Geometry of Matroids

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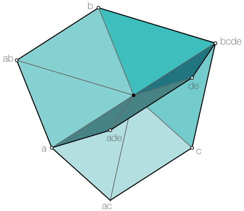
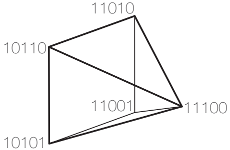
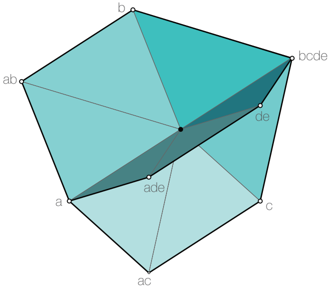
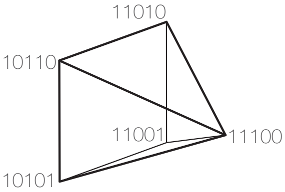
Geometry of Polynomials
Simons Institute, February 13, 2019









- Thank you for the invitation!

Geometry and Combinatorics. Two visionary remarks.

example is so beautiful that we decided to publish it independently of the applications. We believe that combinatorial methods will play an increasing role in the future of geometry and topology.

We consider the Grassmann manifold G^k_{n-k} of all $(n-k)$-dimensional

Gelfand–Goresky–MacPherson–Serganova, 1987

of dedication and lasting achievements, we were struck by one remark, which to our minds was later to prove prophetic: "We combinatorialists have much to gain from the study of algebraic geometry, if not by its direct applications to our field, at least by the analogies between the two subjects."

R. C. Bose (quoted by Kelly–Rota, 1973)

Summary.

Matroids are geometric.

Geometry and matroid theory help each other a lot.

Geometry can prove log concavity.

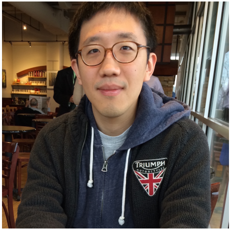
Summary.

Matroids are geometric.

Geometry and matroid theory help each other a lot.

Geometry can prove log concavity.

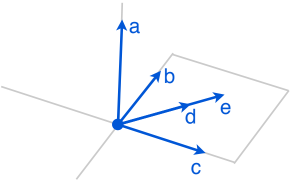
My work here is joint with
Carly Klivans (06), Graham Denham + June Huh (17-19).



Matroids

Goal: Capture the combinatorial essence of independence.

E= set of vectors spanning $\mathbb{R}^d$.
$\mathcal{B}$ = collection of subsets of E which are bases of $\mathbb{R}^d$.



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$\mathcal{B} = \{abc, abd, abe, acd, ace\}$

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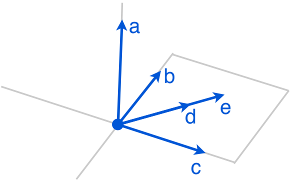

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Properties:
(B1) $\mathcal{B} \neq \emptyset$
(B2) If $A, B \in \mathcal{B}$ and $a \in A - B$,
then there exists $b \in B - A$
such that $(A - a) \cup b \in \mathcal{B}$.

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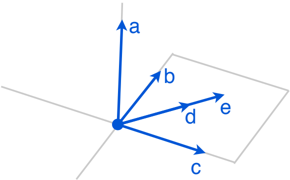

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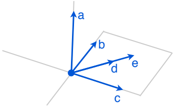
Definition. (Nakasawa, Whitney, 35)
A set E and a collection $\mathcal{B}$ of subsets of E are a **matroid** if they satisfies properties (B1) and (B2).

Many matroids in “nature":

$E = abcde$
$\mathcal{B} = \{abc, abd, abe, acd, ace\}$

1. Linear matroids
E= set of vectors spanning $\mathbb{R}^d$.
$\mathcal{B}$ = bases of $\mathbb{R}^d$ in E.



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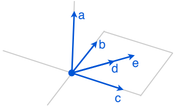

2. Graphical matroids
E= edges of a connected graph G.
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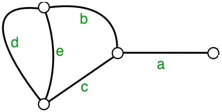

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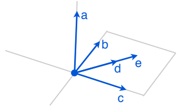

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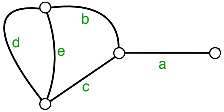

3. Algebraic matroids (field extensions)

4. Transversal matroids (matchings)

5. Gammoids (routings)

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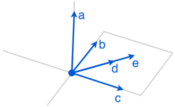

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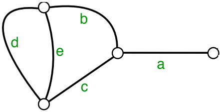

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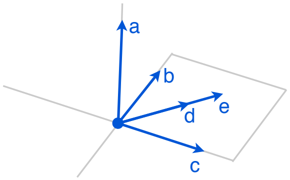
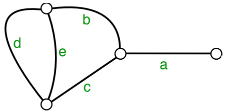
Thm for matroids $\mapsto$ Thms for vectors, graphs, field exts, matchings, routings…

Many points of view.

