP?

The Quantum PCP conjecture: There is $\varepsilon > 0$ s.t. the following problem is QMA-complete: Given $(2, 2, n, m)$-qcsp H determine whether

locality local dim

(i) $\mathrm{unsat}(H)=0$ or (ii) $\mathrm{unsat}(H) > \varepsilon$.

- (Bravyi, DiVincenzo, Loss, Terhal '08) Equivalent to conjecture for (k, d, n, m)-qcsp for any constant k, d.
- At least **NP-hard** (by PCP Thm) and **inside QMA**
- Open *even* for commuting qCSP ($[P_i, P_j] = 0$)

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  no good ansatz for *any* low-energy state

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- For more motivation see review (Aharonov, Arad, Vidick '13) and Thomas recorded talk on bootcamp week

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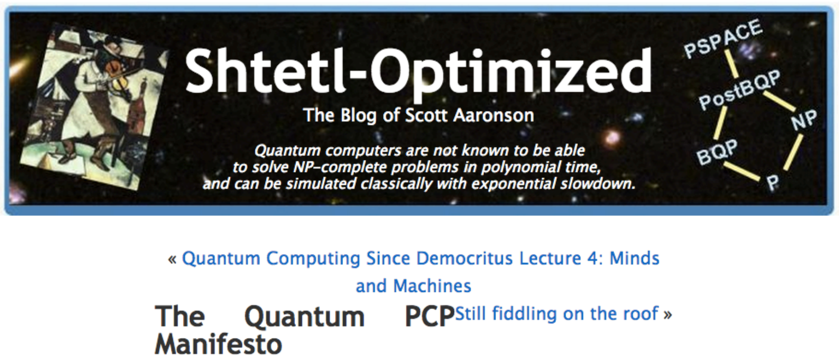

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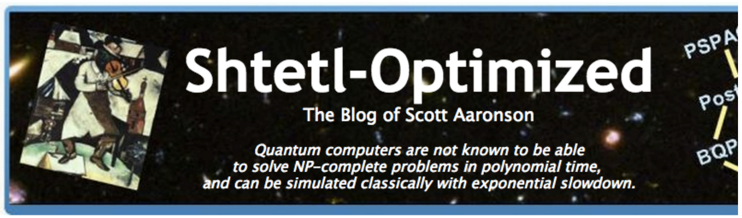

« Quantum Computing Since Democritus Lecture 4: Minds and Machines

The Quantum PCP Manifesto

Still fiddling on the roof »

I’m 99% sure that this theorem (alright, conjecture) or something close to it is true. I’m 95% sure that the proof will require a difficult adaptation of classical PCP machinery (whether Iritean or pre-Iritean), in much the same way that the Quantum Fault-Tolerance Theorem required a difficult adaptation of classical fault-tolerance machinery. I’m 85% sure that the proof is achievable in a year or so, should enough people make it a priority. I’m 75% sure that the proof, once achieved, will open up heretofore undreamt-of vistas of understanding and insight. I’m 0.01% sure that I can prove it. And that is why I hereby bequeath the actual proving part to you, my readers.

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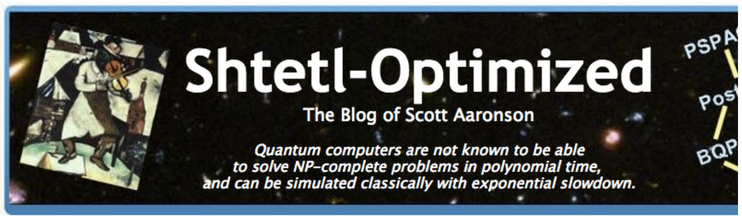

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I'm quite certain that a Quantum PCP Theorem will require significant new ideas. Recently I spent a day or two studying Irit's proof of the classical PCP theorem (which I hadn't done before), and I found about 20 violations of the No-Cloning Theorem on every page. 😃

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- (B. Harrow '13) Approx. in NP for 2-local non-commuting qCSP

this talk

"Blowing up" maps

prop For every $t \geq 1$ there is an efficient mapping from $(2, \Sigma, n, m)$-csp C to $(2, \Sigma_t, n_t, m_t)$-csp C_t s.t.

