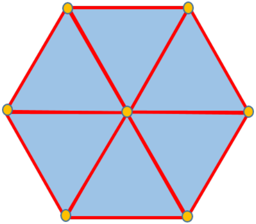
Random Walk on Simplicial Complexes

Tali Kaufman (BIU) and Izhar Oppenheim (BGU)

Simplicial Complexes



Simplicial Complexes - Abstract Definition

A d-dimensional simplicial complex X is defined as follows:

1. V is a set of vertices
2. For every $-1 \leq k \leq d$, the set of k-simplices of X, denoted $X(k)$, is a subset of $\binom{V}{k+1}$ and we denote $X = \bigcup_k X(k)$
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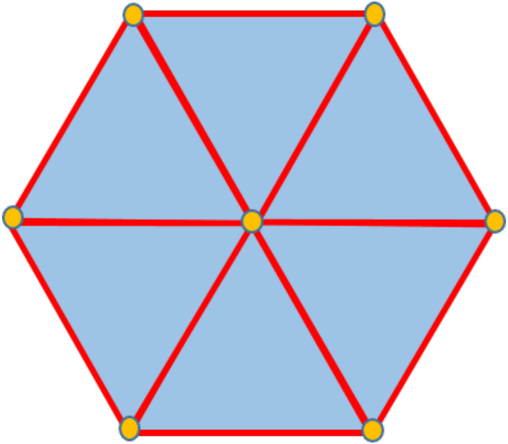
Below X is always assumed to be finite ($|V| < \infty$) and pure d-dimensional (every k-simplex is contained in a d-dimensional simplex).

Geometric interpretation



$C^k(X)$

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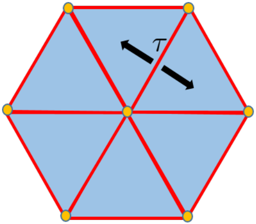
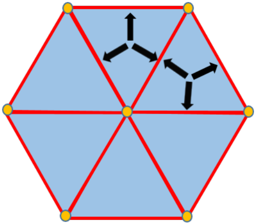
Define the following inner-product on $C^k(X)$:

$$\langle \phi, \psi \rangle = \sum_{\eta \in X(k)} w(\eta)\phi(\eta)\psi(\eta),$$

where w is a weight function which "takes into account" the higher dimensional structure (explicitly, $w(\tau) = (d-k)! \sum_{\sigma \in X(d), \tau \subseteq \sigma} w(\sigma), \ \forall \tau \in X(k)$).

Random Walks on Simplicial Complexes

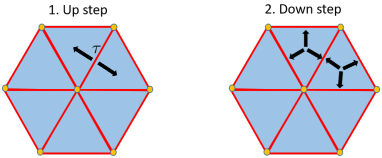




k-th Random walk on X

The k-random walk is a random walk on $X(k)$ defined as follows: for $\tau \in X(k)$

1. Up step: Choose $\eta \in X(k+1)$ such that $\tau \subseteq \eta$ at random (according to the weight function w)
2. Down step: Choose at random $\tau' \in X(k)$ such that $\tau' \subseteq \eta$



We denote by $M_k^+ : C^k(X) \to C^k(X)$ the operator corresponding to this random walk.

Up and Down operators

Define the *Up* operator $U_k : C^k(X) \rightarrow C^{k+1}(X)$: for $\phi \in C^k(X), \eta \in X(k+1)$,

$$(U_k \phi)(\eta) = \sum_{\tau \in X(k), \tau \subseteq \eta} \phi(\tau).$$

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Define the *Down* operator $D_{k+1} : C^{k+1}(X) \rightarrow C^k(X)$: for $\psi \in C^{k+1}(X), \tau \in X(k)$,

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$U_k^* = D_{k+1}$, $M_k^+ = \frac{1}{k+2} D_{k+1} U_k$

The 0-random walk in graphs

Assume that X is a regular graph. What is M_0^+?

