Invariant theory- a gentle introduction for computer scientists (optimization and complexity)

Avi Wigderson, IAS, Princeton

Ankit Garg, Rafael Oliveira

Zeyuan Allen-Zhu, Peter Burgisser, Cole Franks, Leonid Gurvits, Yuanzhi Li, Michael Walter

Prehistory

Linial, Samorodnitsky, W 2000   Cool algorithm
Discovered many times before

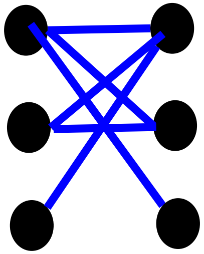
Kruithof 1937 in telephone forecasting,
Deming-Stephan 1940 in transportation science,
Brown 1959 in engineering,
Wilkinson 1959 in numerical analysis,
Friedlander 1961, Sinkhorn 1964 in statistics.
Stone 1964 in economics,

Matrix Scaling algorithm

A non-negative matrix. Try making it doubly stochastic.

Alternating scaling rows and columns

1	1	1
1	1	0
1	0	0

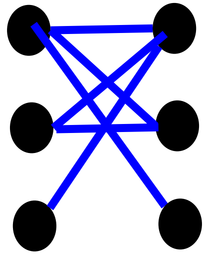


Matrix Scaling algorithm

A non-negative matrix. Try making it doubly stochastic.

Alternating scaling rows and columns

1/3	1/3	1/3
1/2	1/2	0
1	0	0

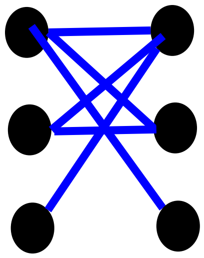


Matrix Scaling algorithm

A non-negative matrix. Try making it doubly stochastic.

Alternating scaling rows and columns

2/11	2/5	1
3/11	3/5	0
6/11	0	0

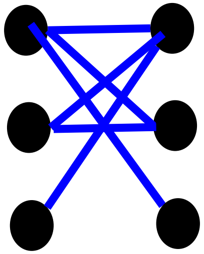


Matrix Scaling algorithm

A non-negative matrix. Try making it doubly stochastic.

Alternating scaling rows and columns

10/87	22/87	55/87
15/48	33/48	0
1	0	0



Matrix Scaling algorithm

A non-negative matrix. Try making it doubly stochastic.

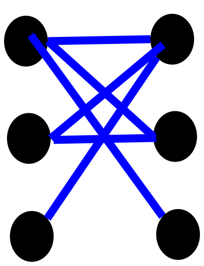
Alternating scaling rows and columns

A very different efficient Perfect matching algorithm

We'll understand it much better with Invariant Theory

Converges (fast) iff Per(A) >0

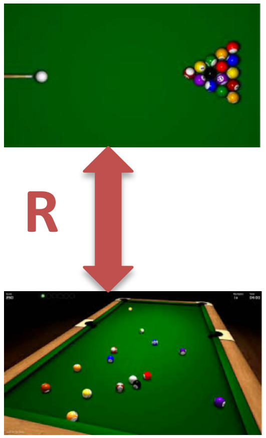
0	0	1
0	1	0
1	0	0



Outline

Main motivations, questions, results, structure

- Algebraic Invariant theory
- Geometric invariant theory
- Optimization & Duality
- Moment polytopes
- Algorithms
- Conclusions & Open problems



- Energy
- Momentum

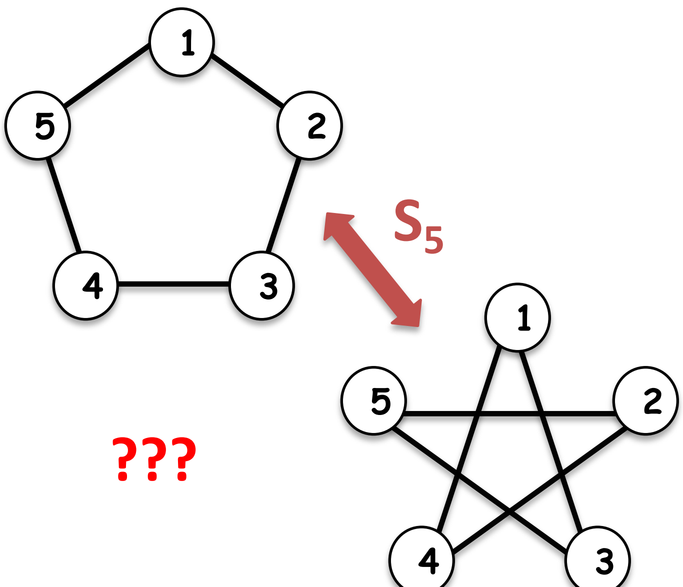
$i, s := 0, 0;$
do $i \neq n \rightarrow$
$\quad i, s := i + 1, s + b[i]$
od

