Counting Hypergraph Colourings in the Local Lemma Regime

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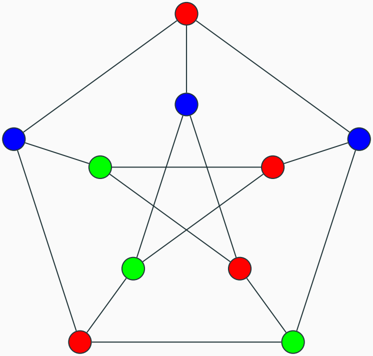
Colourings

Graph (proper) colouring



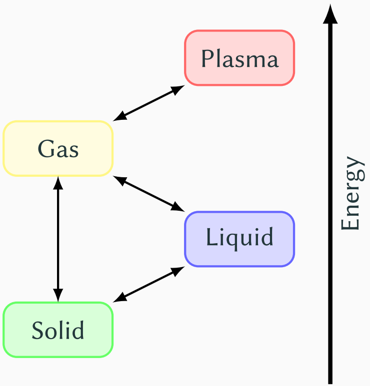
3-colouring of the Petersen graph

Phase transitions

Phase transitions:
as some parameter changes, macroscopic behaviours of the whole system change drastically.

E.g. ice $\to$ water $\to$ water vapor

Computational complexity may also have transitions.



Computational phase transitions

As parameters change, the computational complexity of a problem may change drastically.

Determine whether a graph is q-colourable (or find one if it exists):

- $q = 1, 2$: trivial;
- $q \geqslant 3$: **NP**-hard.

What about graphs with maximum degree Δ?

- $q \geqslant \Delta + 1$: colourable by simple greedy algorithm;
- $q \geqslant \Delta - k_\Delta + 1$: polynomial-time (Molloy, Reed '01 '14);
- $q \leqslant \Delta - k_\Delta$ ($k_\Delta \approx \sqrt{\Delta} - 2$) : **NP**-hard (Embden-Weinert, Hougardy, and Kreuter '98).

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  $(k_\Delta \approx \sqrt{\Delta} - 2)$

Thresholds for randomly colouring a graph

Can we generate a uniform proper colouring at random efficiently?
(closely related to approximately count the number of colourings)

- $q > 2\Delta$: rapid mixing of Glauber dynamics by Jerrum (1995); Salas and Sokal (1997);
- $q > \frac{11}{6}\Delta$: rapid mixing of WSK dynamics by Vigoda (2000);
  improved by Chen and Moitra (2019); Delcourt, Perarnau, and Postle (2019) to $q > \left(\frac{11}{6} - \varepsilon\right)\Delta$ for a small constant ε;
- $q < \Delta$ (even q) : **NP**-hard by Galanis, Štefankovič, and Vigoda (2015);

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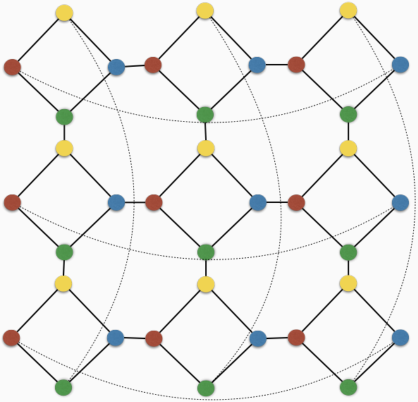
Frozen

Sometimes you just cannot let it go.

$$q = \Delta + 1 = 4$$



credit: Chihao Zhang

Deterministic counting of colourings

MCMC is not the only method to count.

Correlation decay:

- Gamarnik, Katz (2012): $q \geqslant \alpha\Delta + \beta$ for large β and $\alpha \approx 2.84$;
- Lu, Yin (2013): $q \geqslant \alpha\Delta + 1$ for $\alpha \approx 2.58$.

Zeros of the chromatic polynomial:

- Patel, Regts (2017): $q > \alpha\Delta$ for $\alpha \approx 6.91$;
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Cannot match randomised algorithms, yet.

