Sublinear time local access random generators

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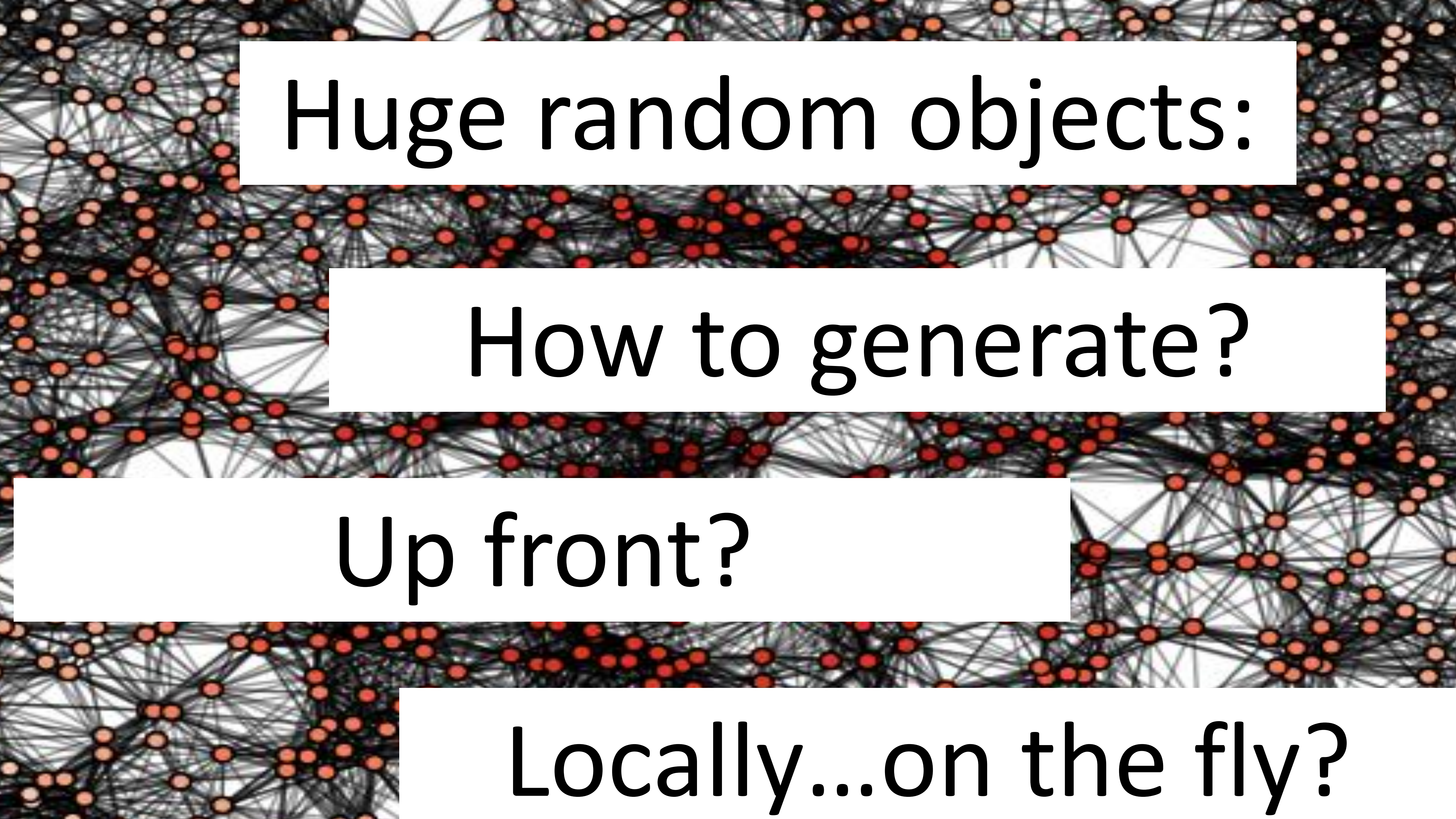


Huge random objects:

How to generate?

Up front?

Locally…on the fly?

Generating large random graph

	1	2	3	4	5	6	7	8	9	10
1			0	1	0		0			
2			0	1	0		0	1		1
3	0	0		0	0	0	0	0	1	
4	1	1	0		0		0	1		
5	0	0	0	0		1	1			
6			0		1				0	
7	0	0	0	0	1					
8		1	0	1						0
9			1			0				
10		1						0		

Generate “on the fly”?

What if required to be symmetric? *d*-regular? support “next-neighbor” queries?

A challenge:
How to handle dependencies?

Sources of dependencies:

