The GCT Program: Recent developments and concrete open problems

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The University of Chicago

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The main reference for this talk

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[GCT5][M.]: Geometric Complexity Theory V: Efficient algorithms for Noether Normalization.

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- This difficulty has to be overcome by any approach to the VP vs. VNP problem that seeks to separate VNP from $\overline{\mathrm{VP}}$. We call any such approach a GCT approach in a broad sense.

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- $\Delta[det, m]$ also equals the $GL_{m^2}(K)$-orbit-closure of $\det(Y) \in \mathbb{P}(V)$ under the natural action of $GL_{m^2}(K)$ on $\mathbb{P}(V)$.

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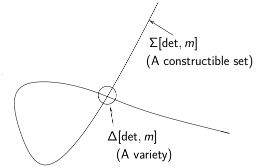

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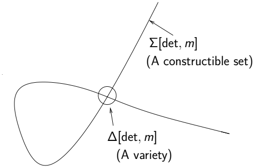

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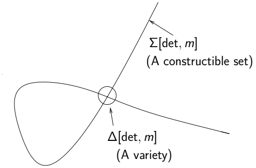

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- Unfortunately, the singularities of $\Delta[\det, m]$ are not even normal [Kumar]. This is the beginning of difficulties [Next].

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- [GCT6]: The $\mathrm{VP}_{\mathrm{ws}}$ vs. $\overline{\mathrm{VP}_{\mathrm{ws}}}$ problem is related to foundational issues in algebraic geometry and representation theory.
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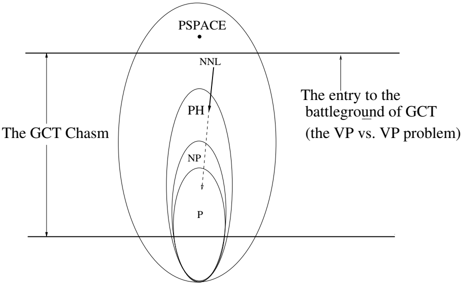

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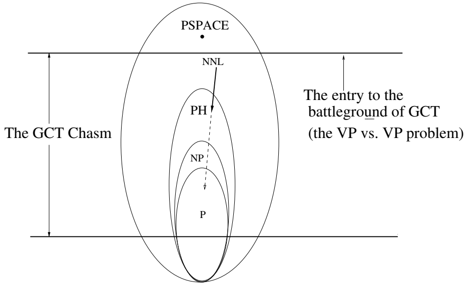

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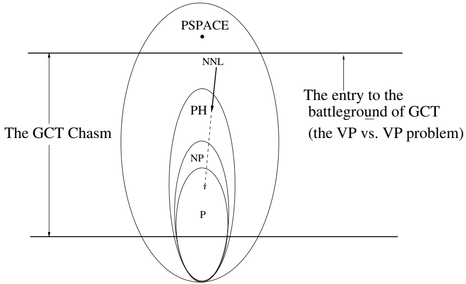

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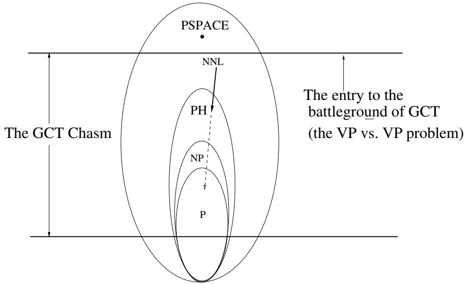

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- GCT5,GCT6, and GCT7 provide a concrete GCT program to cross the GCT chasm.

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The Orbit-Closure-Intersection Hypothesis

Hypothesis (GCT5: M2012; M2017)

The orbit-closure intersection problem is in P, for any finite dimensional representation V of a reductive group G (possibly disconnected).

Expected to be an inherent difficulty underneath white-box PIT.

The status of the hypothesis:

- Holds if G is connected and $\dim(G)$ is constant [GCT5].
- Holds if $V = M_m(K)^n$, with the adjoint action of $G = SL_m(K)$ [GCT5 + Forbes and Shpilka][2012].
- Holds if $V = M_m(K)^n$, with the left-right action of $G = SL_m(K) \times SL_m(K)$ [GGOW; DM; IQS][2016].
  A concrete application of GCT: This special case of the GCT hypothesis above implies a polynomial time algorithm for non-commutative rational identity testing.
- Holds if $V = K^{\binom{n}{2}}$ with the natural action of S_n (Weighted Graph Isomorphism): [Babai][2017].

A special case of the hypothesis for finite groups

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- This is why the techniques such as operator scaling and optimization are unlikely to work for white-box PIT.

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Thank you.