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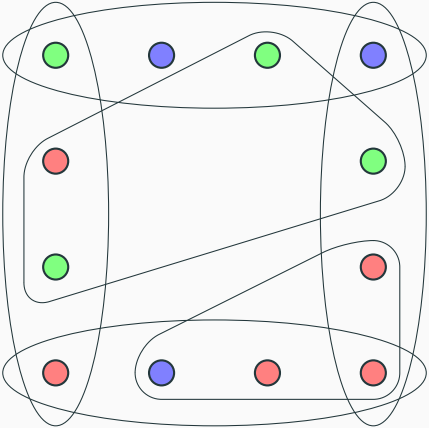
What about hypergraphs?



A proper hypergraph colouring is one where no edge is monochromatic.

Previous results

For k-uniform hypergraphs, Bordewich, Dyer, and Karpinski (2006) show that Glauber dynamics is rapidly mixing if

$$k \geqslant 4 \text{ and } q > \Delta \qquad \text{or} \qquad k = 3 \text{ and } q > 1.5\Delta.$$

However, Lovász local lemma implies the existence of a proper colouring if $q > (ek\Delta)^{1/(k-1)}$.

Frieze and Melsted (2011) gave examples where $q \ll \Delta$, and there exists a colouring so that no move is possible ("frozen").

Frieze and Anastos (2017) showed that Glauber dynamics still converges rapidly if the hypergraph is simple and $q > \max\{C_k \log n, 500k^3\Delta^{1/(k-1)}\}$.

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Our results

Theorem

For $\Delta \geqslant 2$, $k \geqslant 28$, and $q > 315\Delta^{\frac{14}{k-14}}$, there is an FPTAS for the number of q-colourings in k-uniform hypergraphs with maximum degree Δ.

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For $\Delta \geqslant 2$, $k \geqslant 28$, and $q > 798\Delta^{\frac{16}{k-16/3}}$, there is also an almost-uniform polynomial-time sampler.

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Lovász local lemma

(and how it helps with approximate counting)

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The original local lemma (Erdős and Lovász 75) was introduced to show the existence of 3-colourings in hypergraphs.

Let $H = (V, \mathcal{E})$ be the hypergraph, and $\Gamma(e)$ be the set of hyperedges intersecting $e \in \mathcal{E}$. Then $|\Gamma(e)| \leqslant (\Delta - 1)k$.

Theorem (Lovász '77)

If there exists an assignment $x : \mathcal{E} \to (0, 1)$ such that for every $e \in \mathcal{E}$ we have

$$\Pr(e \text{ is monochromatic}) \leqslant x(e) \prod_{e' \in \Gamma(e)} (1 - x(e')), \quad (1)$$

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Reducing to computing marginal probabilities

For approximate counting, we use the (algorithmic) local lemma to find a partial colouring τ so that every hyperedge is satisfied by the first k_1 vertices.

(This will succeed as long as $q > (ek_1\Delta)^{\frac{1}{k_1-1}}$. k_1 will eventually be set to $\frac{k}{14}$.)

Then we compute the probability of τ by "pinning" vertices one by one.

Let $U = \{u_1, \dots, u_r\}$ be the support of τ, and $\mu(\cdot)$ be the Gibbs (uniform) distribution on all proper colourings.

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Thus the key is to estimate marginal probabilities under partial colourings (up to $1 \pm \frac{\varepsilon}{n}$ error), where at least $k - k_1$ vertices are uncoloured in every edge.

Local lemma controls the conditional distribution

Let $\mathcal{C}$ be the set of all proper colourings.

Let $\mu(\cdot)$ be the Gibbs (uniform) distribution on all proper colourings (namely the product distribution conditioned on no monochromatic edge).

The local lemma also gives an upper bound for any event under $\mu(\cdot)$.