(i) $n_t \leq n^{O(t)}$, $m_t \leq m^{O(t)}$

(ii) $\deg(C_t) \geq \deg(C)^t$

(iii) $|\Sigma_t| = |\Sigma|^t$

(iv) $\mathrm{unsat}(C_t) \geq \mathrm{unsat}(C)$

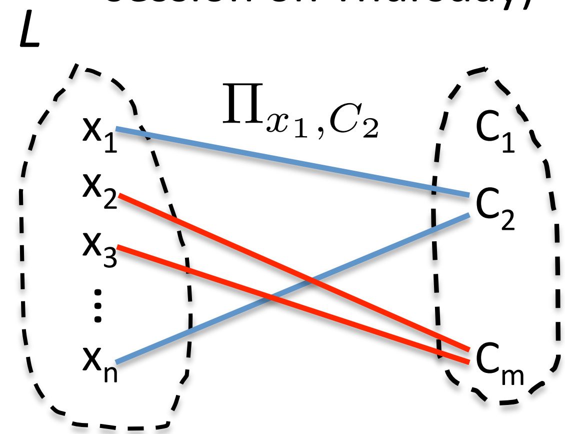
(v) $\mathrm{unsat}(C_t) = 0$ if $\mathrm{unsat}(C) = 0$

Example: Parallel Repetition (for kids)

(see parallel repetition session on Thursday)

1. write C as a *cover label instance* L on $G(V, W, E)$ with function $\Pi_{v,w} : [N] \rightarrow [M]$

Labeling $l : V \rightarrow [N], W \rightarrow [M]$ covers edge (v, w) if $\Pi_{v,w}(l(w)) = l(v)$



2. Define L_t on graph $G'(V', W', E')$ with $V' = V^t$, $W' = W^t$, $[N'] = [N]^t$, $[M'] = [M]^t$

Edge set: $(v' = \{v_{i_1}, \ldots, v_{i_t}\}, w' = \{w_{i_1}, \ldots . w_{t_t}\}) \in E$ iff $(v_{i_j}, w_{i_j}) \in E, \ \forall\, i \in [n], 0 \leq j \leq t$

Function: $\Pi_{v',w'}(b_1, \ldots, b_t) = \{\Pi_{v_1,w_1}, \ldots, \Pi_{v_t,w_t}\}$

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Labeling
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C_1
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Easy to see:

(i) $n_t \leq n^{O(t)}$, $m^{O(t)}$

(ii) $\mathrm{Deg}(L_t) \geq \deg(C)^t$,

(iii) $\mathrm{unsat}(L_t) \geq \mathrm{unsat}(C)$,

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(vi) $\mathrm{unsat}(L_t) \geq \mathrm{unsat}(C)$

In fact: (Raz '95) If $\mathrm{unsat}(C) \geq \delta$, $\mathrm{unsat}(L_t) \geq 1 - \exp(-\Omega(\delta^3 t))$

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Quantum “Blowing up” maps + Quantum PCP?

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then the quantum PCP conjecture is *false*.

Formalizes difficulty of "quantizing" proofs of the PCP theorem

(e.g. Dinur's proof; see (Aharonov, Arad, Landau, Vazirani '08))

Obs: Apparently *not* related to parallel repetition for quantum games

(see session on Thursday)

Entanglement Monogamy…

…is the main idea behind the result.

Entanglement cannot be freely shared

Ex. 1 $\rho_{AB} = |\phi^+\rangle\langle\phi^+|_{AB}, \quad |\phi^+\rangle = (|0,0\rangle + |1,1\rangle)/\sqrt{2}$, $\rho_{ABC} = \rho_{AB} \otimes \rho_C$

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Ex. 2 $|\mathrm{CAT}\rangle_{A_1,\ldots,A_n} = (|0,\ldots,0\rangle + |1,\ldots,1\rangle)/\sqrt{2}$

$$\rho_{A_i A_j} := \mathrm{tr}_{\backslash A_i A_j}(|\mathrm{CAT}\rangle\langle\mathrm{CAT}|) = (|0,0\rangle\langle 0,0| + |1,1\rangle\langle 1,1|)/2$$

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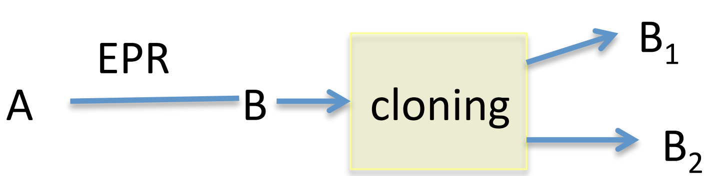
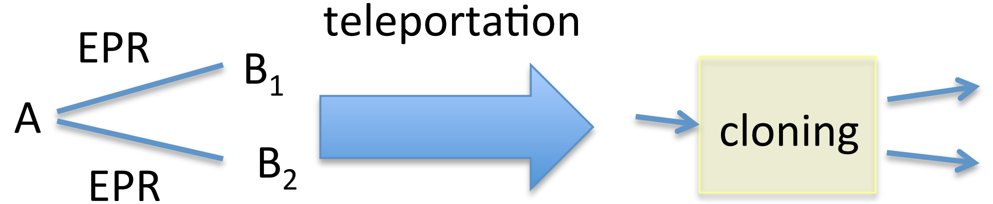
Monogamy vs cloning:



