Sparse Polynomial Factorization: Structural and Algorithmic results

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Sparsity

Given, $f \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ and $\forall\, i,\ deg_{x_i} f \leq d$.

- Sparsity of f (denoted by $\|f\|$) $:=$ number of monomials in f(with non-zero coeff.).
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- Sparsity is natural complexity measure and was studied in [GK85, KS01, Zip79, SW05, SSS13 and ....*many more*].

Are factors of sparse polynomials sparse?

Example (von zur Gathen-Kaltofen'85)

Let

$$f(x) = \prod_{i=1}^{n} (x_i^d - 1),$$
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Conjecture [von zur Gathen-Kaltofen'85]

Whether a quasi-polynomial bound holds for the sparsity of factors of sparse polynomials?

Are factors of sparse polynomials sparse?

Example (Volkovich'15, Dvir-Oliveira'15)

Let $f \in \mathbb{F}_p[x_1, \dots x_n]$, p-prime and let $0 < d < p$.

$$f(x) = x_1^p + x_2^p + \ldots + x_n^p,$$
$$g(x) = \left(x_1 + x_2 + \ldots x_n\right)^d$$

$\|f\| = n$ and $\|g\| = \binom{n+d-1}{d} \approx n^d \implies \|g\| = \|f\|^d$.

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- Constant depth circuits [Oliviera’15] (small individual degree)
- VNP [DSS’18, CKS’18]
- ABP/Formulas [DSS’18, CKS’18] (quasi-polynomial blowup)

Our Result

Factor Sparsity Bound

Let $\mathbb{F}$ be an arbitrary field and let $f \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ be a polynomial of sparsity s and individual degrees at most d, then the sparsity of every factor of f is bounded by $s^{\mathcal{O}(d^2 \log n)}$.

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Remark: Our bound is *field oblivious* and thus "somewhat tight".

Proof of Sparsity Bound

Suppose that $f, g, h \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ such that

$$f = g \cdot h.$$

Want to show, f is s-sparse and with bounded individual degree d, then g and h are both at most s' sparse, where $s' = s^{\mathcal{O}(d^2 \log n)}$.

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We instead show the following slightly more general result.

Suppose that g is any polynomial of individual degree d such that $\|g\| = s$, and suppose that $f = g \cdot h$ *(with no assumptions on the degrees of f and h)*, then

$$\|f\| \geq s^{\frac{1}{\mathcal{O}(d^2 \log n)}}.$$

In particular, there is no polynomial h that one can multiply g with, so that the product $g \cdot h$ has an overwhelming cancellation of monomials.

Newton Polytopes

Let,

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Consider the set,

$$\mathrm{Supp}(f) = \{(i_1, i_2, \dots, i_n) \mid a_{i_1 i_2 \dots i_n} \neq 0\} \subseteq \mathbb{R}^n$$

of exponent vectors of f.

One can then associate a polytope $P_f \subseteq \mathbb{R}^n$, called the Newton polytope of f, which is the convex hull of points in $\mathrm{Supp}(f)$.

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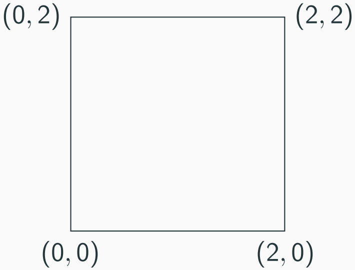

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Notice,

$$\|f\| \geq |V(P_f)| = |V(P_g + P_h)| \geq \max\{|V(P_g)|, |V(P_h)|\}.$$

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arbitrary nonzero polynomial

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$$\|f\| \geq |V(P_g)| \longrightarrow f = g \cdot h$$

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Showing a lower bound on $|V(P_g)|$ will be the main technical core of our proof of the sparsity bound.

This connection between Newton polytopes and sparsity bounds was first made in [Dvir-Oliveira'14] and indeed it inspired the approach taken in this paper.

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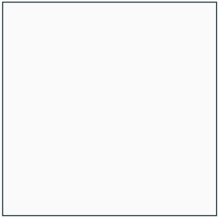

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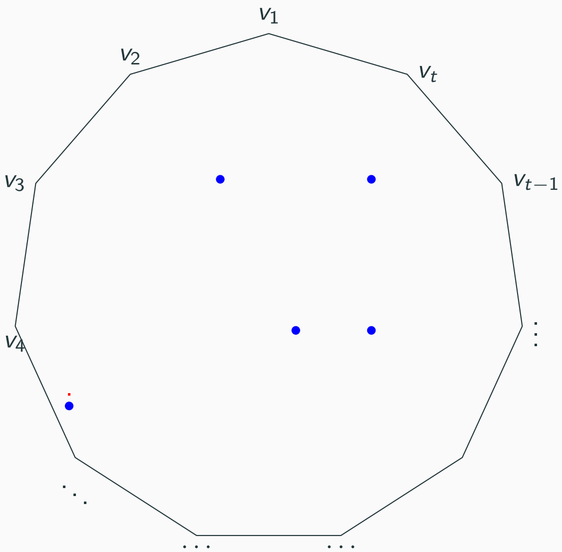
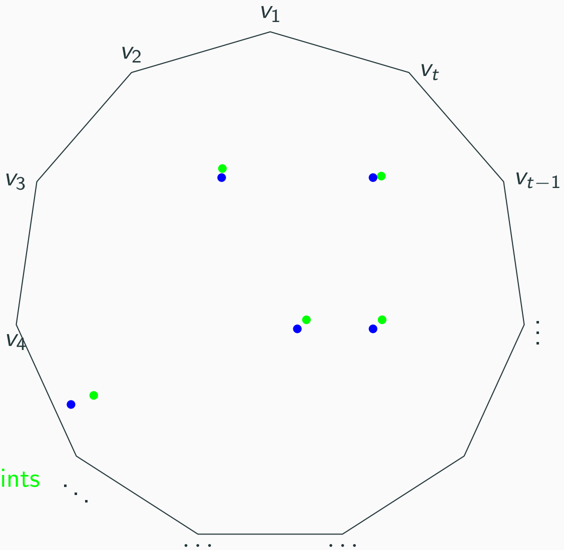


