Certifying polynomial nonnegativity via hyperbolic optimization

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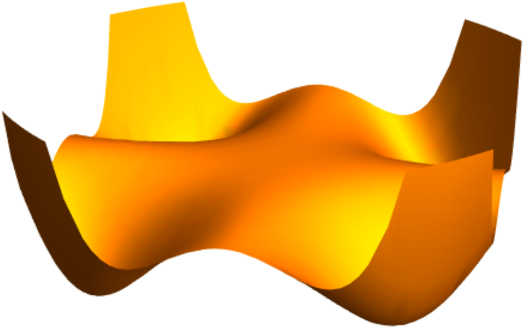
Problem: If f is a polynomial of degree $2d$ in n variables, decide whether $f(x) \geq 0$ for all $x \in \mathbb{R}^n$

Polynomial nonnegativity $\longleftarrow$ Hyperbolic optimization



This talk: Find tractable sufficient conditions for nonnegativity of f based on hyperbolic programming

Hyperbolic polynomials

A polynomial p homogeneous of degree d in n variables is hyperbolic with respect to $e \in \mathbb{R}^n$ if

- $p(e) > 0$
- for all $x \in \mathbb{R}^n$, all roots of $t \mapsto p(x - te)$ are real

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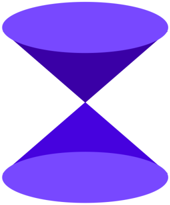
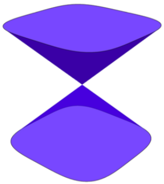


$p(x, y, z) = -x^2 - y^2 + z^2$

hyperbolic w.r.t. $e = (0, 0, 1)$



$p(x, y, z) = -x^4 - y^4 + z^4$

not hyperbolic

Hyperbolic polynomials

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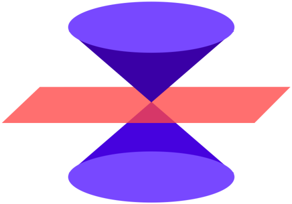

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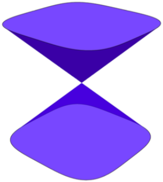

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Hyperbolicity cones

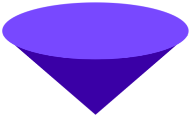
If p is hyperbolic w.r.t. $e \in \mathbb{R}^n$ define hyperbolicity cone as

$$\Lambda_+(p, e) = \{x \in \mathbb{R}^n : \text{all roots of } t \mapsto p(te - x) \text{ non-negative}\}$$

> **Theorem (Gårding 1959)**
> If p is hyperbolic w.r.t. e then $\Lambda_+(p, e)$ is convex.

Example

$$p(x, y, z) = -x^2 - y^2 + z^2$$



- hyperbolic w.r.t. $e = (0, 0, 1)$
- Hyperbolicity cone is second-order/Lorentz/ice-cream cone

Hyperbolic programming

$$\text{minimize}_x \langle c, x \rangle \ \text{ subject to } \ \begin{cases} Ax = b \\ x \in \Lambda_+(p, e) \end{cases}$$

Theorem (Güler 1997)

$-\log_e(p)$ is a self-concordant barrier for $\Lambda_+(p, e)$

Consequence: if can evaluate p, get ‘efficient’ algorithms

Special cases

- Linear programming: $p(x) = x_1 x_2 \cdots x_n$, $e = \mathbf{1}$
- Second-order cone programming
- Semidefinite programming: $p(X) = \det(X)$, $e = \text{Identity}$

Testing for hyperbolicity

Hermite matrix: entries are power sums of eigenvalues

$$[H_{p,e}(x)]_{ij} = \sum_{\ell=1}^{d} (-\lambda_\ell(x))^{i+j-2}$$

Netzer-Plaumann-Thom (2013):
p hyperbolic with respect to $e \iff H_{p,e}(x) \succeq 0$ for all x

(See Dey-Plaumann (2018) for more tests for hyperbolicity)

Testing for hyperbolicity

Hermite matrix: entries are power sums of eigenvalues

$$[H_{p,e}(x)]_{ij} = \sum_{\ell=1}^{d} (-\lambda_\ell(x))^{i+j-2} = h_{i+j-1}(x)$$

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Alternative view: Expand univariate rational function at infinity

$$\frac{D_e p(x+te)}{p(x+te)} = \sum_{i=1}^{d} \frac{1}{t+\lambda_i(x)} = \sum_{k\geq 1} h_k(x) t^{-k}$$

Corresponding Hankel matrix is $H_{p,e}(x)$.

