Geometric Complexity Theory: No Occurrence Obstructions for Determinant vs Permanent

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Problem and Main Result

Permanent versus determinant

- How many arithmetic operations are sufficient to evaluate the permanent of an m by m matrix (x_{ij})?

$$\mathrm{per}_m := \sum_{\pi \in S_m} x_{1\pi(1)} \cdots x_{m\pi(m)}$$

- Best known algorithm: $O(m2^m)$ operations
- The determinant $\det_n$ can be evaluated with $\mathrm{poly}(n)$ operations

$$\det_n := \sum_{\pi \in S_n} sgn(\pi)\, x_{1\pi(1)} \cdots x_{n\pi(n)}$$

- Work over $\mathbb{C}$

Valiant's Conjecture

- Are there linear forms $a_{ij} = a_{ij}(x, z)$ in x_{ij} and z such that $(n \geq m)$

$$z^{n-m}\mathrm{per}_m = \det \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \qquad ? \qquad\qquad (*)$$

- Impossible for $n = m > 2$ (Polya)
- Possible for $n \leq 2^m - 1$ (Valiant, Grenet)
- $n \geq \frac{1}{2}m^2$ (Mignon & Ressayre 2004)

- $\boxed{\text{Valiant's Conjecture (1979): (*) impossible for } n = poly(m)}$

- Conjecture equivalent to the separation $\mathrm{VP}_{ws} \neq \mathrm{VNP}$ of complexity classes
- $\mathrm{P} \neq \mathrm{NP}$ implies $\mathrm{VP}_{ws} \neq \mathrm{VNP}$ under GRH (B, 2000)

Orbit closure of $\det_n$

- Approach by Mulmuley and Sohoni (2001) based on algebraic geometry and representation theory
- Idea of orbit closures already in Strassen (1987) for tensor rank
- nth symmetric power $\mathrm{Sym}^n V^*$ of dual space V^* with natural action of group $G := \mathrm{GL}(V)$
- Orbit $G \cdot f := \{gf \mid g \in G\}$ of $f \in \mathrm{Sym}^n V^*$
- Take $V := \mathbb{C}^{n\times n}$, $N = n^2$, view $\det_n$ as element of $\mathrm{Sym}^n V^*$
- Orbit closure w.r.t. Euclidean or Zariski topology

$$\Omega_n := \overline{\mathrm{GL}_{n^2} \cdot \det_n} \subseteq \mathrm{Sym}^n(\mathbb{C}^{n\times n})^*$$

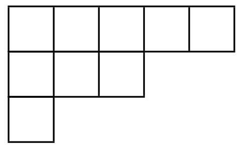
- $\Omega_2 = \mathrm{Sym}^2(\mathbb{C}^{2\times 2})^*$; Ω_3 known (Hüttenhain & Lairez '16); Ω_4 already unknown

Orbit Closure Conjecture

- Padded permanent $X_{11}^{n-m}\mathrm{per}_m \in \mathrm{Sym}^n(\mathbb{C}^{n\times n})^*$, where $n > m$
- **Orbit Closure Conjecture (M-S 2001)**

  For all $c \in \mathbb{N}_{\geq 1}$ we have $X_{11}^{m^c-m}\mathrm{per}_m \notin \Omega_{m^c}$ for infinitely many m.
- The Orbit Closure Conjecture implies Valiant's Conjecture

Splitting into irreps

- Action of group $G = \mathrm{GL}(V)$ on $\mathrm{Sym}^n V^*$ induces action on its graded coordinate ring $\mathbb{C}[\mathrm{Sym}^n V^*] = \oplus_{d \in \mathbb{N}} \mathrm{Sym}^d \mathrm{Sym}^n V$
- The plethysms $\mathrm{Sym}^d \mathrm{Sym}^n V$ splits into irreducible G-representations $\mathcal{W}_\lambda$ (Weyl modules), labeled by partitions $\lambda \vdash dn$ into at most $\dim V = n^2$ parts
- Visualize partition as Young diagram: $(5, 3, 1) \vdash 9$ write as 
- Size $|(5, 3, 1)| := 9$ is number of boxes; length $\ell(5, 3, 1) = 3$ is number of parts
- $\mathbb{C}[\Omega_n]$ denotes coordinate ring of Ω_n
- Restriction of polynomial maps to Ω_n gives surjective G-equivariant linear map:

$$\mathrm{Sym}^d \mathrm{Sym}^n V = \mathbb{C}[\mathrm{Sym}^n V^*] \twoheadrightarrow \mathbb{C}[\Omega_n]_d$$