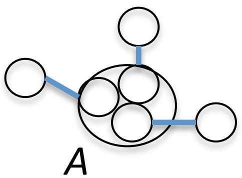


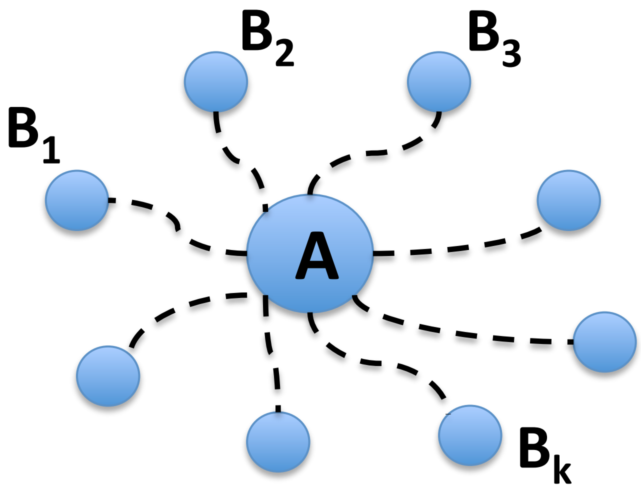
A maximally entangled with B_1 and B_2

Entanglement Monogamy…

…intuition:

- *A* can only be substantially entangled with a few of the *B*s
- How entangled it can be depends on the *size* of *A*.

Ex. 



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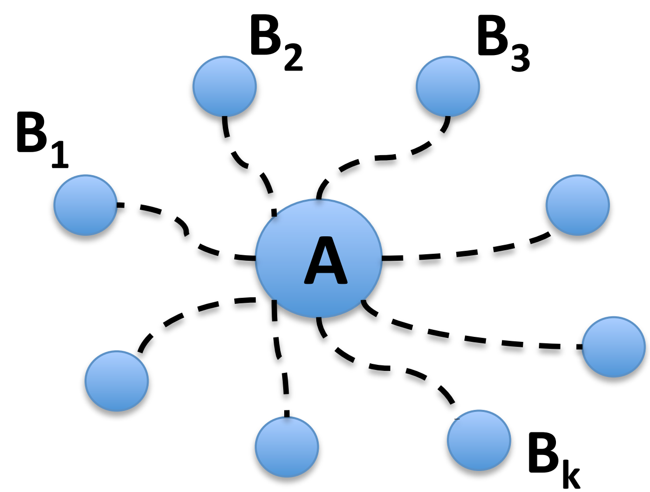

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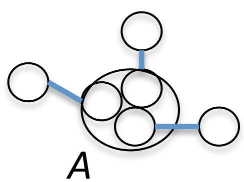
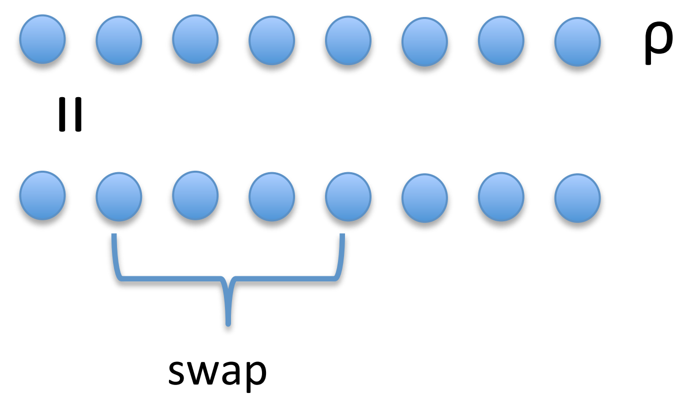
Ex. 

How to make it quantitative?

1. Study behavior of entanglement measures (distillable entanglement, squashed entanglement, …) (see Patrick's talk)
2. Study specific tasks (QKD, MIP*, …) (see sessions on MIP and device independent crypto)
3. Quantum de Finetti Theorems (see also Aram's talk)

Quantum de Finetti Theorems

Let $\rho_{1,\ldots,n}$ be permutation-symmetric, i.e.



Quantum de Finetti Thm:

$$\rho_{1,\ldots,l} \approx \sum_k p_k \rho_k^{\otimes l}$$

$$\frac{d^2 l}{n}$$

(Christandl, Koenig, Mitchson, Renner '05)

- In complete analogy with de Finetti thm for symmetric probability distributions
- But much more remarkable: *entanglement* is destroyed

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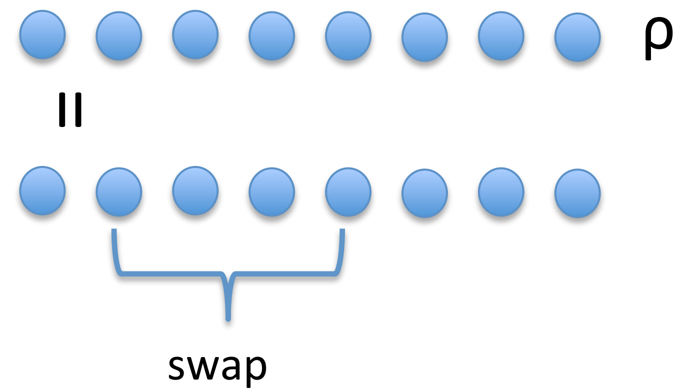

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- Final installment in a long sequence of works: (Hudson, Moody '76), (Stormer '69), (Raggio, Werner '89), (Caves, Fuchs, Schack '01), (Koenig, Renner '05), …
- Can we improve on the error? (see Aram's and Patrick's talk)
- Can we find a more general result, beyond permutation-invariant states?