Characterization of hyperbolicity cones

For any $u \in \mathbb{R}^n$

$$\frac{D_u p(x+te)}{p(x+te)} = \sum_{k\geq 1} h_k(x)[u]t^{-k}$$

Corresponding Hankel matrix:

$$[H_{p,e}(x)[u]]_{ij} = h_{i+j-1}(x)[u]$$

If p hyperbolic w.r.t. e then

$$H_{p,e}(x)[u] \succeq 0 \text{ for all } x \quad \iff \quad u \in \Lambda_+(p,e)$$

- Equivalent formulation in terms of
  Bézoutian of $D_u p(x+te)$ and $p(x+te)$
- Very closely related to Kummer-Plaumann-Vinzant (2015)

Example: symmetric determinant

If $p(X) = \det(X)$ and $e = I$

$$H_{p,e}(X)[U]_{ij} = \operatorname{tr}((-X)^{i+j-2}U)$$

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- $U \succeq 0$ get Gram matrix

$$H_{p,e}(X)[U]_{ij} = \langle (-X)^{i-1}U^{1/2}, (-X)^{j-1}U^{1/2} \rangle$$

- $U \not\succeq 0$, explicitly construct y s.t.

$$y^T H_{p,e}(U)[U]y = \lambda_{\min}(U) < 0$$

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Can use this to prove general case via Helton-Vinnikov theorem

Hyperbolicity cones $\longrightarrow$ non-negative polynomials

If p hyperbolic with respect to e define

$$\phi_{p,e}(x,y)[u] = y^T H_{p,e}(x)[u]y$$

(polynomial in x, y, linear in u)

- Globally nonnegative if and only if u in hyperbolicity cone
- Convex set of nonnegative polynomials that is linearly isomorphic to hyperbolicity cone

$$\{\phi_{p,e}(x,y)[u] \;:\; u \in \Lambda_+(p,e)\}$$

Hyperbolic certificates of nonnegativity

Suppose

- p is hyperbolic with respect ot $e \in \mathbb{R}^n$
- $f : \mathbb{R}^m \to \mathbb{R}^n$ polynomial
- $g : \mathbb{R}^m \to \mathbb{R}^d$ polynomial

If there exists $u \in \Lambda_+(p, e)$ such that

$$q(z) = \phi_{p,e}(f(z), g(z))[u] \quad \text{for all } z$$

say q has hyperbolic certificate of nonnegativity

Get convex set of non-negative polynomials:

$$\{\phi_{p,e}(f(z), g(z))[u] \ : \ u \in \Lambda_+(p, e)\}$$

- Is projection of the hyperbolicity cone
- Can search over these using hyperbolic programming

Sum-of-squares certificates of nonnegativity

If can write q as a sum of squares (SOS)

$$q(z) = \sum_{i=1}^{n} [q_i(z)]^2 \quad \text{then } q(z) \geq 0 \text{ for all } z$$

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Can search for SOS certificate via semidefinite optimization

- q polynomial of degree $2d$ in n variables
- $m_d(z)$ vector of monomials of degree at most d

$q(z)$ is a sum of squares

$$\iff$$

$$\exists Q \succeq 0 \text{ such that } q(z) = m_d(z)^T Q m_d(z)$$

Hyperbolic certificates capture sums of squares

$$q \text{ SOS:} \quad q(z) = m_d(z)^T Q m_d(z) \quad \text{with } Q \succeq 0$$

Data for hyperbolic certificates

- $p(X) = \det(X)$, $e = \text{identity}$
-

$$f(z) = \begin{bmatrix} 0 & m_d(z)^T \\ m_d(z) & 0 \end{bmatrix} \quad g(z) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \end{bmatrix}^T$$

Using these choices...

$$\phi_{p,e}(f(z), g(z)) \left[\begin{smallmatrix} 0 & 0 \\ 0 & Q \end{smallmatrix}\right] = \operatorname{tr}\left(f(z)^2 \left[\begin{smallmatrix} 0 & 0 \\ 0 & Q \end{smallmatrix}\right]\right) = m_d(z)^T Q m_d(z).$$

Can we go beyond sums of squares?