1. Bases (polytope)

$\mathcal{B} = \{abc, abd, abe, acd, ace\}$





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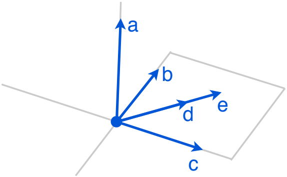
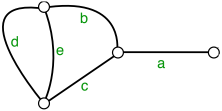


2. Independent sets (simplicial complex)

$\mathcal{I} = \{abc, abd, abe, acd, ace,$
$ab, ac, ad, ae, bc, bd, be, cd, ce,$
$a, b, c, d, e,$
$\emptyset\}$



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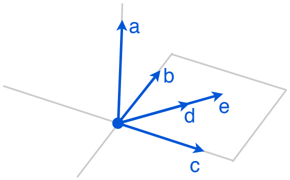

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3. (Broken) Circuits – minl dependences (simplicial complex.)

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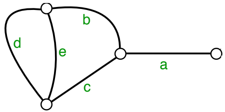

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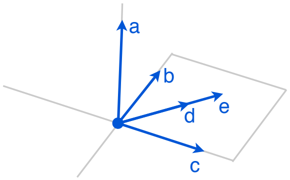

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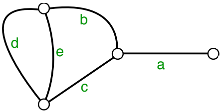

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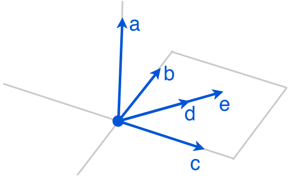

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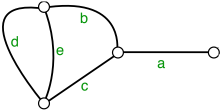

4. Flats – spanned sets (lattice)
$\mathcal{F} = \{abcde$
$ab, ac, ade, bcde,$
$a, b, c, de,$
$\emptyset\}$

Log-concavity: f-vectors.

$$
\begin{aligned}
IN(M) &= \{\text{independent sets}\} \\
\overline{BC}_<(M) &= \{\text{independent sets containing no broken circuit}\}
\end{aligned}
$$

***f*-vector**: $f_i(\Delta)$ = # of sets $F \in \Delta$ with $|F| = i + 1$.

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Conjectures. (Welsh 71 Mason 72, Rota 71 Heron 72 Welsh 76)
The sequences $\{f_i(IN(M))\}$ and $\{f_i(\overline{BC}_<(M))\}$ are

unimodal $\Leftarrow$ log-concave $\Leftarrow$ strongly log-concave.

Remark. $IN(M) = \overline{BC}_<(M * e)$ so it is enough to prove it for $\overline{BC}$.

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Theorems. The sequences $\{f_i(IN(M))\}$ and $\{f_i(\overline{BC}_<(M))\}$ are
- log-concave. (Adiprasito–Huh–Katz 2015)

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Corollaries: – approximating the number of bases of a matroid
– expansion of basis exchange graph is at least 1

Log-concavity: *h*-vectors.

$$IN(M) = \{\text{independent sets}\}$$
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***f*-vector**: |coeffs| of $\overline{\chi}(q)$ $\implies$ ***h*-vector**: |coeffs| of $\overline{\chi}(q+1)$

Log-concavity: h-vectors.

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f-vector: |coeffs| of $\overline{\chi}(q)$ $\Longrightarrow$ **h-vector**: |coeffs| of $\overline{\chi}(q+1)$

Conjectures. (Brylawski 82, Dawson 83, Hibi 89)
The sequences $\{h_i(IN(M))\}$ and $\{h_i(\overline{BC}_<(M))\}$ are

unimodal $\Leftarrow$ log-concave.

Remark. (Oveis Gharan) They are not strongly log-concave.
Remark. (Lenz) $h(\Delta)$ log-concave $\Rightarrow$ $f(\Delta)$ strictly log-concave .

Log-concavity: *h*-vectors.

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Theorem. The sequences $\{h_i(IN(M))\}$ and $\{h_i(\overline{BC}_<(M))\}$ are log-concave. (FA – Denham – Huh)

$\longrightarrow$ Lagrangian theory of matroids

Log-concavity: Why does alg. combinatorics care?

There are **many** combinatorial sequences that are (sometimes conjecturally) positive, unimodal, or log-concave.