Theorem (Haeupler, Saha, and Srinivasan '11)
If LLL condition (1) holds for every $e \in \mathcal{E}$, then for any event B, it holds that

$$\mu(B) \leqslant \Pr(B) \prod_{e \in \Gamma(B)} (1 - x(e))^{-1}.$$

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Theorem (Haeupler, Saha, and Srinivasan '11)
If LLL condition (1) holds for every $e \in \mathcal{E}$, then for any event B, it holds that

$$\mu(B) \leqslant \Pr(B) \prod_{e \in \Gamma(B)} (1 - x(e))^{-1}.$$

Local uniformity

Lemma

If $\forall e \in \mathcal{E}, k' \leqslant |e| \leqslant k, t \geqslant k$ *and* $q \geqslant (et\Delta)^{\frac{1}{k'-1}}$*, then for any* $v \in V$ *and any colour* $c \in [q]$,

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Coupling

Say we want to compute the marginal probability of v.

Let $\mathcal{C}_i$ be the set of colourings where v is coloured i, and μ_i be uniform over $\mathcal{C}_i$. We want to couple μ_1 and μ_2.

Start: $V_1 = \{v\}$, $V_{col} = \{v\}$. Maintain $V_2 = V \setminus V_1$.

Body:

1. For any hyperedge e intersecting both V_1 and V_2, let u be its first vertex. Couple u maximally assuming its marginal probabilities are known.
2. Remove all hyperedges that are satisfied in both copies.
3. If an edge has k_2 vertices coloured, put all remaining vertices in V_1 (failed) and remove the edge.

Stop: all hyperedges intersecting V_1 are removed.

(The constant k_2 is eventually set to $\frac{3k}{7}$ for approximate counting.)

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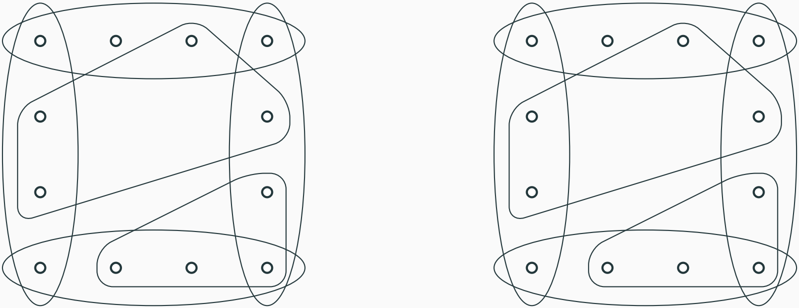

At any time, there are at least $k' - k_2$ empty vertices in any hyperedge.

If $q > C\Delta^{\frac{3}{k'-k_2}}$, then the coupling stops in $O(\log n)$ steps with probability $1 - O\left(\frac{1}{n^c}\right)$.

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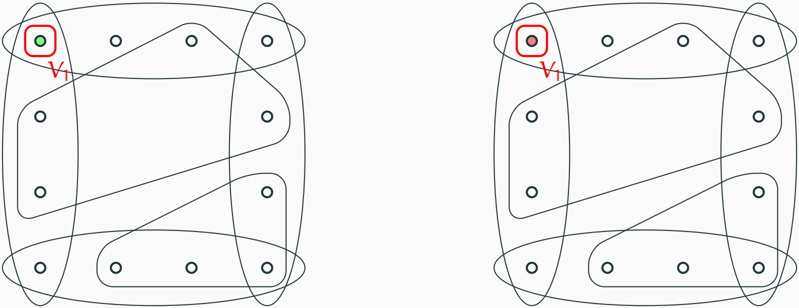

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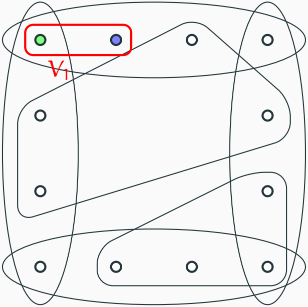