General Quantum de Finetti

thm (B., Harrow '13) Let G = (V, E) be a D-regular graph with n = |V|. Let $\rho_{1,\dots,n}$ be a n-qudit state. Then there exists a *globally separable* state $\sigma_{1,\dots,n}$ such that

$$\mathop{\mathbb{E}}_{(i,j)\in E} \|\rho_{i,j} - \sigma_{i,j}\|_1 \leq 12 \left(\frac{d^2 \ln(d)}{D} \right)^{1/3}$$

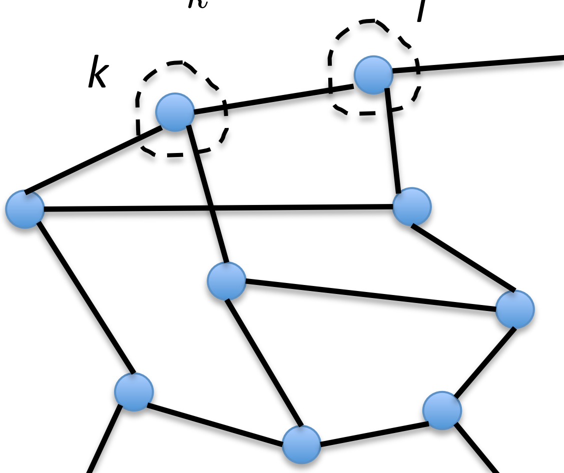
Globally separable (unentangled):

$$\rho_{ij} \approx \sigma_{ij} = \sum_k p_k \sigma_{k_i} \otimes \sigma_{k_j}$$

$$\sigma = \sum_k p_k \sigma_{k_1} \otimes \dots \otimes \sigma_{k_n}$$

probability distribution

local states



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Ex 1. "Local entanglement":

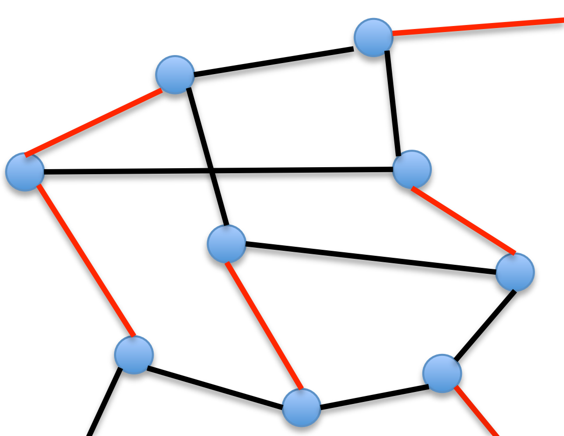
For (i, j) red: $\|\rho_{i,j} - \sigma_{i,j}\|_1 \geq 1/4$

EPR Separable

But for all other (i, j): $\rho_{i,j} = \rho_i \otimes \rho_j$

$\sigma = \rho_1 \otimes \ldots \otimes \rho_n$ gives good approx.

Red edge: EPR pair



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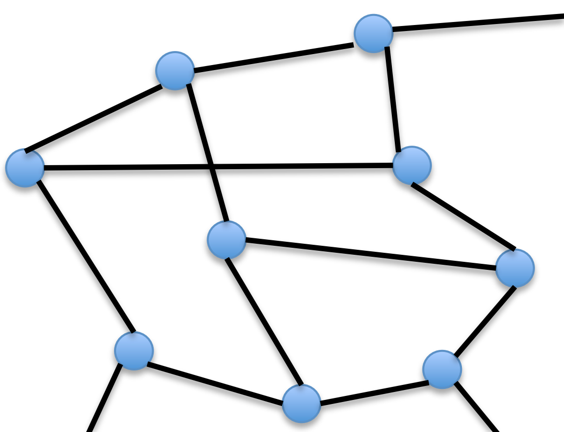
Ex 2. "Global entanglement":

Let ρ = |φ><φ| be a Haar random state

|φ> has a lot of entanglement (e.g. for every region *X*, S(X) ≈ number qubits in *X*)

But: $\rho_{i,j} \approx \frac{\mathrm{id} \otimes \mathrm{id}}{d^2}$



General Quantum de Finetti

thm (B., Harrow '13) Let $G = (V, E)$ be a D-regular graph with $n = |V|$. Let $\rho_{1,\ldots,n}$ be a n-qudit state. Then there exists a *globally separable* state $\sigma_{1,\ldots,n}$ such that

$$\mathop{\mathbb{E}}_{(i,j)\in E} \|\rho_{i,j} - \sigma_{i,j}\|_1 \leq 12 \left(\frac{d^2 \ln(d)}{D} \right)^{1/3}$$

Ex 3.

