Linear preservers of stable and Lorentzian polynomials and deformations of hyperbolicity cones

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Prelude

- Let h be hyperbolic with hyperbolicity cone Λ_+.
- If $\mathbf{v} \in \Lambda_+$, then the **directional derivative**

$$D_{\mathbf{v}} h = v_1 \frac{\partial h}{\partial x_1} + \cdots + v_n \frac{\partial h}{\partial x_n}$$

  is hyperbolic with hyperbolicity cone $\Lambda_+^{(1)} \supseteq \Lambda_+$.
- Hence we get a sequence of relaxations $\Lambda_+ \subseteq \Lambda_+^{(1)} \subseteq \Lambda_+^{(2)} \subseteq \cdots \subseteq \Lambda_+^{(d-1)}$, $d = \deg h$.
- The map $h \mapsto D_{\mathbf{v}} h$ defines a linear operator which preserves hyperbolicity, under which hyperbolicity cones behave "nicely".
- Questions. Are there other such preservers? Can we determine how they deform hyperbolicity cones?

Stable polynomials

- Let $K = \mathbb{R}$ or $\mathbb{C}$, and $\mathbf{x} = (x_1, \ldots, x_n)$ a tuple of variables.
- A polynomial $f \in K[\mathbf{x}]$ is **stable** if

$$\text{Im}(z_j) > 0 \text{ for all } j \quad \Longrightarrow \quad f(\mathbf{z}) \neq 0.$$

- We also consider the identically zero polynomial to be stable.
- $x_1 - 2 + 5i$, $-1 + x_1 + 5x_2 + 3x_3$ and $1 - 2x_1x_2$ are stable.
- A polynomial $f \in \mathbb{R}[x_1]$ is stable iff f is **real-rooted**.
- Let

$$h = x_0^d f(x_1/x_0, \ldots, x_n/x_0).$$

- Lemma. Let $f \in \mathbb{R}[\mathbf{x}]$. Then f is stable iff h is hyperbolic with respect to $(0, 1, 1, \ldots, 1)$, and $\Lambda_+(F) \supseteq \{0\} \times \mathbb{R}^n_{\geq 0}$.
- Question. Which linear operators **preserve stability**?

Brief history

- RH is equivalent to that

$$\Xi(t) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s), \qquad s = \frac{1}{2} + it,$$

  may be approximated, uniformly on compacts, by real-rooted polynomials.
- This motivated Hermite, Laguerre, Jensen, Pólya, Schur, De Bruijn, ... to study linear operators preserving real-rootedness.
- Lee and Yang (1952) used ideas and results of Pólya to prove their celebrated Lee-Yang theorem of multivariate stability of the partition function of the Ising model.
- Choe-Oxley-Sokal-Wagner, Gurvits, B., Borcea-B.-Liggett studied stability (preservers) from a combinatorial point of view.
- Marcus, Srivastava, Spielman used stability preservers in their work on Ramanujan graphs and the Kadison-Singer problem.
- Anari and Oveis Gharan used stability in computer science (e.g. for Traveling salesman problem).

Stability preservers

- Example. $T = \frac{\partial}{\partial x_j}$ preserves stability (Gauss-Lucas theorem).
- For $\kappa \in \mathbb{N}^n$, let

$$K_\kappa[\mathbf{x}] = \{f \in K[\mathbf{x}] : \deg_{x_j}(f) \leq \kappa_j \text{ for all } j\}.$$

- The **symbol** of a linear operator $T : K_\kappa[\mathbf{x}] \to K[\mathbf{x}]$ is

$$G_T = T\big((\mathbf{x}+\mathbf{y})^\kappa\big) = \sum_{0 \leq \alpha \leq \kappa} \binom{\kappa}{\alpha} T(\mathbf{x}^\alpha)\mathbf{y}^{\kappa-\alpha},$$

  where $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\binom{\kappa}{\alpha} = \binom{\kappa_1}{\alpha_1} \cdots \binom{\kappa_n}{\alpha_n}$.
- $G_{\frac{\partial}{\partial x_1}} = \kappa_1(x_1+y_1)^{\kappa_1-1}(x_2+y_2)^{\kappa_2} \cdots (x_n+y_n)^{\kappa_n}$