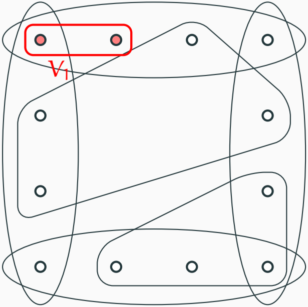

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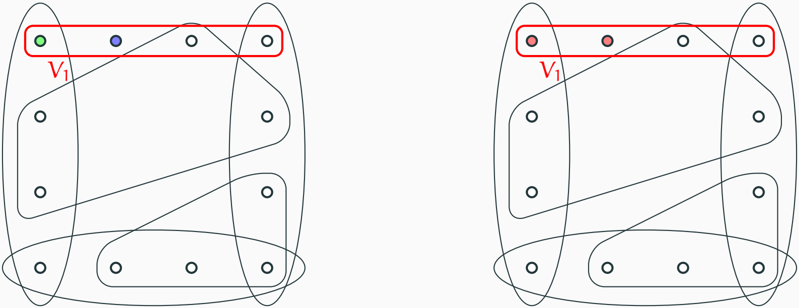

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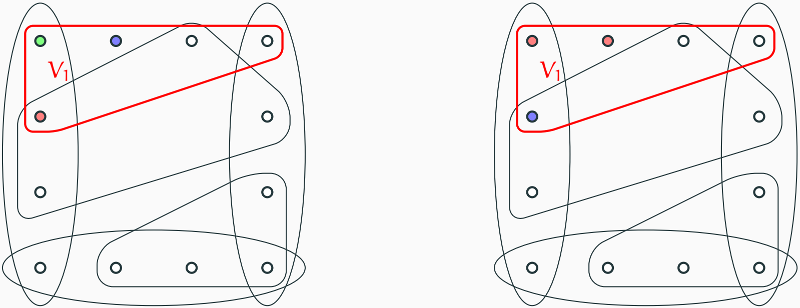

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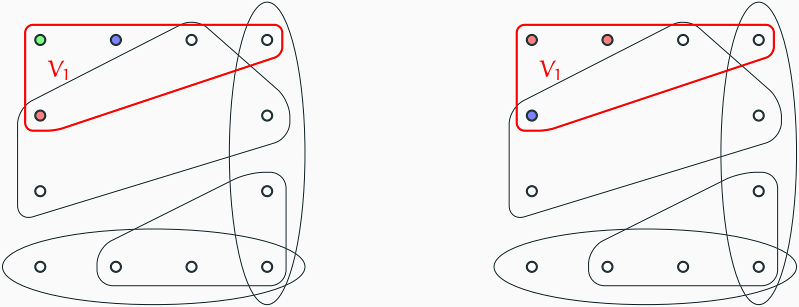

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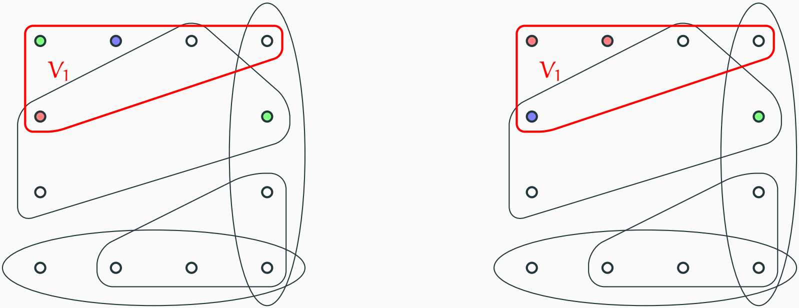

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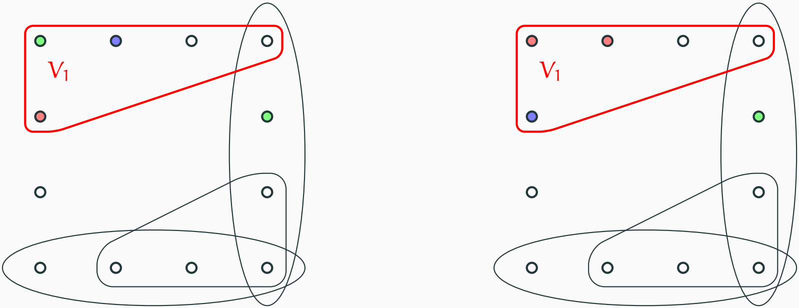