• lattice points



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Approx. Carathéodory's Theorem

Theorem (Barman'15)

Given a set of vectors $V = \{v_1, v_2, \ldots, v_t\} \subseteq \mathbb{R}^n$ *with* $\max_{v \in U} \|u\|_\infty \leq d$, *and* $\epsilon > 0$. *For every* $\mu \in CS(U)$ *there exists an* $\mathcal{O}\left(\frac{d^2 \log n}{\epsilon^2}\right)$ *uniform vector* $\mu' \in CS(U)$ *such that* $\|\mu - \mu'\|_\infty \leq \epsilon$.

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- each lattice point can be "associated" by atleast one $\mathcal{O}\left(d^2 \log n\right)$-uniform vector.
- $\#\left(d^2 \log n\right)$-uniform vectors $\approx t^{\left(d^2 \log n\right)}$

This proves our Sparsity Bound. $\square$

Deterministic Factoring Algorithm

Polynomial factorization

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- Either give a factorization

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Applications: list decoding [Sud97, GS99], derandomization [KI04] and cryptography [CR88].

Representing a Polynomial

- **White-box Arithmetic Circuit**
- **Black-box Arithmetic Circuit**

A brief history on Polynomial factorization

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All of them are Randomized Algorithms

A natural *Algorithmic question*, Can we derandomize this?

Equivalence of PIT and Polynomial factorization

Polynomial Identity Testing(PIT)

Given $f \in \mathcal{C}$, determine whether $f \equiv 0$.

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What about models we already knew PIT about? In particular, Sparse polynomials [Klivans-Spielman'01].

Vague overview of [KSS'14]

All mentioned [von zur Gathen-Kaltofen'85, Kaltofen'87, Kaltofen'89, Kaltofen-Trager'90] need randomness at multiple stages.

Randomized Factoring Algorithm:-

1. Step 1 requires randomness r_1
2. Step 2 requires randomness r_2
3. Step 3 requires randomness r_3

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[KSS'14] Factoring Algorithm:-

1. Step 1 requires ~~randomness r_1~~ PIT for polynomial $p_1(x)$
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Our Results

Main Theorem

There exists a deterministic algorithm that given a polynomial $f \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ of sparsity s and individual degrees at most d, computes the complete factorization of f, using $s^{\mathcal{O}(d^7 \log n)} \cdot \mathrm{poly}(d, |\mathbb{F}|)$ field operations.

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Remark: If one could improve the sparsity bound from *quasi-polynomial* to *polynomial* then this will directly improve the run time of our deterministic factoring algorithm.

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Generic Factoring Algorithm

- Preprocessing
- Hilbert Irreducibility Theorem
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Our Approach

- Preprocessing
- "Brute force" for Hilbert Irreducibility
- Reconstructing the factors
- Test the factorization.

- For simplicity (and WLOG.),

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- Notice, factorization of f looks like

$$f = (x_1^{e_1} + g_1) \cdots (x_1^{e_k} + g_k)$$

- Consequently, f has at most d factors (total).

Projection to few variables

Randomized World

$$
\begin{array}{l}
f = g_1 g_2 \cdots g_5 \in \mathbb{F}[x_1, x_2, \ldots, x_n] \\
\quad \Big\downarrow \text{Hilbert Irrreducibility} \\
\quad \tilde{f} = \tilde{g_1} \tilde{g_2} \cdots \tilde{g_5} \in \mathbb{F}[x_1, y]
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A. We don't need to! We can test our factorization.

Black-box trick [Kaltofen-Trager'90]

How can one evaluate the factors?
Suppose you want to evaluate your factor at $(\alpha, \bar{\beta})$. Notice,

$$h(x, t) := f(x, \bar{b} + (\bar{\beta} - \bar{b})t)$$

Factorize $h(x, t)$ and substitute $x = \alpha, t = 1$.

Lemma (Klivans-Spielman'01)

Given an oracle access to an s-sparse polynomial $f \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ of degree d, we can deterministically reconstruct f in $\mathrm{poly}(n, s, d, \log|\mathbb{F}|)$ time.

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Remaining steps:-

- Reconstruct the factors.
- Test the factorization.

This concludes our factoring algorithm.

Summary

Theorem (Sparsity Bound)

Let $\mathbb{F}$ be an arbitrary field and let $f \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ be a polynomial of sparsity s and individual degrees at most d, then the sparsity of every factor of f is bounded by $s^{\mathcal{O}(d^2 \log n)}$.

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Theorem (Deterministic factoring Algorithm)

Given a polynomial $f \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ of sparsity s and individual degrees at most d, we can computes the factorization of f, using $s^{\mathcal{O}(d^7 \log n)} \cdot \mathrm{poly}(d, |\mathbb{F}|)$ field operations deterministically.

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3. Are ROABPs with bounded individual degree closed under factoring?

Thank you.