Precondition: $n \geq 0$
Postcondition: $s = (\sum j : 0 \leq j < n : b[j])$

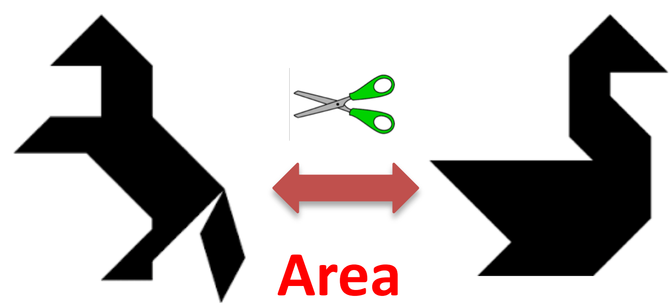
Invariant Theory

symmetries, group actions, orbits, invariants

Physics, Math, CS



???



Linear actions of groups

Group G acts *linearly* on vector space V $(= \mathbf{F}^d)$. [$\mathbf{F} = \mathbf{C}$]

Action: Matrix-Vector multiplication

G reductive.
$g \to M_g$ rational.

$M: G \to GL(V)$ $(d \times d$ matrices$)$ group homomorphism.

$M_g: V \to V$ invertible linear map $\forall\, g \in G$.

$M_{g_1 g_2} = M_{g_1} M_{g_2}$ and $M_{id} = id$.

Ex 1 $G = S_n$ acts on $V = \mathbf{C}^n$ by *permuting coordinates*.

$$M_\sigma\,(x_1, \dots, x_n) = \left(x_{\sigma(1)}, \dots, x_{\sigma(n)}\right).$$

Ex 2 $G = GL_n(\mathbf{C})$ acts on $V = M_n(\mathbf{C})$ by *conjugation*.

$M_A\, X = AXA^{-1}$ $(d = n^2 \gg$ n variables$)$.

Objects of study

Group G acts *linearly* on vector space $V = \boldsymbol{C}^d$), and also on polynomials $\boldsymbol{C}[V] = \boldsymbol{C}[x_1, \ldots, x_d]$

- Invariant polynomials: under action of G:
  p s.t. $p(M_g v) = p(v)$ for all $g \in G$, $v \in V$.

- Orbits: Orbit of vector v, $O_v =. \{M_g v : g \in G\}$

- Orbit-closures: An orbit O_v may not be closed.
  Take its closure in Euclidean topology.
  $\overline{O_v} = \mathrm{cl}\,\{M_g v : g \in G\}$.

Example 1

$G = S_n$ acts on $V = \boldsymbol{C}^n$ by permuting coordinates.

$$M_\sigma\ (x_1, \ldots, x_n) \rightarrow \left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right).$$

- Invariants: symmetric polynomials.
- Orbits: x, y in same orbit iff they are of *same type*.
  $\forall c \in \boldsymbol{C}, |\{i: x_i = c\}| = |\{i: y_i = c\}|$.
- Orbit-closures: same as orbits.

Example 2

$G = GL_n(\boldsymbol{C})$ acts on $V = M_n(\boldsymbol{C}) = \boldsymbol{C}^{n^2}$ by *conjugation*.

$$M_A\, X = AXA^{-1}.$$

- Invariants: trace of powers: $\mathrm{tr}(X^i)$.
- Orbits: Characterized by *Jordan normal form*.
- Orbit-closures: differ from orbits.

1. $\overline{O_X} \neq O_X$ iff X is *not diagonalizable*.

2. $\overline{O_X}$ and $\overline{O_Y}$ intersect iff X, Y have the *same eigenvalues*.

Orbits and orbit-closures in TCS

- *Graph isomorphism*: Whether orbits of two graphs the same. Group action: permuting the vertices.
- *Border rank*: Whether a tensor lies in the orbit-closure of the diagonal unit 3-tensor. [Special case: Matrix Multi exponent] Group action: Natural action of $GL_n(\boldsymbol{C}) \times GL_n(\boldsymbol{C}) \times GL_n(\boldsymbol{C})$.
- *PIT* Does an $n \times n$ symbolic determinant on m variables vanish? Group action: Natural action of $GL_n(\boldsymbol{C}) \times GL_n(\boldsymbol{C}) \times GL_m(\boldsymbol{C})$.
- *Property testing*: Graphs – Group action: Symmetric group,

  Codes - Group action: Affine group
- *Arithmetic circuits*: The VP vs VNP question via GCT program: Whether permanent lies in the orbit-closure of the determinant. Group action = **Reductions** : Action on polynomials induced by linear transformation on variables.

Invariant ring

Group G acts *linearly* on vector space V.