Pólya-Schur master theorem

- Pólya and Schur (1914) characterized diagonal operators $x^n \to \lambda_n x^n$ preserving real-rootedness.
- Theorem (Borcea, B., 2009). Let $T : \mathbb{C}_\kappa[\mathbf{x}] \to \mathbb{C}[\mathbf{x}]$ be a linear operator of rank > 1. Then T preserves stability iff G_T is stable.
- $G_{\frac{\partial}{\partial x_1}} = \kappa_1(x_1+y_1)^{\kappa_1-1}(x_2+y_2)^{\kappa_2}\cdots(x_n+y_n)^{\kappa_n}$ is stable.
- Theorem (Borcea, B., 2009). Let $T : \mathbb{R}_\kappa[\mathbf{x}] \to \mathbb{R}[\mathbf{x}]$ be a linear operator of rank > 2. Then T preserves real stability iff
  - $G_T(\mathbf{x},\mathbf{y})$ is stable, or
  - $G_T(-\mathbf{x},\mathbf{y})$ is stable.
- Example. $T(f)(\mathbf{x}) = f(-\mathbf{x})$ preserves real stability, and $G_T(-\mathbf{x},\mathbf{y}) = (\mathbf{x}+\mathbf{y})^\kappa$.

Transcendental characterization

- The **Laguerre–Pólya class**, $\mathcal{L}\text{–}\mathcal{P}_n$, is the class of entire functions which are limits, uniformly on compacts, of real stable polynomials.
- Example.

$$e^{-\mathbf{x}\cdot\mathbf{y}} = e^{-(x_1y_1+\cdots+x_ny_n)} = \lim_{k\to\infty}\Big(1-\frac{x_1y_1}{k}\Big)^k \cdots \Big(1-\frac{x_ny_n}{k}\Big)^k$$

- The **symbol** of a linear operator $T : \mathbb{R}[\mathbf{x}] \to \mathbb{R}[\mathbf{x}]$ is

$$\bar{G}_T(\mathbf{x},\mathbf{y}) = T(e^{-\mathbf{x}\cdot\mathbf{y}}) = \sum_{\alpha\in\mathbb{N}^n} T(\mathbf{x}^\alpha)\frac{(-\mathbf{y})^\alpha}{\alpha!}.$$

- Theorem (Borcea, B., 2009). Let $T : \mathbb{R}[\mathbf{x}] \to \mathbb{R}[\mathbf{x}]$ be a linear operator of rank > 2. Then T preserves real stability iff
  - $\bar{G}_T(\mathbf{x},\mathbf{y}) \in \mathcal{L}\text{–}\mathcal{P}_{2n}$, or
  - $\bar{G}_T(-\mathbf{x},\mathbf{y}) \in \mathcal{L}\text{–}\mathcal{P}_{2n}$.

Transcendental characterization

- Example. If $T = -1 + 2\frac{\partial}{\partial x_1} + 3\frac{\partial}{\partial x_2}$, then

$$\bar{G}_T(\mathbf{x}, \mathbf{y}) = T(e^{-\mathbf{x}\cdot\mathbf{y}}) = (-1 + 2y_1 + 3y_2)e^{-\mathbf{x}\cdot\mathbf{y}} \in \mathcal{L}\text{–}\mathcal{P}_4.$$

- Corollary. Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator of rank > 2. Then T preserves real-rootedness iff
  - $\bar{G}_T(x, y) \in \mathcal{L}\text{–}\mathcal{P}_2$, or
  - $\bar{G}_T(-x, y) \in \mathcal{L}\text{–}\mathcal{P}_2$.
- Question. Is there a transcendental Helton–Vinnikov theorem? If $f(x, y) \in \mathcal{L}\text{–}\mathcal{P}_2$, then

$$f(x, y) = \det(A + xB + yC),$$

where A, B, C?? and $\det$??

Discrete convexity

- A finite subset M of $\mathbb{Z}^n$ is a **polymatroid** (or **M-convex**) if

  $\alpha, \beta \in M$ and $\alpha_i > \beta_i \Longrightarrow$
  there is a j such that $\beta_j > \alpha_j$ and $\alpha - e_i + e_j \in M$.

- The **support** of a polynomial $f = \sum_{\alpha \in \mathbb{N}^n} a(\alpha)\mathbf{x}^\alpha$ is

$$\mathrm{supp}(f) = \{\alpha \in \mathbb{N}^n : a(\alpha) \neq 0\}.$$

- Theorem[Choe-Oxley-Sokal-Wagner, 2004]. The support of a homogeneous and stable polynomial is a polymatroid.
- Theorem[B., 2007]. The set of bases of the Fano matroid F_7 is **not** the support of any stable polynomial.