Sometimes the proofs are quite easy.

If they are not easy, they are often quite **hard**, and require a fundamentally new construction or connection.

We:

- understand something new about our structures.
- derive the conjectures as a consequence.

Are matroids geometric?

A **linear** matroid comes from a set of vectors. Are they all linear?

(linear matroids) vs. (all matroids):

- Almost any matroid we think of is linear.
- (Nelson, 2018) 100% of matroids are not linear.

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- “Missing axiom" for linear matroids? No. (Mayhew et al, ’14)
- This is not a flaw! Matroids are natural geometric objects.

The geometry of matroids.

My main point today.
Matroids are natural geometric objects.

When I wrote my book on 'matroids, I changed the name. I called it "Combinatorial Geometries" – but it didn't take. They said "that's really matroids, isn't it?"

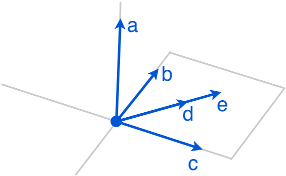
Gian-Carlo Rota, Combinatorial Theory, Fall 1998. (Thanks to John Guidi.)

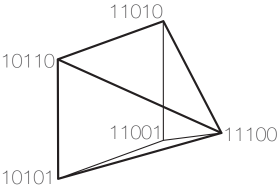
Model 1: Matroid polytopes

Def. (Edmonds 70; Gelfand Goresky MacPherson Serganova 87)
The **matroid polytope** of a matroid M on E is

$$P_M = \text{conv}\{e_B : B \text{ is a basis of } M\} \subset \mathbb{R}^E$$

where e_B is the $0-1$ indicator vector of B.





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The **matroid polytope** of M is

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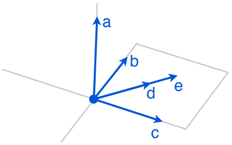

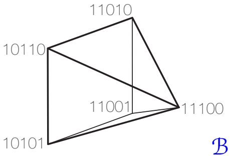

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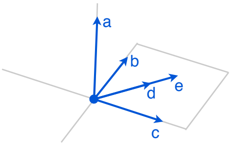

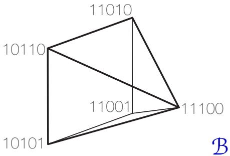

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Matroid polytopes in “nature":

1. Optimization. (Edmonds 70) For a cost function $c : E \to \mathbb{R}$, find the bases $\{b_1, \ldots, b_r\}$ of minimal cost $c(b_1) + \cdots + c(b_r)$.

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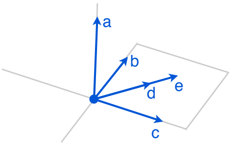

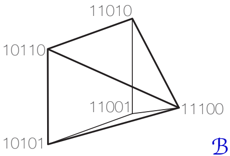

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2. Algebraic geometry. (Gelfand Goresky MacPherson Serganova 87) Understand the action of the torus $(\mathbb{C}^*)^n$ on the Grassmannian $\operatorname{Gr}(k, n)$.

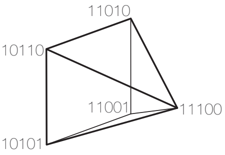
A Coxeter–geometric characterization of matroids

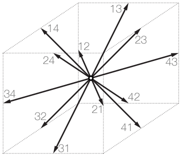
Theorem. (GGMS 87) A collection $\mathcal{B}$ of r-subsets of $[n]$ is a matroid if and only if every edge of the polytope

$$P_M = \text{conv}\{e_B : B \in \mathcal{B}\} \subset \mathbb{R}^n$$

is a translate of vectors $e_i - e_j$ for some i, j.

Def. A **matroid** is a 0-1 polytope with edge directions $e_i - e_j$.





$ij : e_i - e_j$

A Coxeter–geometric characterization of matroids

Theorem. (GGMS 87) A collection $\mathcal{B}$ of r-subsets of $[n]$ is a matroid if and only if every edge of the polytope

$$P_M = \text{conv}\{e_B : B \in \mathcal{B}\} \subset \mathbb{R}^n$$

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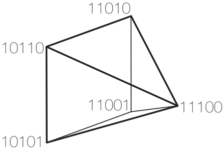

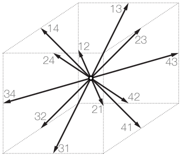

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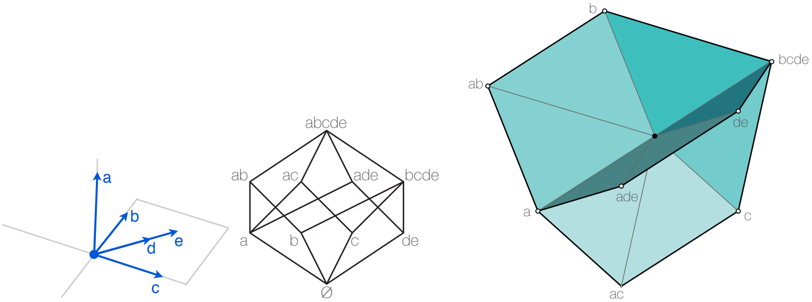
From this geometric viewpoint, all matroids are equally natural. Matroids provide the correct level of generality!