$\boldsymbol{C}[V]^G$: ring of invariant polynomials.

[Hilbert 1890, 93]: $\boldsymbol{C}[V]^G$ is *finitely generated* !

Nullstellansatz, Finite Basis Theorem etc. proved in these papers as "lemmas"! Also origin of *Grobner Basis Algorithm*

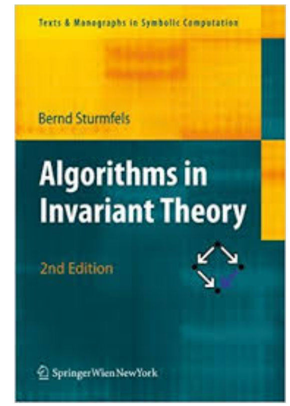
1. $G = S_n$ acts on $V = \boldsymbol{C}^n$ by permuting coordinates.
   $\boldsymbol{C}[V]^G$ generated by *elementary symmetric* polynomials.
2. $G = GL_n(\boldsymbol{C})$ acts on $V = M_n(\boldsymbol{C})$ by conjugation.
   $\boldsymbol{C}[V]^G$ generated by $\mathrm{tr}(X^i)$, $1 \le i \le n$.

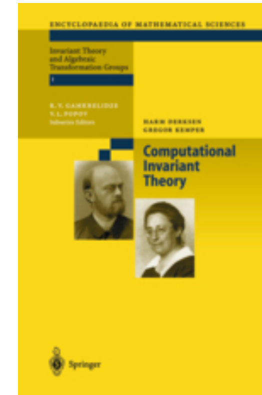
Degree bound $\le$ n

[Derksen 2000]: $\boldsymbol{C}[V]^G$ is *generated* by degree $\exp(n)$

Computational invariant theory

Highly *algorithmic field*.
Algorithms sought and well developed.
Polynomial eq sys solving, ideal bases,
comp algebra, FFT, MM via groups,…



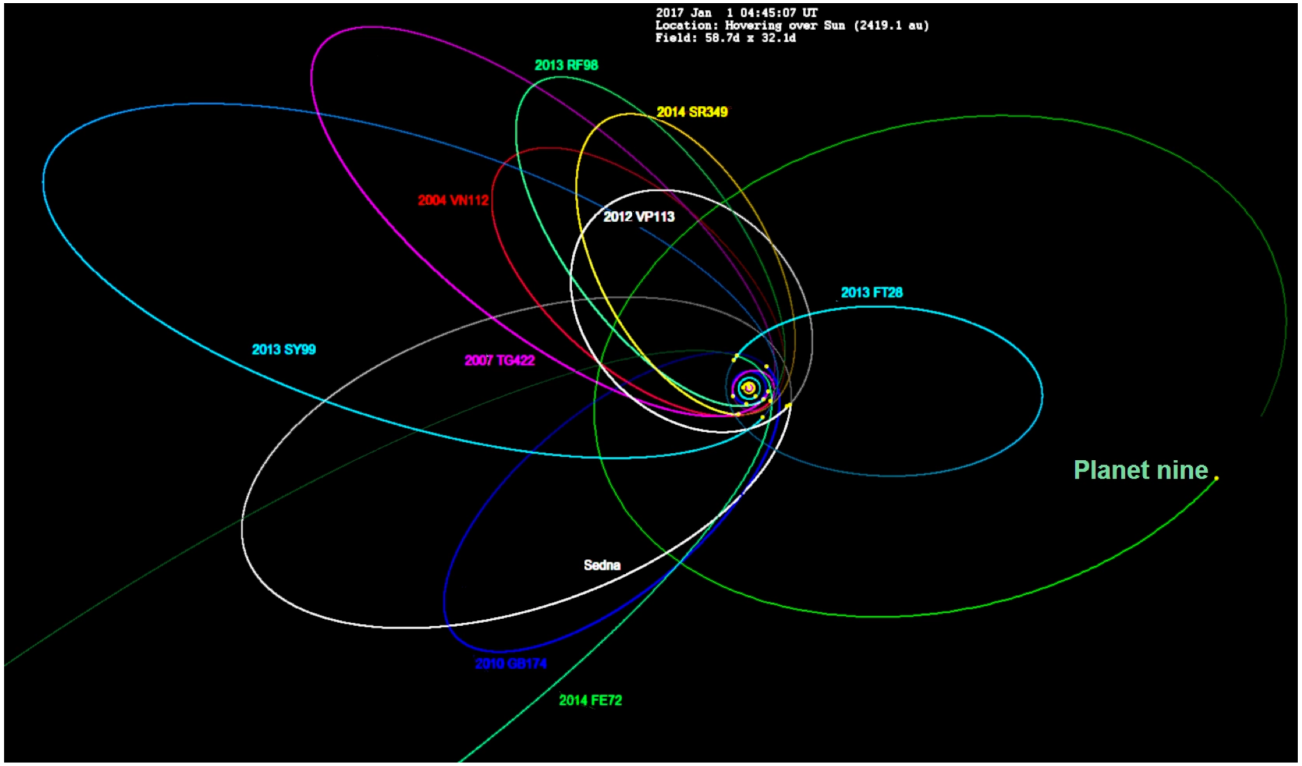


Main problems:

- Describe all invariants (*generators, relations*).
- Simpler: *degree bounds* for generating set.
- *Isomorphism*/*Word problem:* When are two objects the "same"?