Let $\rho = |CAT\rangle\langle CAT|$ with $|CAT\rangle = (|0, \ldots, 0\rangle + |1, \ldots, 1\rangle)/\sqrt{2}$

$$\rho_{i,j} = \frac{1}{2}|0,0\rangle\langle 0,0| + \frac{1}{2}|1,1\rangle\langle 1,1|$$

$$\sigma = \frac{1}{2}|0,\ldots,0\rangle\langle 0,\ldots,0| + \frac{1}{2}|1,\ldots,1\rangle\langle 1,\ldots,1|$$

gives a good approximation

Product-State Approximation

cor Let G = (V, E) be a D-regular graph with n = |V|. Let

$$H = \sum_{(i,j)\in E} P_{i,j}$$

Then there exists $|\phi\rangle = |\phi_1\rangle \otimes \ldots \otimes |\phi_n\rangle$ such that

$$\frac{2}{nD}\langle\phi|H|\phi\rangle \leq \text{unsat}(H) + 12\left(\frac{d^2\log(d)}{D}\right)^{1/3}$$

- The problem is in NP for $\varepsilon = O(d^2\log(d)/D)^{1/3}$ (φ is a classical witness)
- Limits the range of parameters for which quantum PCPs can exist
- For any constants c, α, β > 0 it's NP-hard to tell whether

  unsat = 0 or unsat ≥ c $|\Sigma|^{\alpha}/D^{\beta}$

Product-State Approximation

From **thm** to **cor**:

Let ρ be optimal assignment (aka groundstate) for $H = \sum_{(i,j)\in E} P_{i,j}$

By **thm**:

$$\exists\ \sigma = \sum_k p_k \sigma_{k_1} \otimes \ldots \otimes \sigma_{k_n} \quad \text{s.t.} \quad \mathop{\mathbb{E}}_{(i,j)\in E} \|\rho_{i,j} - \sigma_{i,j}\|_1 \le 12 \left(\frac{d^2 \ln(d)}{D} \right)^{1/3}$$

Then

$$\frac{2}{nD}\mathrm{tr}(\sigma H) - \underbrace{\frac{2}{nD}\mathrm{tr}(\rho H)}_{\text{unsat}(H)} = \mathop{\mathbb{E}}_{(i,j)\in E} \mathrm{tr}(P_{i,j}(\sigma - \rho)) \le \mathop{\mathbb{E}}_{(i,j)\in E} \|\rho_{i,j} - \sigma_{i,j}\|_1$$

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So

$$\frac{2}{nD}\sum_k p_k \operatorname{tr}(\sigma_{k_1} \otimes \ldots \otimes \sigma_{k_n} H) = \frac{2}{nD}\operatorname{tr}(\sigma H) \leq \text{unsat}(H) + 12\left(\frac{d^2\ln(d)}{D}\right)^{1/3}$$

Coming back to quantum "blowing up" maps + qPCP

thm If for every $t \geq 1$ there is an efficient mapping from $(2, d, n)$-qcsp H to $(2, d_t, n_t)$-qcsp H_t s.t.

(i) $n_t \leq n^{O(t)}$
(ii) $\mathrm{Deg}(H_t) \geq \deg(H)^t$
(iii) $|d_t| = |d|^t$
(iv) $\mathrm{unsat}(H_t) \geq \mathrm{unsat}(H)$
(v) $\mathrm{unsat}(H_t) = 0$ if $\mathrm{unsat}(H) = 0$

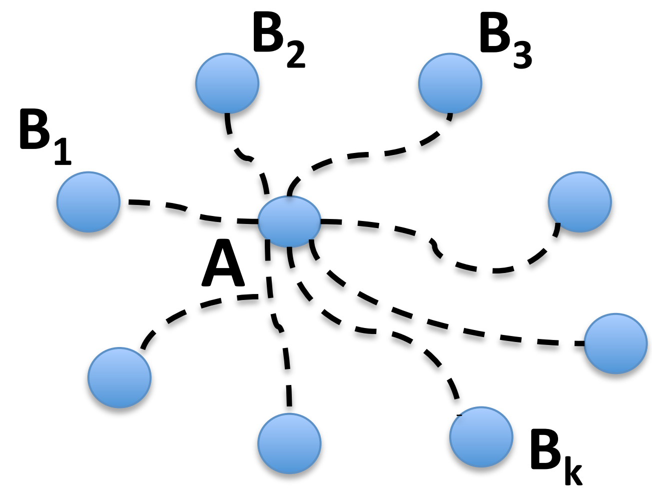
then the quantum PCP conjecture is *false*.