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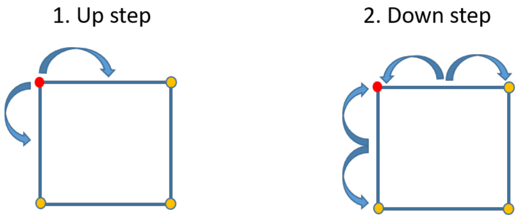

Note: This is not the usual random walk, but a lazy RW (has probability 0.5 to stay at the vertex).

Motivating questions

Note:

- $M_k^+ \mathbb{1} = \mathbb{1}$.
- M_k^+ is self-adjoint and all its eigenvalues are in $[0, 1]$.
- Under mild connectivity conditions on X, every eigenfunction $\phi \perp \mathbb{1}$ has eigenvalue < 1.

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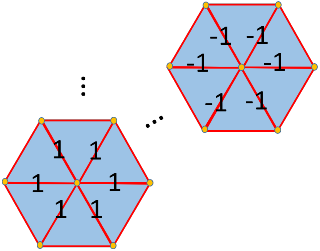
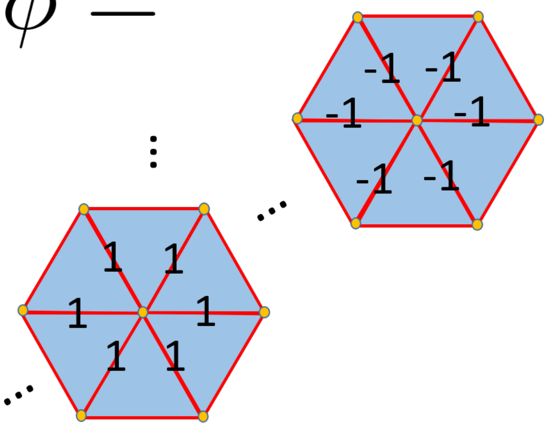
2. What can we say about $\langle M_k^+ \phi, \phi \rangle$ for a specific ϕ beyond the bound on the second eigenvalue?

How well can the RW mix



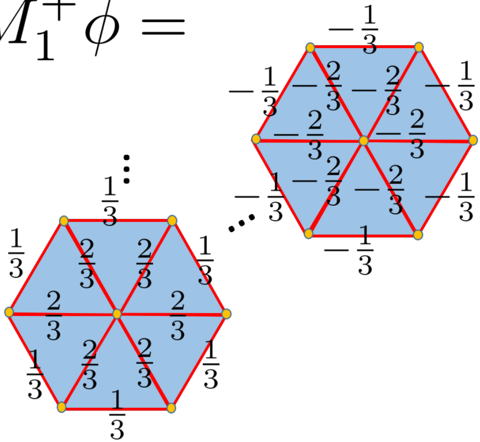
How well can the RW mix? (1)

$\phi =$



How well can the RW mix? (2)

$$M_1^+ \phi =$$



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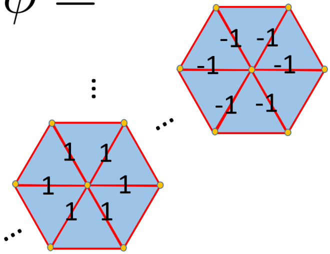

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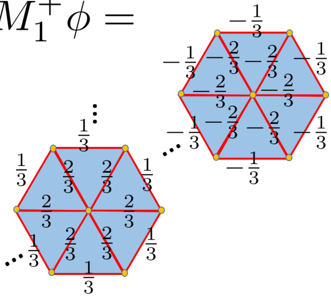
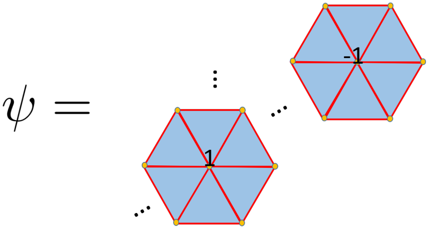


$$\langle M_1^+ \phi, \phi \rangle = \frac{2}{3} \|\phi\|^2$$

Observe that the obstruction to $\frac{1}{3}$-mixing came from "below": $\phi = U_0\psi$ where ψ is 1 on one vertex and -1 on the other (0 everywhere else)