1. Orbit intersection.
2. Orbit-closure intersection.
3. Noether normalization, Mulmuley's GCT5,…
4. Orbit-closure containment.
5. Simpler: null cone. When is an object ``like" 0? Is $0 \in \overline{O_v}$?

Geometric invariant theory (GIT)



Null cone

Captures many interesting questions.

Group G acts *linearly* on vector space V.

Null cone: Vectors v s.t. 0 lies in the orbit-closure of v.

$$N_G(V) = \{v: \ 0 \in \overline{O_v}\}.$$

Sequence of group elements $g_1, \ldots, g_k, \ldots$ s.t. $\lim_{k\to\infty} M_{g_k} v = 0$.

Problem: Given $v \in V$, decide if it is in the null cone.

Optimization/Analytic: Is $\inf_{g\in G} ||M_g v|| = 0$?

Algebraic:[Hilbert 1893; Mumford 1965]: v in null cone iff

$p(v) = 0$ for *all* homogeneous invariant polynomials p.

- One direction clear (polynomials are continuous).
- Other direction uses *Nullstellansatz* and algebraic geometry.

analytic ↔ algebraic

optimization ↔ complexity

Example 1

$G = S_n$ acts on $V = \boldsymbol{C}^n$ by permuting coordinates.

$$M_\sigma(x_1, \dots, x_n) \to \left(x_{\sigma(1)}, \dots, x_{\sigma(n)}\right).$$

Null cone = $\{0\}$.

No closures (same for all finite group actions).

Example 2

$G = GL_n(\boldsymbol{C})$ acts on $V = M_n(\boldsymbol{C})$ by *conjugation*.

$$M_A\, X = AXA^{-1}.$$

- Invariants: generated by $\mathrm{tr}(X^i)$.

- Null cone: nilpotent matrices.

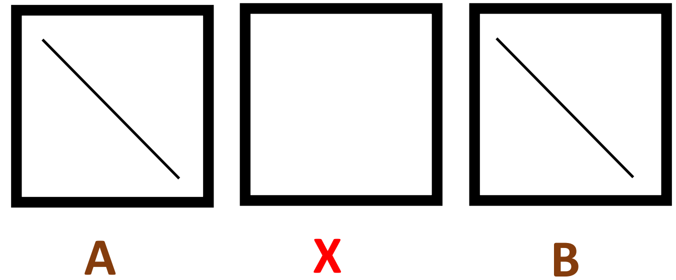
Example 3

$G = SL_n(\boldsymbol{C}) \times SL_n(\boldsymbol{C})$ acts on $V = M_n(\boldsymbol{C})$ by left-right multiplication.

$$M_{(A,B)}\, X = AXB.$$

- Invariants: generated by $\mathrm{Det}(X)$.
- Null cone: Singular matrices.

Example 4: Matrix Scaling



ST_n: group of $n \times n$ diagonal matrices with determinant 1.

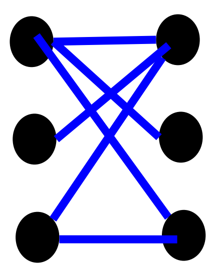
$G = ST_n \times ST_n$ acts on $V = M_n(\mathbb{C})$ by left-right multiplication.

$$M_{(A,B)}\, X = AXB.$$

- Invariants: generated by matchings $X_{1,\sigma(1)} X_{2,\sigma(2)} \cdots X_{n,\sigma(n)}$.
- Null cone $\leftrightarrow$ Per(X)=0
- A_H is in null cone $\leftrightarrow$ H has no perfect matching.