Suppose w.l.o.g. $d^2\log(d)/D < ½$ for C. Then there is a product state φ s.t.

$$\frac{2}{n_t D_t}\langle\phi|H_t|\phi\rangle \leq \mathrm{unsat}(H_t) + 12\left(\frac{d_t^2\log(d_t)}{D_t}\right)^{1/3} \leq \mathrm{unsat}(H_t) + 12\left(\frac{d^2\log(d)}{D}\right)^{t/3}$$

Proving de Finetti Approximation

For simplicity let's consider a *star* graph



Want to show: there is a state

$$\sigma_{AB_1,\ldots,B_D} = \sum_k p_k \sigma_{A,k} \otimes \sigma_{B_1,k} \otimes \ldots \otimes \sigma_{B_D,k}$$

s.t. $\mathbb{E}_i \|\rho_{AB_i} - \sigma_{AB_i}\|_1 \leq 12 \left(\frac{d^2 ln(d)}{D} \right)^{1/3}$

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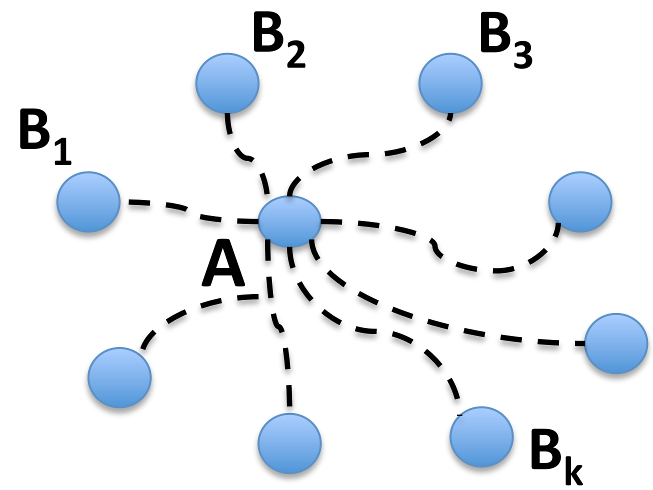

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Idea: Use *information theory*. Consider $\mathop{\mathbb{E}}_{i_1,\ldots,i_D} I(A : B_{i_1}, \ldots, B_{i_D})$

mutual info:
I(X:Y) = H(X) + H(Y) – H(XY)

(i) $I(A : B_{i_1}, \ldots, B_{i_D}) \leq 2 \log(d)$

(ii) $I(A : B_{i_1}, \ldots, B_{i_D}) = I(A : B_{i_1}) + \ldots + I(A : B_{i_D} : B_{i_1} \ldots B_{i_{D-1}})$

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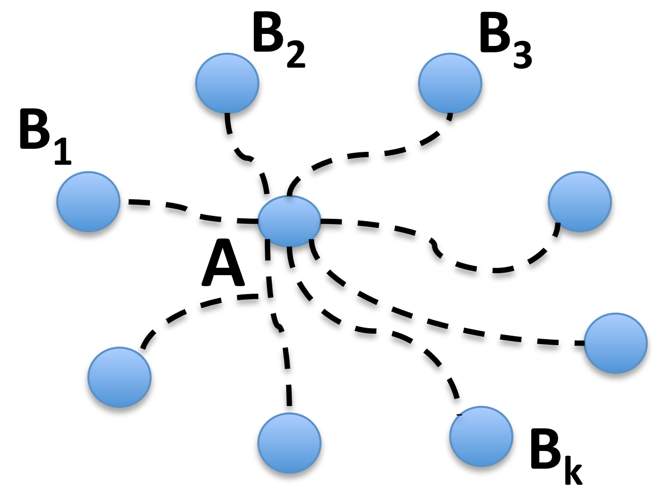

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$$\exists\, s \leq D : \mathop{\mathbb{E}}_{i_1,\dots,i_{s-1}} \mathop{\mathbb{E}}_{i_s} I(A : B_{i_s} | B_{i_1} \dots B_{i_{s-1}}) \leq \frac{2\log(d)}{D}$$

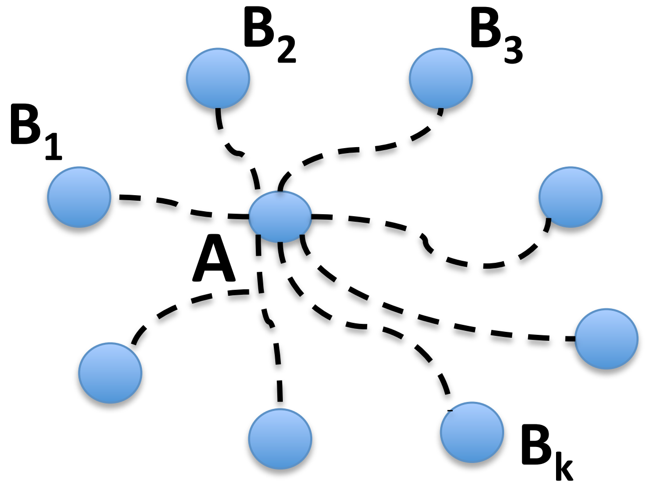
What small conditional mutual info implies?