- Say λ occurs in $\mathbb{C}[\Omega_n]_d$ if it contains a copy of $\mathcal{W}_\lambda$

Obstructions

- $Z_{n,m}$ denotes orbit closure of the padded permanent ($n > m$):

$$Z_{n,m} := \overline{\mathrm{GL}_{n^2} \cdot X_{11}^{n-m} \mathrm{per}_m} \subseteq \mathrm{Sym}^n(\mathbb{C}^{n \times n})^*. \tag{1}$$

- Suppose $X_{11}^{n-m} \mathrm{per}_m \in \Omega_n$
- Then $Z_{n,m} \subseteq \Omega_n$ and restriction gives $\mathbb{C}[\Omega_n] \twoheadrightarrow \mathbb{C}[Z_{n,m}]$
- Schur's lemma: if λ occurs in $\mathbb{C}[Z_{n,m}]$, then λ occurs in $\mathbb{C}[\Omega_n]$
- Partition λ violating this condition is called occurrence obstruction.
- Its existence would prove $Z_{n,m} \not\subseteq \Omega_n$
- Schur's lemma also gives inequality of multiplicities:

$$\mathrm{mult}_\lambda \mathbb{C}[\Omega_n] \;\geq\; \mathrm{mult}_\lambda \mathbb{C}[Z_{n,m}]$$

- Partition λ violating this inequality is called multiplicity obstruction.

Main Result

M-S suggested the following conjecture

Occurrence Obstruction Conjecture (M-S 2001)

For all $c \in \mathbb{N}_{\geq 1}$, for infinitely many m, there exists a partition λ occurring in $\mathbb{C}[Z_{m^c,m}]$ but not in $\mathbb{C}[\Omega_{m^c}]$.

Occurrence Obstruction Conjecture implies Orbit Closure Conjecture

Unfortunately, the Occurrence Obstruction Conjecture is false!

Thm. (B, Ikenmeyer, Panova, FOCS 16, J. AMS '18)

Let n, d, m be positive integers with $n \geq m^{25}$ and $\lambda \vdash nd$. If λ occurs in $\mathbb{C}[Z_{n,m}]$, then λ also occurs in $\mathbb{C}[\Omega_n]$. In particular, the Occurrence Obstruction Conjecture is false.

Before this, [IP16] (Ikenmeyer, Panova FOCS 16) had a similar result showing that the Orbit Closure Conjecture cannot be resolved via Kronecker coefficients

No occurrence obstructions for Waring rank

- Waring rank (symmetric tensor rank) of $p \in \mathrm{Sym}^n V^*$: minimum r s.t. $p = \varphi_1^n + \ldots + \varphi_r^n$ for linear forms $\varphi_i \in V^*$
- Can prove exponential lower bound on Waring rank of $\det_n$, per_n
- May think of proving lower bounds on Waring rank by studying orbit closure

$$\mathrm{PS}_n := \overline{\mathrm{GL}_{n^2} \cdot (X_1^n + \cdots + X_{n^2}^n)} \subseteq \mathrm{Sym}^n(\mathbb{C}^{n^2})^*.$$

Corollary

Let n, d, m be positive integers with $n \geq m^{25}$ and $\lambda \vdash nd$. If λ occurs in $\mathbb{C}[Z_{n,m}]$, then λ also occurs in $\mathbb{C}[\mathrm{PS}_n]$. Moreover, the permanent can be replaced by any homogeneous polynomial p of degree m in m^2 variables.

Hence strategy of occurrence obstructions cannot even be used in weak model of PS_n against padded polynomials!