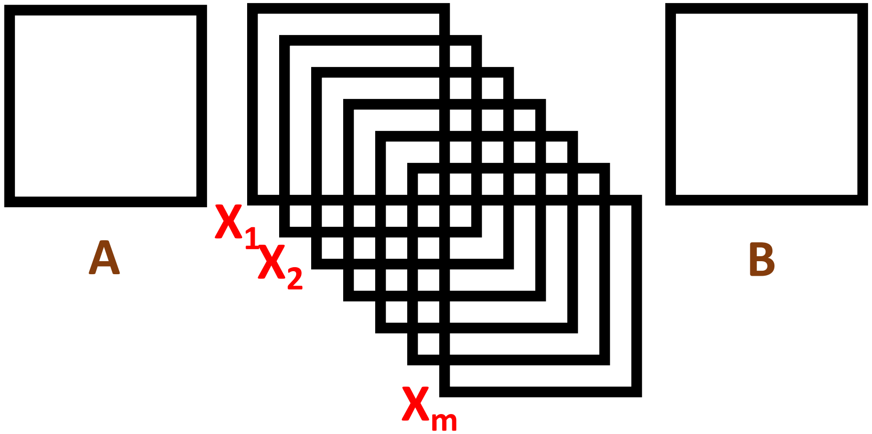
1	1	1
1	0	0
1	0	1

A_H



H

Example 5: Operator Scaling



$G = SL_n(\boldsymbol{C}) \times SL_n(\boldsymbol{C})$ acts on $V = M_n(\boldsymbol{C})^{\oplus m}$

by *simultaneous left-right* multiplication.

$$M_{(A,B)}(X_1, \dots, X_m) = (AX_1B, \dots, AX_mB).$$

PIT problem

- Invariants [DW 00, DZ 01, SdB 01]: generated by $\mathrm{Det}(\sum_i D_i \otimes X_i)$.
- Null cone ↔ Non-commutative singularity of symbolic matrices.

↔ Non-commutative rational identity testing ↔ …

[GGOW 16, IQS 16]: Deterministic polynomial time algorithms.

[DM 16]: Polynomial degree bounds on generators.

Example 6: Linear programming

$G = T_n$: (Abelian!) group of $n \times n$ diagonal matrices.

V: Laurent polynomials. $q \in V$ (poly w/some exponents negative).

G acts on V by scaling variables. $\boldsymbol{t} \in T_n$, $\boldsymbol{t} = \text{diag}(t_1, \ldots, t_n)$.

$$M_{\boldsymbol{t}}\, q(x_1, \ldots, x_n) = q(t_1 x_1, \ldots, t_n x_n).$$

$q = \sum_{\alpha \in \Omega} c_\alpha x^\alpha$. $\text{supp}\,(q) = \{\alpha \in \Omega : c_\alpha \neq 0\}$.

Null cone ↔ Linear Programming

q *not in* null cone ↔ $0 \in$ *conv*$\{\text{supp}(q)\}$. (=Newton polytope (q))

In non-Abelian groups, the null cone (membership) problem is a *non-commutative* analogue of Linear Programming.

GIT: computational perspective

What is *complexity* of *null cone* membership?
GIT puts it in $NP \cap coNP$ (morally).

- Hilbert-Mumford criterion:
  how to certify membership in null cone.
- Kempf-Ness theorem:
  how to certify non-membership in null cone.

Many mathematical characterizations have this flavor.
Begs for complexity theoretic quantification
(e.g proof complexity approach to Nullstellensatz, Positivstellensatz…)

Hilbert-Mumford

Group G acts linearly on vector space V.

How to *certify* $v \in N_G(V)$ (null cone)?

Sequence of group elements $g_1, \ldots, g_k, \ldots$

such that $\lim_{k \to \infty} M_{g_k} v = 0$.

Compact description of the sequence?

Given by *one-parameter subgroups*.

[Hilbert 1893; Mumford 1965]: $v \in N_G(V)$ iff $\exists$ one-parameter subgroup $\lambda: \boldsymbol{C}^* \to G$ s.t. $\lim_{t \to 0} M_{\lambda(t)} v = 0$.

One-parameter subgroups

One-parameter subgroup: *Group homomorphism* $\lambda: \boldsymbol{C}^* \to G$.
Also this map is *algebraic*.

- $G = \boldsymbol{C}^*$: $\lambda(t) = t^a, \quad a \in \boldsymbol{Z}$.
- $G = T_n = (\boldsymbol{C}^*)^{\times n}$: $\lambda(t) = \mathrm{diag}(t^{a_1}, \dots, t^{a_n}), \quad a_i \in \boldsymbol{Z}$.
- $G = ST_n$: $\lambda(t) = \mathrm{diag}(t^{a_1}, \dots, t^{a_n}), \quad a_i \in \boldsymbol{Z}, \ \sum_i a_i = 0$.
- $G = GL_n$: $\lambda(t) = S\,\mathrm{diag}(t^{a_1}, \dots, t^{a_n})S^{-1}, S \in GL_n, a_i \in \boldsymbol{Z}$.