$$\exists\, s \leq D : \mathop{\mathbb{E}}_{i_1,\ldots,i_{s-1}} \mathop{\mathbb{E}}_{i_s} I(A : B_{i_s} | B_{i_1} \ldots B_{i_{s-1}}) \leq \frac{2\log(d)}{D}$$

For X, Y, Z random variables

$$I(X:Y|Z)_p = \mathbb{E}_z I(X:Y)_{p_z}$$

$$p_z(x,y) = p(x,y,z)/p(z)$$



No similar interpretation is known for I(X:Y|Z) with *quantum* Z

Solution: Measure sites $i_1, \ldots, i_{s-1}$

Proof Sktech

Consider a measurement $\Lambda(X) := \sum_k \mathrm{tr}(M_k X)|k\rangle\langle k|$

and $\pi = \mathrm{id}_A \otimes \Lambda^{\otimes D}(\rho)$

POVM

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There exists s ≤ D s.t. $\mathop{\mathbb{E}}_{i_1,\ldots,i_{s-1}} \mathop{\mathbb{E}}_{i_s} I(A : B_{i_s} | B_{i_1} \ldots B_{i_{s-1}})_{\pi_r} \leq \frac{2\log(d)}{D}$

So $\mathop{\mathbb{E}}_{i_1,\ldots,i_{s-1}} \mathop{\mathbb{E}}_{r_1,\ldots,r_{s-1}} \mathop{\mathbb{E}}_{i_s} I(A : B_{i_s})_{\pi_r} \leq \frac{2\log(d)}{D}$

with π_r the postselected state conditioned on outcomes *(r_1, ..., r_{s-1})*.

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with π_r the postselected state conditioned on outcomes *(r_1, …, r_{s-1})*. Thus:

$$\mathop{\mathbb{E}}_{i_1,\ldots,i_{s-1}} \mathop{\mathbb{E}}_{r_1,\ldots,r_{s-1}} \mathop{\mathbb{E}}_{i_s} \|(\pi_r)_{AB_{i_s}} - (\pi_r)_A \otimes (\pi_r)_{B_{i_s}}\|_1 \leq \left(\frac{4\ln(2)\log(d)}{D}\right)^{1/2}$$

(by Pinsker inequality)

Proof Sktech

Again:

$$\mathop{\mathbb{E}}_{i_1,\ldots,i_{s-1}} \mathop{\mathbb{E}}_{r_1,\ldots,r_{s-1}} \mathop{\mathbb{E}}_{i_s} \| (\pi_r)_{AB_{i_s}} - (\pi_r)_A \otimes (\pi_r)_{B_{i_s}} \|_1 \leq \left(\frac{4 \ln(2) \log(d)}{D} \right)^{1/2}$$

But $(\pi_r)_{A,B_i} = \mathrm{id}_A \otimes \Lambda_{B_i}(\rho_r)$. Choosing Λ an **informationally-complete** measurement:

$$\mathop{\mathbb{E}}_{i_1,\ldots,i_{s-1}} \mathop{\mathbb{E}}_{r_1,\ldots,r_{s-1}} \mathop{\mathbb{E}}_{i_s} \| (\rho_r)_{AB_{i_s}} - (\rho_r)_A \otimes (\rho_r)_{B_{i_s}} \|_1 \leq 12 \left(\frac{d^2 \log(d)}{D} \right)^{1/2}$$

Conversion factor from info-complete meas.

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Conversion factor from info-complete meas.

Separable state: $\sigma = \mathop{\mathbb{E}}_{i_1,\ldots,i_{s-1}} \mathop{\mathbb{E}}_{r_1,\ldots,r_{s-1}} (\rho_{\vec{r},\vec{i}})_A \otimes \left(\bigotimes_{k\in[D]} (\rho_{\vec{r},\vec{i}})_{B_k}\right)$