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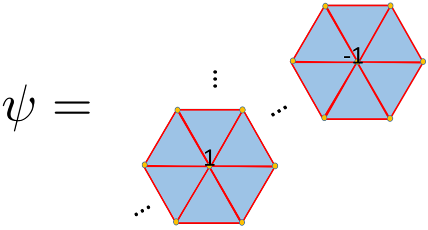
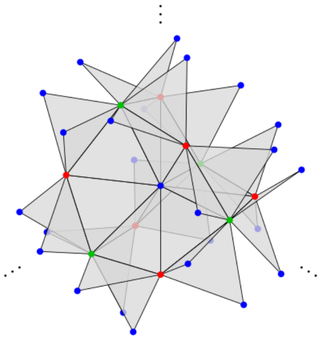


This is a general phenomenon: in general, for $k > 0$, in the k-walk we should expect to see $\frac{2}{k+2}, \ldots, \frac{k+1}{k+2}$ "obstructions" coming from dimensions $k-1, \ldots, 0$.

High dimensional local spectral expanders



Links

Given a simplex $\tau \in X$, the *link* of τ is the subcomplex of X, denoted X_τ and defined as

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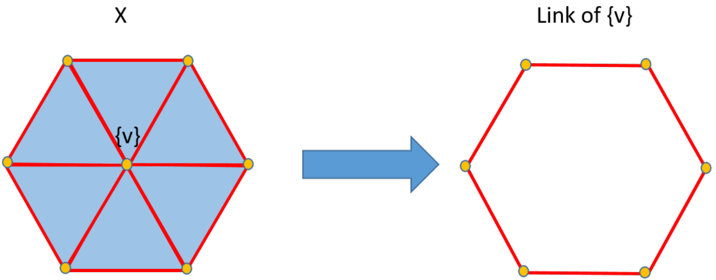
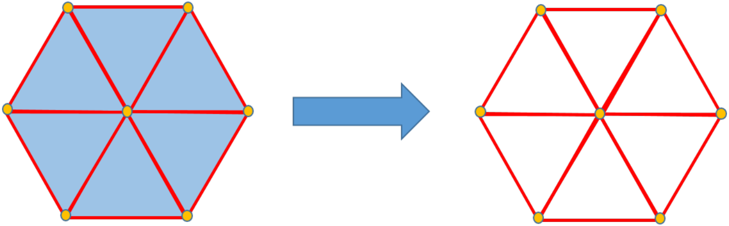


1-Skeleton

The 1-Skeleton of a complex is the graph $(X(0), X(1))$:



High dimensional local spectral expanders - definition

For a constant $0 < \lambda < 1$, X is called a one-sided (two sided) λ-local spectral expander if:

1. The 1-skeleton of X is connected and normalized spectrum of the 1-skeleton of X is contained in $[-1, \lambda] \cup \{1\}$ (two-sided: $[-\lambda, \lambda] \cup \{1\}$).
2. For every $\tau \in X(k)$, $k < d - 1$, 1-skeleton of X_τ is connected and normalized spectrum of the 1-skeleton of X_τ is contained in $[-1, \lambda] \cup \{1\}$ (two-sided: $[-\lambda, \lambda] \cup \{1\}$).

Normalized spectrum = normalized according to the weight function w.

Local spectral expansion can be deduces “very” locally

Theorem (O.): If X and all the links (of dim. ≥ 1) are connected and the second e.v. for all the 1-dimensional links is $\leq \frac{\lambda}{1+(d-1)\lambda}$, then X is λ-local spectral expander.