  (Abelian, up to a basis change S)

Example: Matrix Scaling & Perfect Matching

$G = ST_n \times ST_n$ (ST_n: $n \times n$ diagonal matrices with det 1)

acts on $V = M_n$ (X an $n \times n$ matrix)

$$M_{(A,B)} X = AXB.$$

X in *null cone* $\Leftrightarrow \exists\, a_1, \dots, a_n, b_1, \dots, b_n \in \boldsymbol{Z}$:

$$\sum_i a_i = \sum_j b_j = 0$$

$$\text{s.t. } a_i + b_j > 0 \qquad \forall\, (i,j) \in \text{supp}(X).$$

$\Leftrightarrow \text{Supp}(X)$ has no perfect matching (Hall's theorem)

$\text{Supp}(X) = \{(i,j) \in [n] \times [n]: X_{i,j} \neq 0\}$ (adjacency matrix of X)

1-parameter subgroups: $\lambda(t) = \left((t^{a_1}, \dots, t^{a_n}), (t^{b_1}, \dots, t^{b_n}) \right)$

$$a_i, b_j \in \boldsymbol{Z}: \quad \sum_i a_i = \sum_j b_j = 0.$$

$\lambda(t)$ sends X to 0 $\Leftrightarrow$ $a_i + b_j > 0 \quad \forall\, (i,j) \in \text{supp}(X)$

Kempf-Ness

Group G acts linearly on vector space V.

How to *certify* v is not in null cone?

Algebraic: Exhibit *invariant* polynomial p s.t. $p(v) \neq 0$.

Typically doubly exponential time...

Invariants hard to find, high degree, high complexity etc.

Analytic: Kempf-Ness provides a more efficient way.

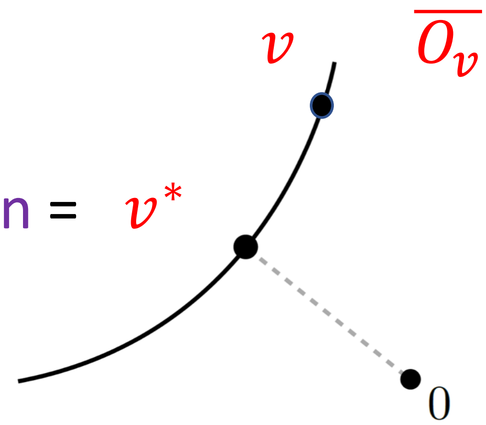
An optimization perspective (+ duality!)

Finding *minimal norm* elements in orbit-closures!

Group G acts linearly on vector space V.

$\mathrm{cap}(v) = \inf_{g \in G} \|M_g\, v\|_2^2$.

argmin = v^*



$\mathrm{cap}(v) = 0 \;\leftrightarrow\; v \in$ Null cone

$\mathrm{cap}(v) > 0 \;\leftrightarrow\; v \notin$ Null cone

$\leftrightarrow\; \mu_G(v^*) = 0$ $\quad \mu_G$ moment map (gradient)

$\leftrightarrow\; v$ can be ``scaled”

Minimizing μ_G is a dual optimization problem.

Moment map

Group G acts linearly on vector space V.

Moment map $\mu_G(v)$: gradient of $\|M_g\, v\|_2^2$ at $g = id$.

How much *norm* of v decreases by *infinitesimal action* near id.

$\mu_G(v)$: a linear function (like the familiar gradient),

on a linear space called the Lie algebra of the group G.

μ_G can be defined in more general contexts.

Moment → *momentum*.

Fundamental in *symplectic geometry* and *physics*.

Minimizing $\mu_G(v)$=0 (finding $\mu_G(v^*)$=0) is a scaling problem!

Example 1: Matrix Scaling

$G = ST_n \times ST_n$ acts on $V = M_n$. $\quad M_{(A,B)}\, X = AXB$.

Consider only $\quad w: \sum_j w(j) = 0$

$A(s) = \text{diag}\exp(s\, q_1)$, $\quad B(s) = \text{diag}\exp(s\, q_2)$

Directional derivative: action of $\big(A(s), B(s)\big)$ on X, $s \approx 0$.

$\mu_G(X) = (p_1, p_2)$, $\quad \sum_i p_1(i) = \sum_j p_2(j) = 0$ s.t.

$$\langle p_1, q_1 \rangle + \langle p_2, q_2 \rangle = \frac{d}{ds}\left[\left\| M_{(A,B)}\, X \right\|_F^2\right]\Big|_{s=0}$$

$$= \langle r_X, q_1 \rangle + \langle c_X, q_2 \rangle$$

$$= \langle r_X - \alpha \mathbf{1}, q_1 \rangle + \langle c_X - \alpha \mathbf{1}, q_2 \rangle$$

$\mu_G(X) = (r_X - \alpha \mathbf{1}, c_X - \alpha \mathbf{1})$, $\quad$ ($\alpha = \langle r_X, \mathbf{1} \rangle = \langle c_X, \mathbf{1} \rangle$)

r_X, c_X vectors of *row* and *column* ℓ_2^2 norms of X.