Finally:

$$\mathop{\mathbb{E}}_{i} \|\rho_{AB_i} - \sigma_{AB_i}\|_1 \leq \mathop{\mathbb{E}}_{i_1,\ldots,i_{s-1}} \mathop{\mathbb{E}}_{r_1,\ldots,r_{s-1}} \mathop{\mathbb{E}}_{i_s} \|(\rho_s)_{AB_{i_s}} - (\rho_r)_A \otimes (\rho_r)_{B_{i_s}}\|_1$$

$$\leq 12\left(\frac{d^2\log(d)}{D}\right)$$

Product-State Approximation: General Theorem

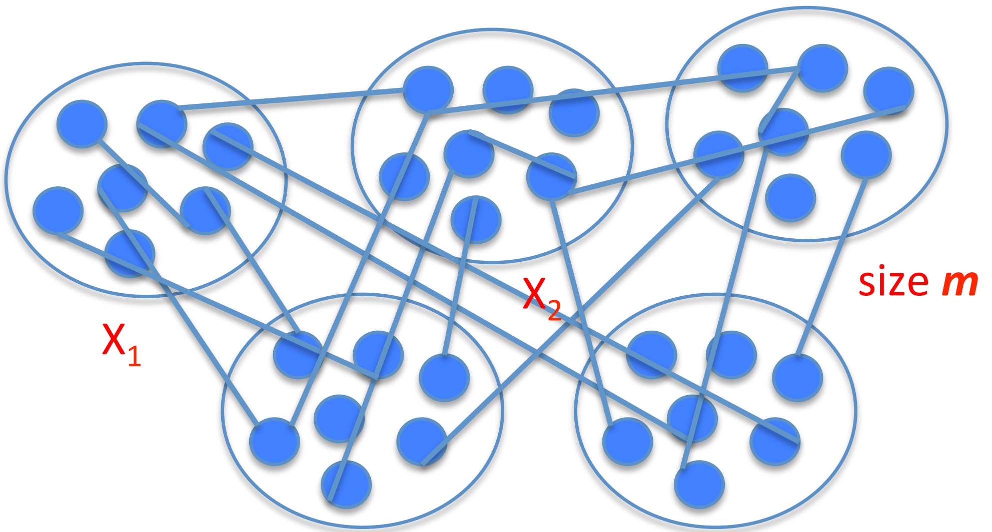
thm Let H be a 2-local Hamiltonian on qudits with D-regular interaction graph G(V, E) and |E| local terms.

Let $\{X_i\}$ be a partition of the sites with each X_i having m sites. Then there are states φ_i in X_i s.t.

$$\frac{2}{nD}\langle\phi_1,\ldots,\phi_{n/m}|H|\phi_1,\ldots,\phi_{n/m}\rangle \leq \text{unsat}(H) + 9\left(\frac{d^2\ln(d)\Phi_G}{D}\frac{\mathbb{E}_i S(X_i)}{m}\right)^{1/6}$$

Φ_G : average expansion

$S(X_i)$: entropy of groundstate in X_i



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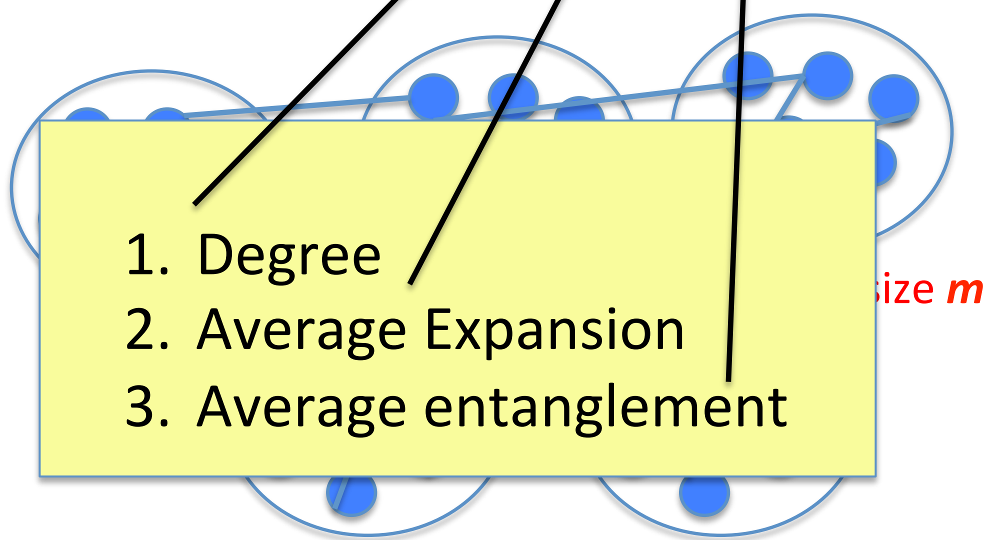

1. Degree
2. Average Expansion
3. Average entanglement

Summary and Open Questions

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Entanglement monogamy puts limitations on quantum PCPs *and* on approaches for proving them.

Open questions:

- Can we combine (BH '13) with (Aharonov, Eldar '13)? I.e. approximation for highly expanding non-commuting *k*-local models?
  (Needs to go beyond both product-state approximations and Bravyi-Vyalyi)
- Relate quantum "blowing up" maps to quantum games?
- Improved clock-constructions for better gap? (Daniel's talk)
- Understand better power of tensor network states (product states 1st level)
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Thanks!