Lorentzian (CLC, SLC) polynomials

- Consider a quadratic f written as $f = \sum_{i,j=1}^{n} a_{ij} x_i x_j$, where $A = (a_{ij})_{i,j=1}^{n}$ is a symmetric matrix with nonnegative entries.
- Lemma. f is stable iff A has exactly one positive eigenvalue.
- Definition. A homogeneous degree d polynomial $f \in \mathbb{R}[\mathbf{x}]$ with positive coefficients is **strictly Lorentzian** if for all $i_1, i_2, \ldots, i_{d-2}$, the quadratic

$$\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_{d-2}}} f$$

has Lorentz signature $(+, -, -, \ldots)$, i.e., exactly one positive eigenvalue and $n-1$ negative eigenvalues.
- Definition. A polynomial is **Lorentzian** if it is the limit of strictly Lorentzian polynomials.

Lorentzian polynomials

- Theorem[B., Huh, 2019]. A homogeneous degree d polynomial $f \in \mathbb{R}_{\geq 0}[\mathbf{x}]$ is **Lorentzian** if
  - $\mathrm{supp}(f)$ is a polymatroid, and
  - For all $i_1, i_2, \ldots, i_{d-2}$, the quadratic

$$\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_{d-2}}} f$$

    is stable.

- Example. A polynomial $\sum_{k=M}^{N} a_k x^k y^{d-k}$, $a_k > 0$, is Lorentzian iff

$$\frac{a_k^2}{\binom{d}{k}^2} \geq \frac{a_{k-1}}{\binom{d}{k-1}} \cdot \frac{a_{k+1}}{\binom{d}{k+1}}, \qquad M < k < N.$$

Lorentzian polynomials

- Example. If M is a polymatroid, then $\sum_{\alpha \in M} \frac{\mathbf{x}^\alpha}{\alpha!}$, is Lorentzian.
- Example. If $r : 2^{[n]} \to \mathbb{N}$ is the rank function of a matroid and $0 < q \leq 1$, then

$$\sum_{A \subseteq [n]} q^{-r(A)} x_0^{n-|A|} \prod_{i \in A} x_i$$

is Lorentzian.
- A matrix $A \in \mathbb{R}^{n \times n}$ is an M-matrix if all off-diagonal entries are nonpositive and all principal minors are nonnegative.
- Theorem[B., Huh, 2019]. If A is an M-matrix, then

$$\det(x_0 I + \operatorname{diag}(x_1, \ldots, x_n) A) = \sum_{S \subseteq [n]} A(S) x_0^{n-|S|} \prod_{i \in S} x_i$$

is Lorentzian.

Lorentzian preservers

- Question. Which linear operators preserve the Lorentzian property?
- Theorem (B., Huh, 2019+). Let $T : \mathbb{R}_\kappa[\mathbf{x}] \to \mathbb{R}[\mathbf{x}]$. If G_T is Lorentzian, then T preserves the Lorentzian property.
- Corollary. If G_T is homogeneous and stable, then T preserves the Lorentzian property.
- Example. Let $\alpha \leq \beta \in \mathbb{N}^n$. The operator

$$T\left(\sum_{\gamma \in \mathbb{N}^n} a(\gamma)\mathbf{x}^\gamma\right) = \sum_{\alpha \leq \gamma \leq \beta} a(\gamma)\mathbf{x}^\gamma$$

  preserves the Lorentzian property (but not stability).
- Nonexample. The operator $T : \mathbb{R}_\kappa[\mathbf{x}] \to \mathbb{R}_\kappa[\mathbf{x}]$ defined by $T(f)(\mathbf{x}) = \mathbf{x}^\kappa f(1/x_1, \dots, 1/x_n)$ preserves real stability but **not** the Lorentzian property.

How do zeros move under preservers?

- The **symmetric additive convolution** of two univariate polynomials $p, q \in \mathbb{R}_d[x]$ is

$$(f \boxplus_d g)(x) = \frac{1}{d!} \sum_{k=0}^{d} f^{(k)}(0) \cdot g^{(d-k)}(x).$$

- Let $\lambda_{\max}(f)$ be the largest zero of a real-rooted polynomial f.
- For $\alpha \geq 0$, let $U_\alpha = 1 - \alpha \frac{d}{dx}$.
- Theorem (Marcus, Srivastava, Spielman, 2015).

$$\lambda_{\max}(U_\alpha(f \boxplus_d g)) \leq \lambda_{\max}(U_\alpha(f)) + \lambda_{\max}(U_\alpha(g)) - d\alpha.$$

- Theorem (Leake, Ryder, 2018).