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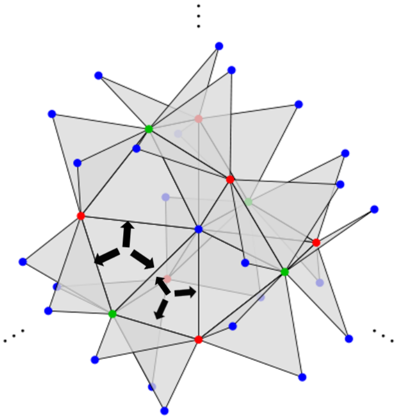
If in addition the smallest e.v. all the 1-dimensional links is $\geq \frac{-\lambda}{1+(d-1)\lambda}$, then X is a two-sided λ-local spectral expander.

Previous work on high order walks

- First introduced by Kaufman and Mass, who studied it for ONE sided local spectral expanders; they got $1 - \frac{1}{(k+2)^2} + f(\lambda, k)$ on second e.v of M_k^+.
- Later improved by Dinur and Kaufman who studied it for TWO sided local spectral expanders; they $1 - \frac{1}{k+2} + O(\lambda(k+1))$ on second e.v of M_k^+; This was useful for agreement expansion questions.

Decomposition Theorems for Random Walks on HD expanders



Decomposition Theorem - general idea

If X is λ-local spectral expander and $\phi \in C^k(X)$, $\phi \perp \mathbb{1}$, then

1. ϕ can be "projected" on $C^i(X)$, $0 \leq i \leq k$
2. These projections control how well the random walk mixes: the more ϕ is concentrated at the higher dimensions, the faster the mixing.

Decomposition Theorem - exact formulation

Main Theorem: Let X be a λ-local spectral expander and $0 \leq k \leq d-1$ constant. For any $\phi \in C^k(X), \phi \perp \mathbb{1}$ there are $\phi^k \in C^k(X), \phi^{k-1} \in C^{k-1}(X), ..., \phi^0 \in C^0(X)$ such that

$$\phi^k \perp \mathbb{1}, ..., \phi^0 \perp \mathbb{1},$$

$$\|\phi\|^2 = \|\phi^k\|^2 + \|\phi^{k-1}\|^2 + ... + \|\phi^0\|^2,$$

$$\langle M_k^+ \phi, \phi \rangle \leq \sum_{i=0}^{k} \left(\frac{k+1-i}{k+2} + \lambda f(k,i) \right) \|\phi^i\|^2,$$

$(f(k,i) = \frac{(k+i+2)(k+1-i)}{2(k+2)})$.

Bound on the second eigenvalue

$$\langle M_k^+ \phi, \phi \rangle \leq \sum_{i=0}^{k} \left(\frac{k+1-i}{k+2} + O((k+1)\lambda) \right) \|\phi^i\|^2.$$

When λ is small, we note that the coefficients of the $\|\phi^i\|$'s in the sum above become larger as i becomes smaller. Therefore, the "worst case scenario" is when $\|\phi\|^2 = \|\phi^0\|^2$.

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$$\langle M_k^+ \phi, \phi \rangle \leq \left(\frac{k+1}{k+2} + \lambda \frac{k+1}{2} \right) \|\phi\|^2,$$

and therefore the second eigenvalue is bounded by $\frac{k+1}{k+2} + \lambda \frac{k+1}{2}$.

A more explicit decomposition for 2-sided λ-local spectral expanders

(Inspired by Dikstein, Dinur, Filmus and Harsha)

Assuming 2-sided λ-local spectral gap:

- The non-trivial spectrum of M_k^+ is contained in $[\frac{1}{k+2} - f(k)\lambda, \frac{1}{k+2} + f(k)\lambda] \cup \ldots \cup [\frac{k+1}{k+2} - f(k)\lambda, \frac{k+1}{k+2} + f(k)\lambda]$
- The eigenspaces are $O(\lambda)$-approximated by the Up operators images.

Some words about the proofs (if time permits)

Thank you for listening