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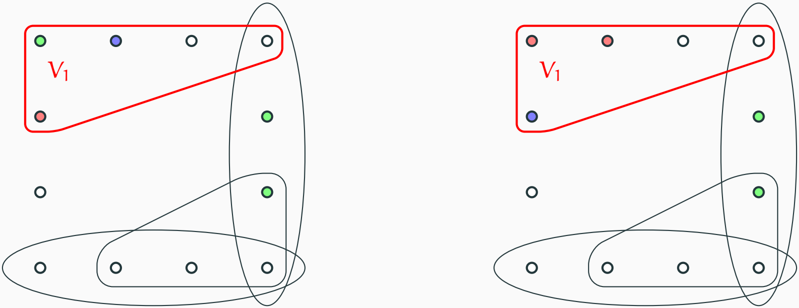

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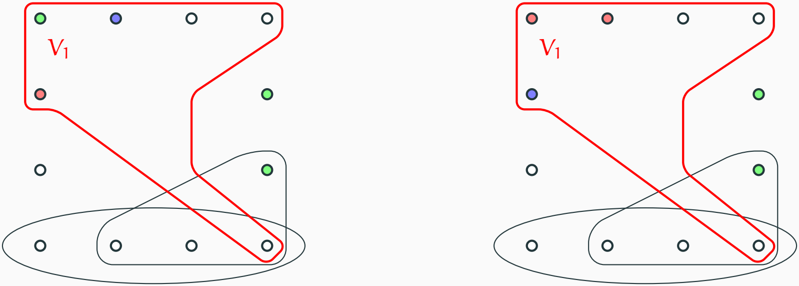

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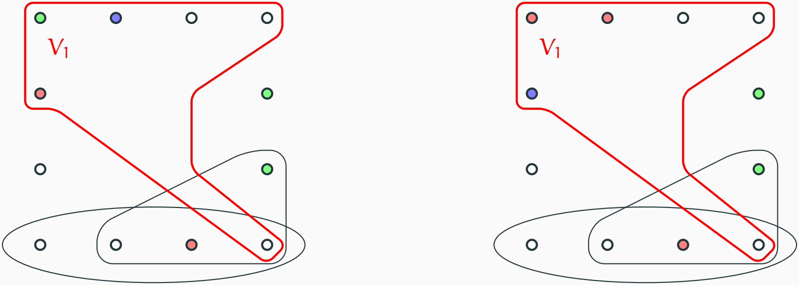

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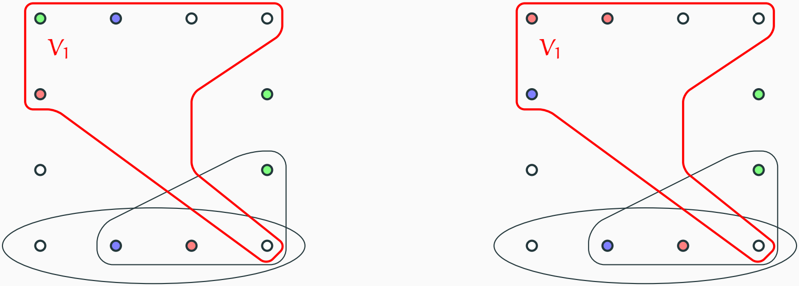

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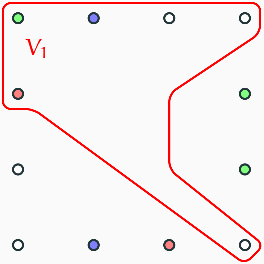

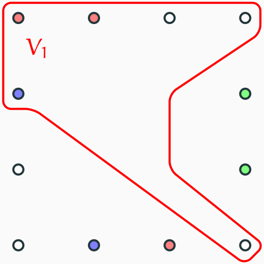

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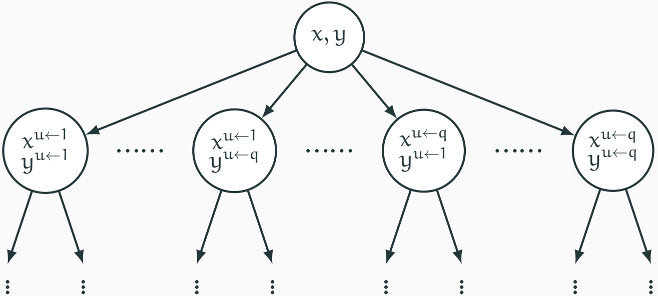
Coupling tree



Coupling tree $\mathcal{T}$: each node is a pair of partial colourings (x, y).