$$\lambda_{\max}(f \boxplus_d g \boxplus_d h) + \lambda_{\max}(h) \leq \lambda_{\max}(g \boxplus_d h) + \lambda_{\max}(f \boxplus_d h).$$

- What is the max-root of a real stable polynomial f?
- Definition. The **hyperbolicity set** of f is

$$\begin{aligned}\mathcal{C}[f] &= \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x})f(\mathbf{y}) > 0 \text{ for all } \mathbf{y} \geq \mathbf{x}\} \\ &= \Lambda_{++}(h) \cap \{x_0 = 1\},\end{aligned}$$

  where $h = x_0^d f(x_1/x_0, \ldots, x_n/x_0)$.
- If λ is the largest zero of $f(x_1)$, then $\mathcal{C}[f] = \{x \in \mathbb{R} : x > \lambda\}$.
- For $f, g \in \mathbb{R}_\kappa[\mathbf{x}]$, let

$$(f \boxplus_\kappa g)(\mathbf{x}) = \frac{1}{\kappa!} \sum_{\alpha \leq \kappa} (\partial^\alpha f)(0) \cdot (\partial^{\kappa-\alpha} g)(\mathbf{x}),$$

  where

$$\partial^\alpha = \prod_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\right)^{\alpha_i}.$$

- Lemma. $\boxplus_\kappa$ preserves stability on $\mathbb{C}_\kappa[\mathbf{x}] \times \mathbb{C}_\kappa[\mathbf{x}]$.

Proof. Fix stable g and consider $T(f) = f \boxplus_\kappa g$. Then

$$G_T = (\mathbf{x} + \mathbf{y})^\kappa \boxplus_\kappa g = g(\mathbf{x} + \mathbf{y})$$

is stable.

- Lemma. If f is stable and $\mathbf{w} \in \mathbb{R}^n_{\geq 0}$, then $f - D_\mathbf{w} f$ is stable, where $D_\mathbf{w} = w_1 \frac{\partial}{\partial x_1} + \cdots + w_n \frac{\partial}{\partial x_n}$.

Proof. Let $T = 1 - D_\mathbf{w}$. Then

$$\bar{G}_T = T(e^{-\mathbf{x}\cdot\mathbf{y}}) = (1 + w_1 y_1 + \cdots + w_n y_n) e^{-\mathbf{x}\cdot\mathbf{y}} \in \mathcal{L}\text{–}\mathcal{P}_n.$$

- Let $\mathcal{C}_\mathbf{w}[f] = \mathcal{C}[f - D_\mathbf{w} f]$. Then $\mathbf{u} \leq \mathbf{w} \implies \mathcal{C}_\mathbf{w}[f] \subseteq \mathcal{C}_\mathbf{u}[f]$.

- Theorem (B., Marcus, 2019+). If $f, g \in \mathbb{R}_\kappa[\mathbf{x}]$, then

$$\begin{aligned} \mathcal{C}_{\mathbf{w}}[P \boxplus_\kappa Q] &\supseteq \mathcal{C}_{\mathbf{w}}[P] + \mathcal{C}_{\mathbf{w}}[Q] - \kappa\mathbf{w} \\ &= \{\mathbf{x} + \mathbf{y} - \kappa\mathbf{w} : \mathbf{x} \in \mathcal{C}_{\mathbf{w}}[P] \text{ and } \mathbf{y} \in \mathcal{C}_{\mathbf{w}}[Q]\}, \end{aligned}$$

  where $\kappa\mathbf{w} = (\kappa_1 w_1, \ldots, \kappa_n w_n)$.
- If $T : \mathbb{R}_\kappa[\mathbf{x}] \to \mathbb{R}_\gamma[\mathbf{x}]$ is a linear operator with symbol G_T, then

$$T(f(\mathbf{x}+\mathbf{y})) = (\mathbf{x}^\gamma f(\mathbf{y})) \boxplus_{\gamma \oplus \kappa} G_T(\mathbf{x}, \mathbf{y}),$$

  where $\gamma \oplus \kappa = (\gamma_1, \ldots, \gamma_m, \kappa_1, \ldots, \kappa_n)$.
- Unifies the proofs for the different convolutions considered by Marcus, Spielman and Srivastava (2015).
- Conjecture(Leake, Ryder, 2018).

$$\mathcal{C}[P \boxplus_\kappa Q \boxplus_\kappa R] + \mathcal{C}[R] \supseteq \mathcal{C}[P \boxplus_\kappa R] + \mathcal{C}[Q \boxplus_\kappa R].$$