In general: is $\phi_{p,e}(x, y)[u]$ always a sum of squares?

Definition: p is SOS-hyperbolic w.r.t. e
if $\phi_{p,e}(x, y)[u]$ is a sum of squares whenever $u \in \Lambda_+(p, e)$

For which n, d are there
hyperbolic polynomials that are not SOS-hyperbolic?

‘Bad’ news

If a power of p has a definite determinantal representation
then p is SOS hyperbolic

(common generalization of Kummer-Plaumann-Vinzant 2015
and Netzer-Plaumann-Thom 2013)

Defnite determinantal representation:

$$p(x) = \det\left(A_1 x_1 + A_2 x_2 + \cdots + A_n x_n\right)$$

where

- $A_1, \ldots, A_n$ are $d \times d$ symmetric
- $\sum_i A_i e_i \succ 0$ (definite)

‘Bad’ news

If a power of p has a definite determinantal representation
then p is SOS hyperbolic

(common generalization of Kummer-Plaumann-Vinzant 2015
and Netzer-Plaumann-Thom 2013)

$\implies$ hyperbolic polynomials in 3 vars are SOS-hyperbolic
(using Helton-Vinnikov 2007, or via a direct argument)

$\implies$ hyperbolic quadratics are SOS-hyperbolic
(using Netzer-Thom 2011, or via direct argument)

$\implies$ hyperbolic cubics in 4 vars are SOS-hyperbolic
(using Buckley-Košir 2007, direct argument??)

‘Good’ news

Theorem (S. 2018)
There are hyperbolic, but not SOS-hyperbolic, polynomials of degree d in n variables whenever

- $d \geq 4$ and $n \geq 4$
- $d = 3$ and $n \geq 43$

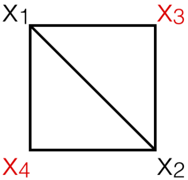
- Case of cubics in $5 \leq n \leq 42$ variables open
- Two key examples:
  - $n = d = 4$
  - $d = 3$ and $n = 43$
- Constructions to increase d or n and preserve being not SOS-hyperbolic

Quartic example: specialized Vámos polynomial

$$p(x_1, x_2, x_3, x_4) = \\ x_3^2 x_4^2 + 4(x_1 + x_2 + x_3 + x_4)(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4)$$

Can show:

- p is hyperbolic w.r.t. $e = (0, 0, 1, 1)$
- $u = (0, 0, 0, 1) \in \Lambda_+(p, e)$
- $\phi_{p,e}(x, y)[u]$ not SOS



Special case of construction due to Amini and Brändén

Hyperbolic cubics

Renaissance fact (16th century):
$t^3 - 3at + 2b$ has real roots if and only if $a^3 - b^2 \geq 0$

- Recover this from determinant of
  Bézoutian/Hermite matrix of p and p'
- Focus on cubics in $n+1$ variables of the form

$$p(x_0, x) = x_0^3 - 3x_0\|x\|^2 + 2q(x)$$

Hyperbolic cubics $\longleftrightarrow$ extreme values on sphere

Consequence:

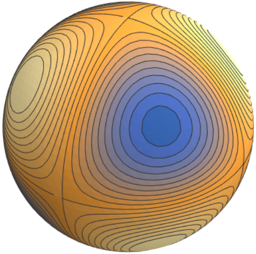
Homogeneous cubic in $n+1$ variables of the form

$$p(x_0, x) = x_0^3 - 3x_0\|x\|^2 + 2q(x)$$

is hyperbolic with respect to $e_0 = (1, 0, \dots, 0)$

$$\iff q(x)^2 \leq \|x\|^6 \quad \forall x \in \mathbb{R}^n$$

$$\iff \max_{\|x\|^2=1} q(x) \leq 1$$



Hardness of deciding hyperbolicity

Given graph $G = (V, E)$ define cubic $q_G(x, y) = \sum_{(i,j)\in E} x_i x_j y_{ij}$