Outline and Ingredients of Proof

Kadish & Landsberg's observation

- body $\bar{\lambda}$ of λ: obtained by removing the first row of λ,

> **Kadish & Landsberg '14**
>
> If $\lambda \vdash nd$ occurs in $\mathbb{C}[Z_{n,m}]_d$, then $\ell(\lambda) \leq m^2$ and $|\bar{\lambda}| \leq md$.

- $|\bar{\lambda}| \leq md$ is equivalent to $\lambda_1 \geq (n-m)d$: λ must have a very long first row if n is substantially larger than m
- This is the only information we exploit about the orbit closure $Z_{n,m}$ of the padded permanent
- Can replace the permanent by any homogeneous polynomial p of degree m in m^2 variables
- Kadish & Landsberg also crucially used in [IP16]

Semigroup property

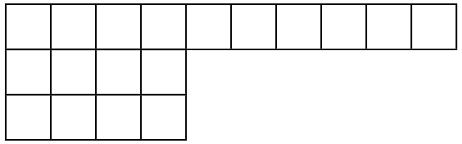
- Need to show that many partitions λ occur in $\mathbb{C}[\Omega_n]$
- For this establish the occurrence of certain basic shapes in $\mathbb{C}[\Omega_n]$
- Then get more shapes by

Semigroup Property

If λ occurs in $\mathbb{C}[\Omega_n]$ and μ occurs in $\mathbb{C}[\Omega_n]$, then $\lambda + \mu$ occurs in $\mathbb{C}[\Omega_n]$.

- Also crucially used in [IP16]

Basic building blocks

- Denote by $(k \times \ell)^{\sharp nk}$ the rectangular diagram $k \times \ell$ with k rows of length ℓ, to which a row has been appended s.t. we ge nk boxes
- $$(3 \times 4)^{\sharp 18} =$$ 

- **Prop. RER (Row Extended Rectangles)**

  Let $n \geq k\ell$ and ℓ be even. Then $(k \times \ell)^{\sharp nk}$ occurs in $\mathbb{C}[\Omega_n]_k$.

- The only property of Ω_n used in the proof is that Ω_n contains many padded power sums (follows from universality of determinant)

Prop. PPS (Padded Power Sums)

- Let $X, \varphi_1, \ldots, \varphi_k$ be linear forms on $\mathbb{C}^{n \times n}$ and assume $n \geq sk$. Then the power sum $X^{n-s}(\varphi_1^s + \cdots + \varphi_k^s)$ of k terms of degree s, padded to degree n, is contained in Ω_n.

Strategy of proof of main result

- Suppose have even $\lambda \vdash nd$ such that $n \geq m^{25}$ and λ occurs in $\mathbb{C}[Z_{n,m}]$. Want to show that λ occurs in $\mathbb{C}[\Omega_n]$.
- By [KL14] we have $\ell(\lambda) \leq m^2$ and $|\bar{\lambda}| \leq md$.
- Distinguish two cases
- CASE 1: If the degree d is large (say $d \geq 24m^6$), we proceed as in [IP16]: we decompose body $\bar{\lambda}$ into a sum of even rectangles
- Since n and d are sufficiently large in comparison with m, can write (!) λ as a sum of row extended rectangles $(k \times \ell)^{\sharp nk}$, where $n \geq k\ell$.
- By Prop. RER the row extended rectangles occur in $\mathbb{C}[\Omega_n]_k$. The semigroup property implies that λ occurs in $\mathbb{C}[\Omega_n]_d$.

Case of small degree

- CASE 2: If the degree d is small, we rely on the following crucial result. Recall $V = \mathbb{C}^{n\times n}$.

Prop. ALL

Let $\lambda \vdash nd$ be such that $|\bar{\lambda}| \leq md$ and $md^2 \leq n$ for some m.

Then every highest weight vector of weight λ in $\mathrm{Sym}^d \mathrm{Sym}^n V$, viewed as a degree d polynomial function on $\mathrm{Sym}^n V^*$, does not vanish on Ω_n.