Scaling = Minimizing $\mu_G(X)$ = DS G-scaling Y of the matrix X.

Example 2: Scaling polynomials

T_n: (Abelian!) group of $n\times n$ diagonal matrices.

V: Laurent polynomials (with negative exponents).

G acts on V by scaling variables. $\boldsymbol{t} \in T_n$, $\boldsymbol{t} = \mathrm{diag}(t_1, \dots, t_n)$.

$$M_{\boldsymbol{t}}\, q(x_1, \dots, x_n) = q(t_1 x_1, \dots, t_n x_n).$$

$T(s) = \mathrm{diag}\, \exp(sw)$, $w \in \boldsymbol{R}^n$

Directional derivative: action of $T(s)$ on q, $s \approx 0$.

$\mu_G(q) = u$, $u \in \boldsymbol{R}^n$ s.t.

$$\langle u, w\rangle = \frac{d}{ds}\left[\left\|M_{T(s)}q\right\|_2^2\right]\Big|_{s=0} = \langle \mathrm{grad}\, \hat{q}(\mathbf{1}), w\rangle$$

$\mu_G(q) = \mathrm{grad}\, \hat{q}(\mathbf{1})$ (the usual gradient)

$q = \sum_{\alpha\in\Omega} c_\alpha x^\alpha$ $\hat{q} = \sum_{\alpha\in\Omega} |c_\alpha|^2 x^\alpha$

Scaling = Minimizing $\mu_G(q)$ = finding extrema of $\hat{q}$

Kempf-Ness

Group G acts linearly on vector space V.

[Kempf, Ness 79]: v not in null cone iff there exist a *non-zero* w in *orbit-closure* of v s.t. $\mu_G(w) = 0$.

w *certifies* v not in null cone.

Easy direction.

- v not in null cone. Take w vector of *minimal norm* in the orbit-closure of v. w non-zero.
- w minimal norm in its orbit. ⇒ Norm does not decrease by infinitesimal action around id. ⇒ $\mu_G(w) = 0$.
- *global* minimum ⇒ *local* minimum.

Kempf-Ness

Hard direction: *local* minimum ⇒ *global* minimum.
Some *"convexity"*.

- *Commutative* group actions – *Euclidean convexity* . (change of variables) [exercise].
- *Non-commutative* group actions: *geodesic convexity*.

Example: Matrix Scaling

$G = ST_n \times ST_n$ acts on $V = M_n$.

$$M_{(A,B)}\, X = AXB.$$

[Hilbert-Mumford]: X in null cone iff *bipartite* graph defined by $\mathrm{supp}(X)$ does not have a *perfect matching*.

[Kempf-Ness]: X not in null cone $\Leftrightarrow$

$\Leftrightarrow$ non-zero Y in orbit-closure s.t. $\mu_G(Y) = 0$ $\Leftrightarrow$

$\Leftrightarrow$ X is *scalable* to ``*Doubly Stochastic*''

Why only *DS*?

Matrix scaling theorem [Rothblum, Schneider 89].

Moment polytopes

Moment polytopes

Group G acts linearly on vector space V.

Δ= {all gradients} = $\{ \mu_G(w) : w \in V \}$

Δ_v= {all gradients in the orbit closure of v} = $\{ \mu_G(w) : w \in \overline{O_v} \}$

[Atiyah, Hilbert, Mumford]: **All "such" are convex polytopes**

(μ_G needs to be normalized, standardized)

Uniform Scaling: Given v, does $0 \in \Delta_v$? (null cone problem)

Non-uniform Scaling: Given $v \in V$, r, does $r \in \Delta_v$?

We have algorithms!

Polyhedral combinatorics!!

Non-uniform matrix scaling

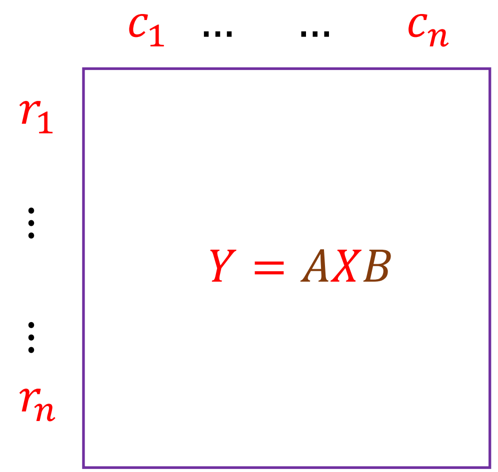
(r, c): probability distributions over $\{1, \ldots, n\}$.

Non-negative $n \times n$ matrix X.