The children of (x, y) are all q^2 ways to extend them to the next vertex.

Linear program

We cannot really run the coupling. Instead, we set up a linear program. The variables are to mimic:

$$p^x_{x,y} = \frac{|\mathcal{C}_1|}{|\mathcal{C}_x|} \cdot \mu_{cp}(x, y),$$
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where $\mathcal{C}_i$ is the set of colourings s.t. $v \leftarrow i$ for $i = 1, 2$, and $\mathcal{C}_x$ (or $\mathcal{C}_y$) is the set of colourings consistent with x (or y).

Note that $\sum_y p^x_{x,y} = 1$ and thus $0 \leqslant p^x_{x,y}, p^y_{x,y} \leqslant 1$.

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From the definition: $\frac{|\mathcal{C}_1|}{|\mathcal{C}_2|} = \frac{p^x_{x,y}}{p^y_{x,y}} \cdot \frac{|\mathcal{C}_x|}{|\mathcal{C}_y|}$.

If (x, y) is a leaf in $\mathcal{T}$, then we can compute $\frac{|\mathcal{C}_x|}{|\mathcal{C}_y|}$ in time $\exp(|V_1 \setminus V_{col}|)$.

Constraints 1: For every leaf (x, y), we have the constraints:

$$\underline{r} \leqslant \frac{p^x_{x,y}}{p^y_{x,y}} \cdot \frac{|\mathcal{C}_x|}{|\mathcal{C}_y|} \leqslant \overline{r}.$$

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Constraints 2

Constraints 2: For the root $(x_0, y_0) \in \mathcal{T}$, we have

$$p^{x_0}_{x_0,y_0} = p^{y_0}_{x_0,y_0} = 1.$$

Moreover, for every non-leaf $(x, y) \in \mathcal{T}$, let u be the next vertex to couple. For every $c \in [q]$,

$$\sum_{c' \in [q]} p^{x^{u\leftarrow c}}_{x^{u\leftarrow c}, y^{u\leftarrow c'}} = \frac{|\mathcal{C}_1|}{|\mathcal{C}_{x^{u\leftarrow c}}|} \cdot \frac{|\mathcal{C}_{x^{u\leftarrow c}}|}{|\mathcal{C}_x|} \cdot \mu_{cp}(x, y) = p^x_{x,y};$$

$$\sum_{c' \in [q]} p^{y^{u\leftarrow c}}_{x^{u\leftarrow c'}, y^{u\leftarrow c}} = \frac{|\mathcal{C}_2|}{|\mathcal{C}_{y^{u\leftarrow c}}|} \cdot \frac{|\mathcal{C}_{y^{u\leftarrow c}}|}{|\mathcal{C}_y|} \cdot \mu_{cp}(x, y) = p^y_{x,y}.$$

Recover the marginals

Due to **Constraints 2**, a simple induction shows that for every $\sigma \in \mathcal{C}_1$,

$$\sum_{(x,y)\in\mathcal{L}(\mathcal{T}):\ \sigma\models x} p_{x,y}^{x} = 1.$$

Rewrite $|\mathcal{C}_1|$:

$$\begin{aligned}|\mathcal{C}_1| &= \sum_{\sigma\in\mathcal{C}_1} 1 = \sum_{\sigma\in\mathcal{C}_1}\ \sum_{(x,y)\in\mathcal{L}(\mathcal{T}):\sigma\models x} p_{x,y}^{x} \\ &= \sum_{(x,y)\in\mathcal{L}(\mathcal{T})}\ \sum_{\sigma\models x} p_{x,y}^{x} \\ &= \sum_{(x,y)\in\mathcal{L}(\mathcal{T})} p_{x,y}^{x}\,|C_x|.\end{aligned}$$