In particular, if λ occurs in $\mathrm{Sym}^d \mathrm{Sym}^n V$, then λ occurs in $\mathbb{C}[\Omega_n]_d$.

- The proof relies on new insights on "lifting highest weight vectors" in plethysms
- This is related to known stability property of plethysms, for which we obtain new proofs
- For treating noneven partitions, need more bulding blocks (row and column extended rectangles) and more tricks

Fundamental Invariants and Lifting of Highest Weight Vectors

Highest weight vectors

- How to show that λ occurs in $\mathbb{C}[\Omega_n]$?
- $F \in \mathrm{Sym}^d \mathrm{Sym}^n \mathbb{C}^N$ called highest weight vector of weight λ if

$$\begin{pmatrix} t_1 & * & * & * \\ & t_2 & * & * \\ & & \ddots & \vdots \\ & & & t_N \end{pmatrix} \cdot F = t_1^{\lambda_1} \cdots t_N^{\lambda_N} F \qquad \text{for all } t_i \in \mathbb{C}^*$$

- F is invariant under SL_N iff λ is rectangular: $\lambda_1 = \ldots = \lambda_N$
- View F as homogeneous degree d polynomial function

$$F \colon \mathrm{Sym}^n(\mathbb{C}^N)^* \to \mathbb{C}, \quad F(p) = \langle F, p^n \rangle$$

- Essential observation:

If $F(p) \neq 0$, then λ occurs in $\mathbb{C}[\overline{\mathrm{GL}_N \cdot p}]$

Fundamental invariants

- Suppose n is even. Howe ('87) showed:
- If $d < N$, then $\mathrm{Sym}^d \mathrm{Sym}^n \mathbb{C}^N$ doesn't have a nonzero SL_N-invariant
- If $d = N$, then $\mathrm{Sym}^d \mathrm{Sym}^n \mathbb{C}^N$ has exactly one SL_N-invariant $F_{n,N}$, up to scaling, the fundamental invariant, already known to Cayley as a "hyperdeterminant"
- View $F_{n,N}$ as a homogeneous degree N polynomial map

$$F_{n,N} \colon \mathrm{Sym}^n (\mathbb{C}^N)^* \to \mathbb{C}$$

- For $p = \sum_{1 \le j_1, \dots, j_n \le N} v(j_1, \dots, j_n) X_{j_1} \cdots X_{j_n}$ with symmetric coefficients

$$F_{n,N}(p) = \sum_{\sigma_1, \dots, \sigma_n \in S_N} \mathrm{sgn}(\sigma_1) \cdots \mathrm{sgn}(\sigma_n) \prod_{i=1}^{N} v(\sigma_1(i), \dots, \sigma_n(i))$$

- For $g \in \mathrm{GL}_N$

$$F_{n,N}(g \cdot p) = \det(g)^n F_{n,N}(p)$$

- Ex. $n = 2$: $F_{2,N}(p) = N! \det(v)$ where v is symmetric matrix

Evaluating fundamental invariants

- [B, Ikenmeyer '17]: systematic investigation of fundamental invariants
- $F_{n,N}$ is a highest weight vector (weight $N \times n$)
- It is not easy to prove $F_{n,N}(p) \neq 0$
- Seemingly simple example (n even)

$$F_{n,n}(X_1 \cdots X_n) = \frac{1}{n!}\big(\#\{\text{col. even latin squares}\} - \#\{\text{col. odd latin squares}\}\big) \stackrel{?}{=} 0$$

- This is unknown: Alon-Tarsi Conjecture!
- Essential for basic building blocks: prove $F_{n,N}(X_1^n + \ldots + X_N^n) \neq 0$ by writing it as sum of squares [B, Christandl, Ikenmeyer '11]

Lifiting in plethysms

- Construct explicit injective linear lifting map for $n \geq m$

$$\kappa^d_{m,n} \colon \mathrm{Sym}^d \mathrm{Sym}^m V \to \mathrm{Sym}^d \mathrm{Sym}^n V$$