Model 2: Bergman fan

Def/Theorem. (FA-Klivans 06)
The **Bergman fan** Σ_M of M is the polyhedral complex with

- rays: $e_F := e_{f_1} + \cdots + e_{f_k}$ for each flat $F = \{f_1, \ldots, f_k\}$
- faces: cone$\{e_F : F \in \mathcal{F}\}$ for each flag $\mathcal{F} = \{\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_l \subsetneq E\}$.



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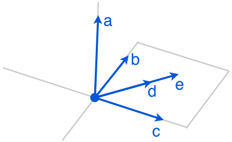

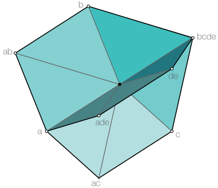

Bergman fans in “nature": Tropical geometry.

algebraic variety $V \mapsto$ Trop(V) polyhedral complex

Trop(V) still knows information about V, and can be studied combinatorially.

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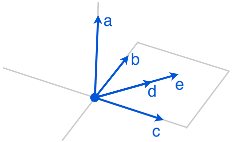

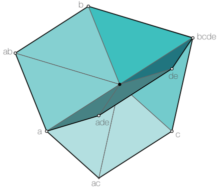

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Trop(V) still knows information about V, and can be studied combinatorially.

Question. (Sturmfels 2002) Describe Trop(linear space).

Theorem. (FA-Klivans 2006)
The tropicalization of a linear space $V \subseteq \mathbb{R}^n$ is the Bergman fan $\Sigma_{M(V)}$.

A tropical characterization of matroids

A **tropical variety** is a polyhedral complex “with zero-tension".
It has a **tropical degree**, and AlgDeg(V) = TropDeg(Trop V).

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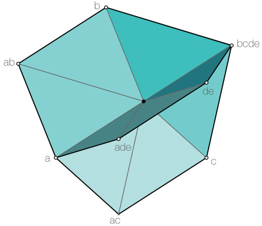
Theorem. (Fink 2013) A tropical variety has degree 1 if and only if it is the Bergman fan of a matroid.

Definition. A **matroid** is a tropical variety of degree 1.



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Towards Model 3: Orthogonality for matroids

Theorem / Definition. If $\mathcal{B}$ is a matroid on E, then

$$\mathcal{B}^{\perp} = \{E - B : B \in \mathcal{B}\}$$

is also a matroid, the **orthogonal** or **dual** matroid $M^{\perp}$.

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This generalizes:

- Dual graphs:

  abe spanning tree of G

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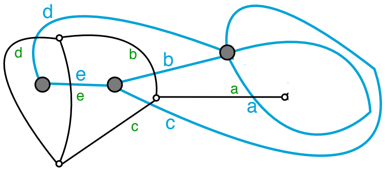

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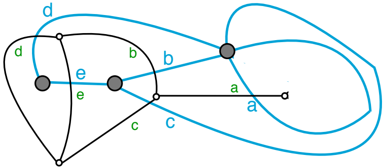

- Orthogonal complements:

  abe basis of W

  $\updownarrow$

  cd basis of $W^\perp$

$$W = \text{rowspace} \begin{bmatrix} 0 & 1 & 0 & .5 & 1 \\ 0 & 0 & 1 & .5 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$W^\perp = \text{rowspace} \begin{bmatrix} 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 2 & -1 \end{bmatrix}$$

Model 3: conormal fan

Definition. (FA-Denham-Huh)
The *conormal fan* $\Sigma_{M,M^\perp}$ is the polyhedral complex in $\mathbb{R}^{E \sqcup E}$ with
- rays $e_F + f_G$ for each flat F and coflat G with $F \cup G = E$
- $cone(\mathcal{F}, \mathcal{G}) := cone\{e_{F_i} + f_{G_i} : 1 \leq i \leq l\}$ for each biflag $(\mathcal{F}, \mathcal{G})$.