Nesterov 2003: if $\omega(G)$ is size of maximum clique in G

$$\max_{\|x\|^2+\|y\|^2=1} q_G(x, y) = \sqrt{\tfrac{2}{27}}\sqrt{1 - \tfrac{1}{\omega(G)}}$$

Corollary: Given $G = (V, E)$ and a positive integer k

$$p(x_0, x) = \tfrac{2k}{k-1}x_0^3 - x_0\|x\|^2 + q_G(x, y)$$

is hyperbolic w.r.t. e_0 if and only if $\omega(G) \leq k$.

$\implies$ co-NP hard to decide hyperbolicity of cubics

Necessary condition for SOS-hyperbolicity

Homogeneous cubic in $n + 1$ variables of the form

$$p(x_0, x) = x_0^3 - 3x_0\|x\|^2 + 2q(x)$$

Recall:

$$p \text{ hyperbolic w.r.t. } e_0 \iff \|x\|^6 - q(x)^2 \geq 0 \text{ for all } x$$

Turns out:

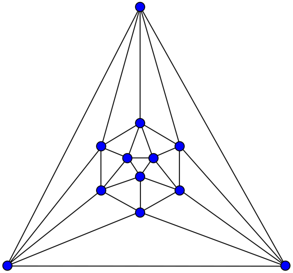
$$p \text{ SOS-hyperbolic w.r.t. } e_0 \implies \|x\|^6 - q(x)^2 \text{ is SOS}$$

A hyperbolic cubic that is not SOS-hyperbolic

If $G = (V, E)$ is the icosahedral graph,

$$p(x_0, x, y) = x_0^3 - 3x_0(\|x\|^2 + \|y\|^2) + 9\sum_{(i,j)\in E} x_i x_j y_{ij}$$

is not SOS-hyperbolic



$|V| = 12$, $|E| = 30$
20 maximum cliques

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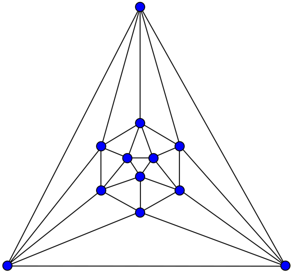

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Corollary: Explicit hyperbolic cubic no power of which has definite determinantal rep.

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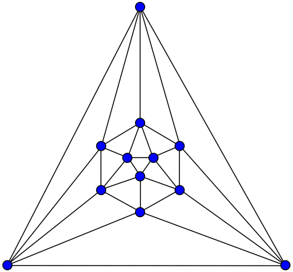

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Corollary: Explicit hyperbolic cubic no power of which has definite determinantal rep.

Conjecture: There is hyp. cubic in 5 variables that is not SOS-hyperbolic

Hyperbolic certificates

Strange 'certificates'

- Proof of nonnegativity relies on proof of hyperbolicity
- But proof of hyperbolicity may not be simple!
- Different from SOS in this regard

Many choices

- Possibility to tailor to problem class
- Too many choices: where to start?

Are these features or bugs?

Summary

- Sufficient conditions for polynomial nonnegativity that can search for via hyperbolic programming
- Hyperbolic polynomials
  - all SOS-hyperbolic if $n = 3$ or $d = 2$ or $(n, d) = (4, 3)$
  - possibly not SOS-hyperbolic if
    $d \geq 4$ and $n \geq 4$ or $d = 3$ and $n \geq 43$
  - Unknown: cubics with $5 \leq n \leq 42$
- On the way...
  - co-NP hard to decide hyperbolicity of cubics
  - example of hyperbolic cubic such that
    no power has definite determinatal rep.

Step toward generic way to obtain hyperbolic programming relaxations of polynomial optimization problems

Preprint:

- ‘Certifying polynomial nonnegativity via hyperbolic optimization’ `https://arxiv.org/abs/1904.00491`

THANK YOU!