- $\kappa^d_{m,n}$ defined as d-fold symmetric power of linear map

$$M \colon \mathrm{Sym}^m V \to \mathrm{Sym}^n V,\ p \mapsto p\, e_1^{n-m}$$

  multiplication with e_1^{n-m}, 1st standard basis vector $e_1 \in V = \mathbb{C}^N$
- Use duality to show for $f \in \mathrm{Sym}^d \mathrm{Sym}^m V$, $q \in \mathrm{Sym}^n V^*$,

$$\langle \kappa^d_{m,n}(f), q^d \rangle = \langle f, M^*(q)^d \rangle$$

  Here $M^* \colon \mathrm{Sym}^n V^* \to \mathrm{Sym}^m V^*$ denotes dual map of M.
- $M^*(q)$ is $(n-m)$-fold partial derivative of q in direction e_1 (times $m!/n!$)

Highest weight vectors in plethysms

- Proved that lifting

$$\kappa^d_{m,n} \colon \mathrm{Sym}^d \mathrm{Sym}^m V \to \mathrm{Sym}^d \mathrm{Sym}^n V \,,$$

  maps highest weight vectors of weight $\mu \vdash md$ to highest weight vectors of weight $\mu^{\sharp dn}$ (μ with extended 1st row)
- Constructed system of generators v_T of space of highest weight vectors of weight μ, labelled by tableaux T of shape $\mu \vdash dm$ with d letters, each occuring m times (no letter appears more than once in a column)
- Proved: $\kappa^d_{m,n}$ maps generator v_T to generator $v_{T'}$ where T' arises from T by adding in the first row $n - m$ copies of each of the d letters
- Side result: new proof of known stability property of plethysms

Corollary on lifting

Cor. Lift

Suppose $\lambda \vdash nd$ satisfies $\lambda_2 \leq m$ and $\lambda_2 + |\bar{\lambda}| \leq md$. Then every highest weight vector of weight λ is obtained as a lifting.

Proof.

- $\lambda_2 + |\bar{\lambda}| \leq md$ is number of boxes of λ that appear in non-singleton columns
- Hence λ is obtained by extending the 1st row of some $\mu \vdash md$
- Let T' be a tableau of shape λ with d letters, each occuring m times. Since no letter appears more than once in a column, each of the d letters appears at least $n - \lambda_2 \geq n - m$ times in singleton columns. Hence T' is obtained from a tableau T of shape μ as before
- From before: $\kappa^d_{m,n}(v_T) = v_{T'}$
- Moreover, the $v_{T'}$ generate space of hwv of weight λ $\square$

Proof of Prop. ALL

Prop. ALL

$\lambda \vdash nd$ s.t. $|\bar{\lambda}| \leq md$ and $md^2 \leq n$. Then every highest weight vector of weight λ in $\mathrm{Sym}^d \mathrm{Sym}^n V$ does not vanish on Ω_n.

Proof.

- Let $h \in \mathrm{Sym}^d \mathrm{Sym}^n V$ be hwv of weight λ
- $\lambda_2 \leq |\bar{\lambda}| \leq md$ and $\lambda_2 + |\bar{\lambda}| \leq 2|\bar{\lambda}| \leq 2md \leq md \cdot d$
- Cor. Lift applied to $\mathrm{Sym}^d \mathrm{Sym}^{md} V \to \mathrm{Sym}^d \mathrm{Sym}^n V$ shows $h = \kappa^d_{md,n}(f)$ for some hwv $f \in \mathrm{Sym}^d \mathrm{Sym}^{md} V$ of weight λ
- Can show that for almost all power sums $p = \varphi_1^{md} + \cdots + \varphi_d^{md}$ we have $\langle f, p^d \rangle \neq 0$ and with $q := X_1^{n-md} p$,
$$\langle f, M^*(q)^d \rangle \neq 0$$
- Using duality
$$\langle h, q^d \rangle = \langle \kappa^d_{m,n}(f), q^d \rangle = \langle f, M^*(q)^d \rangle \neq 0.$$
By Prop. PPS, we have $q \in \Omega_n$ since $n \geq md \cdot d$. $\square$

Thank you for your attention!