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Definition. (FA-Denham-Huh)
A **biflag** of M consists of a flag $\mathcal{F} = \{F_1 \subseteq \cdots \subseteq F_l\}$ of flats and a flag $\mathcal{G} = \{G_1 \supseteq \cdots \supseteq G_l\}$ of coflats (flats of $M^\perp$) such that

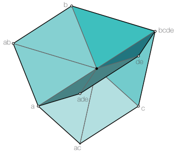
$$\bigcap_{i=1}^{l} (F_i \cup G_i) = E, \qquad \bigcup_{i=1}^{l} (F_i \cap G_i) \neq E.$$

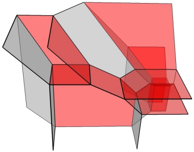
Fact. All maximal biflags have length $n-2$.

(Motivation: toric + tropical geometry, hyperplane arrs, Coxeter combinatorics)

Tropical applications.

1. A **tropical manifold** is a tropical variety that looks locally like a (Bergman fan of a) matroid. (Mikhalkin, Rau, Shaw, ...)





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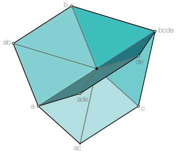

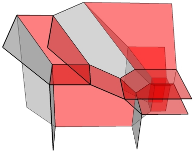

2. The conormal fan is a Lagrangian analog of the Bergman fan. Expectation: Conormal fans should play a similar role for **tropical Lagrangian submanifolds** (Mikhalkin, ...)

Log-concavity strategy 1: geometric models

To prove log-concavity of invariants of a **linear** matroid M:

1. Build an algebro-geometric model $X(M)$ for M.
2. (Combin invariants of M) = (Geom invariants of $X(M)$).
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Good news: This strategy works! (Huh, 2012, 15)

Bad news: ...only when M is a linear matroid.

Log-concavity strategy 2: tropical geometric models

To prove log-concavity of invariants of **any** matroid M:

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$f_i(\overline{BC}_<(M))$: Bergman fan Σ_M.

Sturmfels 02, A.–Klivans 03

$h_i(\overline{BC}_<(M))$: conormal fan $\Sigma_{M,M^\perp}$.

A.–Denham–Huh

Good news: This works even when M is not realizable!

Good? Bad? news: We have to work harder for our inequalities.

Cohomology of the Bergman fan

Def/Thm. (Adiprasito–Huh–Katz 18) The **Chow ring** of Σ_M is

$$A_M = \mathbb{Z}[x_F \,:\, F \text{ proper flat}]\,/\,(I_M + J_M)$$

where

$$I_M = (x_F x_{F'} \,:\, F \text{ and } F' \text{ are incomparable})$$

$$J_M = \left(\sum_{F \ni i} x_F - \sum_{F \ni j} x_F : \quad i,j \in E \right)$$

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It behaves like the Chow ring of a smooth proj. algebraic variety:

Poincaré duality	:	$A = A_0 \oplus \cdots \oplus A_r, \qquad A_i \cong A_{n-1-i}$
Hard Lefschetz theorem	:	$\cdot \ell^{r-2i} : A_i \cong A_{n-1-i}$ for ℓ strictly submodular
Hodge-Riemann relations	:	a bilinear form on A^i is pos. def. on $\operatorname{Ker} \ell^{r-2i+1}$

Tropical intersections and log-concavity

Theorem. (Adiprasito – Huh – Katz 18) In the **Chow ring** of Σ_M

$$A_M = \mathbb{Z}[x_F \,:\, F \text{ proper flat}] \,/\, (I_M + J_M)$$

the classes

$$\alpha = \sum_{i \in F} x_F, \qquad \beta = \sum_{i \notin F} x_F$$

satisfy

$$\alpha^{r-i}\beta^i = f_i(\overline{BC}_<(M)) \qquad (1 \leq i \leq r)$$

Hodge-Riemann relations $\Rightarrow$ $f_0, f_1, \ldots, f_r$ **is log-concave.**

Note: $A_r \cong A_0 = \mathbb{Z}$ $\Rightarrow$ degree r elements are just scalars!

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$$a^i d^{n-2-i} = h_{r-i}(\overline{BC}_<(M)) \qquad (1 \leq i \leq r-1)$$

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• It is much harder to prove $a^i d^{n-2-i} = h_{r-i}$ now.
 – lots of new (intricate, interesting) matroid theory, or
 – new Lagrangian interpretation of CSM classes of matroids.

(Chern-Simons-MacPherson classes (LópezdeMedrano–Rincón–Shaw 2017))

So how can I think about these fans?

(If there is time.)

muchas gracias.