Scaling of X with *row sums* $r_1, \ldots, r_n$

and *column sums* $c_1, \ldots, c_n$?

$$\Delta_X = \{(r, c) \colon r = Y\mathbb{1}, c = Y^t\mathbb{1}\}.$$



[…; Rothblum, Schneider 89]: Δ_X *convex polytope*!

Membership: Linear programming

Δ_X = $\{(r, c)$: $\exists\, Z, \mathrm{supp}(Z) \subseteq \mathrm{supp}(X),\ Z$ marginals $(r, c)\}$.

Commutative group actions: *classical marginal* problems.

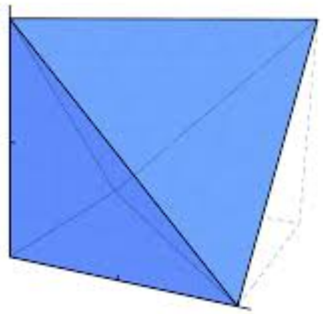
Also related to *maximum entropy* distributions.

Quantum marginals

Pure quantum state $|\psi\rangle_{S_1,\ldots,S_d}$ (d quantum systems): ψ is a d -tensor

Underlying group action: Products of GL's on d $-$*tensors*.
(``local'' basis changes in each system)

Characterize marginals $\rho_{S_1}, \ldots, \rho_{S_d}$ (marginal states on systems)?
Only the spectra of ρ_{S_i} matter (local rotations for free).



- Collection of such spectra Δ_ψ *convex polytope*!
- Follows from theory of *moment polytopes*.
- [BFGOWW 18]: Membership via *non-uniform tensor scaling*.

More examples of moment polytopes

Schur-Horn: A $n \times n$ symmetric matrix.

$$\Delta_A = \{ \text{diag}(B) : B \text{ similar to } A \} \subseteq \boldsymbol{R}^n$$

Horn: $\Delta = \{(\lambda_A, \lambda_B, \lambda_C) : A + B = C\} \subseteq \boldsymbol{R}^{3n}$

Brascamp-Lieb: Feasibility of analytic inequalities

Newton: $q = \sum_{\alpha \in \Omega} c_\alpha x^\alpha \in \boldsymbol{C}[x_1, \dots, x_n]$, homogeneous polynomial

$$\Delta_q = \text{conv}\{\alpha : \alpha \in \Omega\} \subseteq \boldsymbol{R}^n$$

Edmonds: M, M' matroids on $[n]$ (over the Reals).

$$\Delta_{M,M'} = \text{conv}\{ 1_S : S \text{ basis for } M, M'\} \subseteq \boldsymbol{R}^n$$

Algorithms: membership in moment polytopes

Group G acts linearly on vector space V.
$v \in V \leftrightarrow \Delta_v \subseteq \boldsymbol{R}^n$ moment polytope.

Non-uniform scaling: Given $v \in V$, $r \in \boldsymbol{R}^n$, $\epsilon > 0$
does $r \in \Delta_v$
or ϵ–far from Δ_v

For general settings we have efficient:

- Alternating minimization: convergence poly(1/ϵ)
- Geodesic optimization: convergence polylog(1/ϵ)

Conclusions & Open problems

Summary: Invariant Theory + ToC

Lots of similar type questions, notions, results

- Algorithms are important, sought and discovered
- Has both an algebraic and analytic nature
- Quantitative, with many asymptotic notions
- Studies families of objects
- Needs comp theory structure, reductions, completeness
- Symmetry is becoming more central in ToC

Summary: Consequences

New efficient algorithmic techniques, solving classes of:

- non-convex optimization problems
- systems of quadratic equations
- linear programs of exponential size

Applicable (or potentially applicable) in:

- Derandomization (PIT)
- Analysis (Brascamp-Lieb inequalities)
- Non-commutative algebra (word problem)
- Quantum information theory (distillation, marginals, SLOCC)
- Representation theory (asymptotic Kronecker coefficients)
- Operator theory (Paulsen problem)
- Combinatorial optimization (moment polytopes)

Open problems

- PIT in $\boldsymbol{P}$?
- Is PIT a null cone problem?

- Polynomial time algorithms for

1. *Null cone* membership.
2. *Moment polytopes* membership, separation, optimization.

- Extend algorithmic theory to group actions on algebraic varieties, Riemannian/symplectic manifolds

Learn more?

EATCS survey [Garg,Oliveira]

https://arxiv.org/abs/1808.09669

My CCC'17 tutorial:

http://www.computationalcomplexity.org/Archive/2017/tutorial.php

STOC 2018 tutorial:

https://staff.fnwi.uva.nl/m.walter/focs2018scaling/

A week of tutorials:

https://www.math.ias.edu/ocit2018

Mathematics and Computation

New book on my website