Similar equalities hold on the y side, implying:

$$\frac{|\mathcal{C}_1|}{|\mathcal{C}_2|} = \frac{\sum_{(x,y)\in\mathcal{L}(\mathcal{T})} p_{x,y}^{x}\,|C_x|}{\sum_{(x,y)\in\mathcal{L}(\mathcal{T})} p_{x,y}^{y}\,|C_y|}.$$

Recover the marginals (cont.)

$$\frac{|\mathcal{C}_1|}{|\mathcal{C}_2|} = \frac{\sum_{(x,y)\in\mathcal{L}(\mathcal{T})} p^x_{x,y} |C_x|}{\sum_{(x,y)\in\mathcal{L}(\mathcal{T})} p^y_{x,y} |C_y|}$$

Recall **Constraints 1**. For any $(x, y) \in \mathcal{L}(\mathcal{T})$,

$$\underline{r} \leqslant \frac{p^x_{x,y} |C_x|}{p^y_{x,y} |C_y|} \leqslant \overline{r}.$$

It implies that

$$\underline{r} \leqslant \frac{|\mathcal{C}_1|}{|\mathcal{C}_2|} \leqslant \overline{r}.$$

Constraints 3

Unfortunately, the whole linear program is exponentially large. The saving grace is that the coupling stops at $O(\log n)$ size whp.

If we truncate at $O(\log n)$ levels, the error should be small, due to local uniformity.

Constraints 3: For every $c, c' \in [q]$ that $c \neq c'$:

$$p^{x^{u \leftarrow c}}_{x^{u \leftarrow c}, y^{u \leftarrow c'}} \leqslant \frac{5}{t} \cdot p^{x}_{x,y};$$
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The quantity t will eventually be set as $C(k\Delta)^6$.

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Truncation error

Recall that

$$|\mathcal{C}_1| = \sum_{\sigma \in \mathcal{C}_1} \sum_{(x,y) \in \mathcal{L}(\mathcal{T}) : \sigma \models x} p^x_{x,y}.$$

The truncation error from a particular $\sigma \in \mathcal{C}_1$ comes from conditioned on outputing σ, the coupling lasts too long.

Such "bad" colourings do exist (all early vertices are monochromatic).

We prove two things:

1. The fraction of "bad" colourings is small;
2. For every "good" colouring, the truncation error is small because of **Constraints 3**.

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Bound the error

A "bad" colouring must fail many hyperedges during the coupling, but we couple k_2 vertices of every hyperedge.

Thus its fraction is small if k_2 is sufficiently large.

The error allowed by **Constraints 3** is controlled by the number of empty vertices in the coupling process, namely the quantity $k' - k_2$.

The larger $k' - k_2$, the more uniform all vertices are and the smaller coupling errors.

We solve an optimization problem to get the best k_2 balancing the two points above.

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Counting and sampling

So far we are calculating the marginal probability, which requires that there are sufficiently many uncoloured vertices in all hyperedges.

- For approximate counting, we use the local lemma to find a partial colouring so that every hyperedge is satisfied by its first $\frac{k}{14}$ vertices.

  Then we compute the marginal probability of this partial colouring by pinning vertices one by one.

- For sampling, we use the marginal to colour vertices, similar to the coupling process. We colour $\frac{3k}{16}$ vertices of every hyperedge.

  With high probability, every remaining connected component has size $O(\log n)$.

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Concluding remarks

Open problems

- What is the correct threshold for hypergraph colouring?
  - Is it $q \asymp \Delta^{\frac{2}{k}}$?
- What about **NP**-hardness of sampling hypergraph colourings?
- Does this method work for graph colourings?
- Can we show anything about the zeros of hypergraph colourings?

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Thank you!

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