Model, supported queries,…

Some prior work

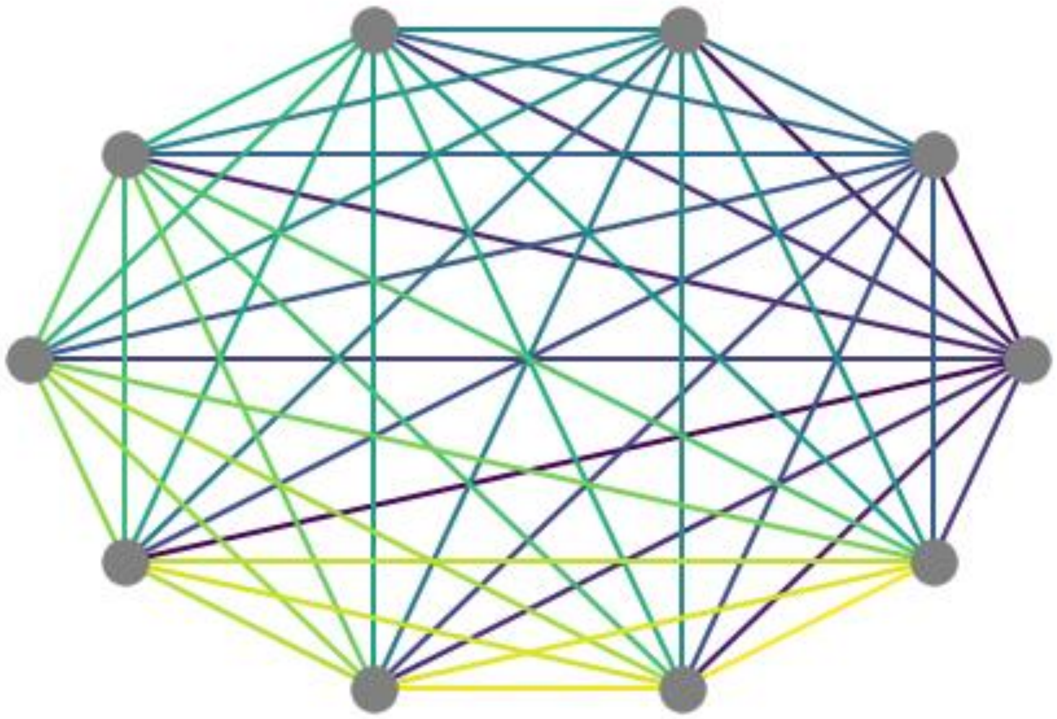
Implementation of Huge Random Objects

- Huge pseudorandom functions/permutations/balls-in-bins [Goldreich-Goldwasser-Micali’86][Luby-Rackoff ‘88][Naor-Reingold ’97][Mansour-Rubinstein-Vardi-Xie ’12]

- Model introduced and formalized in [Goldreich-Goldwasser-Nussboim 2003]
  - Generators for random functions, codes, graphs,…
  - Give important primitives
    - e.g. Sampling from binomial distribution, interval-sum queries for functions (see also [Gilbert, Guha, Indyk, Kotidis, Muthukrishnan, Strauss 2002])
  - Generators provide (limited) queries to random graphs with specified property
    - e.g. Planted Hamiltonian cycle
    - Focus on *indistinguishable* (under small number of queries and poly time) and *truthful* implementations (more on this by [Naor Nussboim Tromer 05] [Alon Nussboim 07])

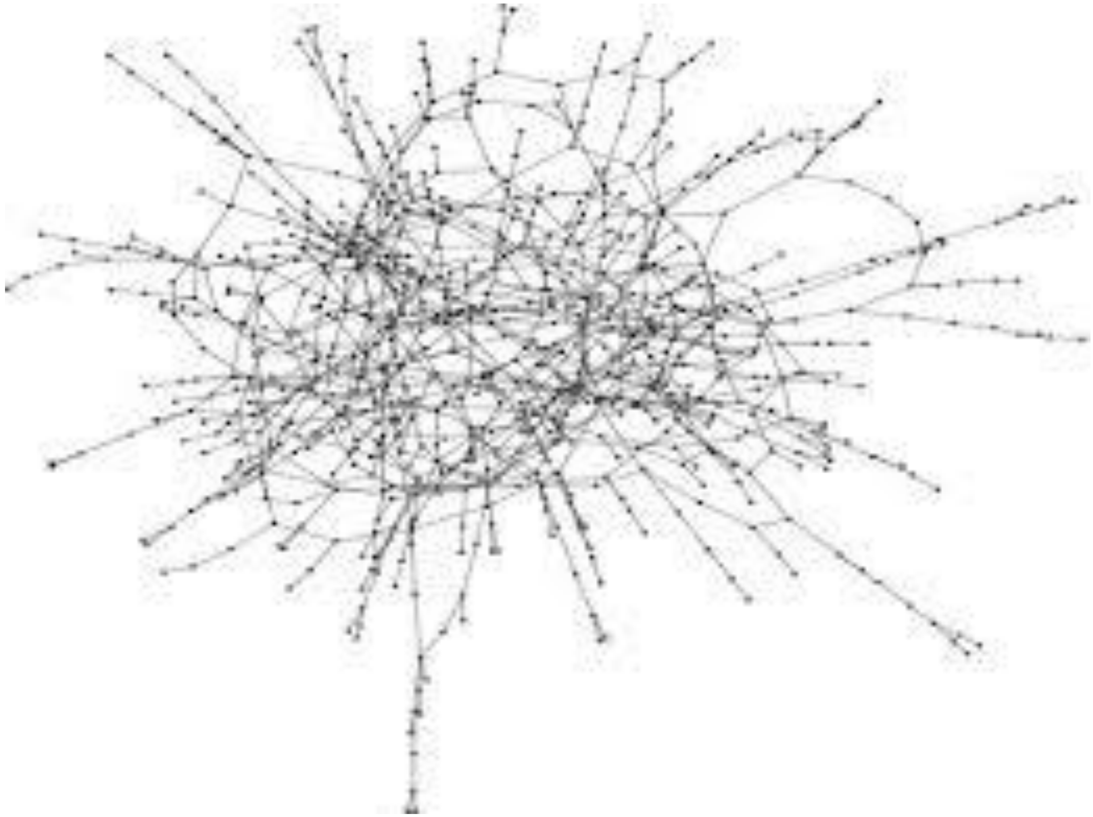
Implementations of random *G(n,p)* graphs [Goldreich Goldwasser Nussboim 03]

- Graphs generated:
  - Have a specific property e.g., colorability, clique, connectedness, bipartiteness
- Queries:
  - Adjacency
  - Up to polylog queries



Implementation I of sparse G(n,p) graph [GGN]

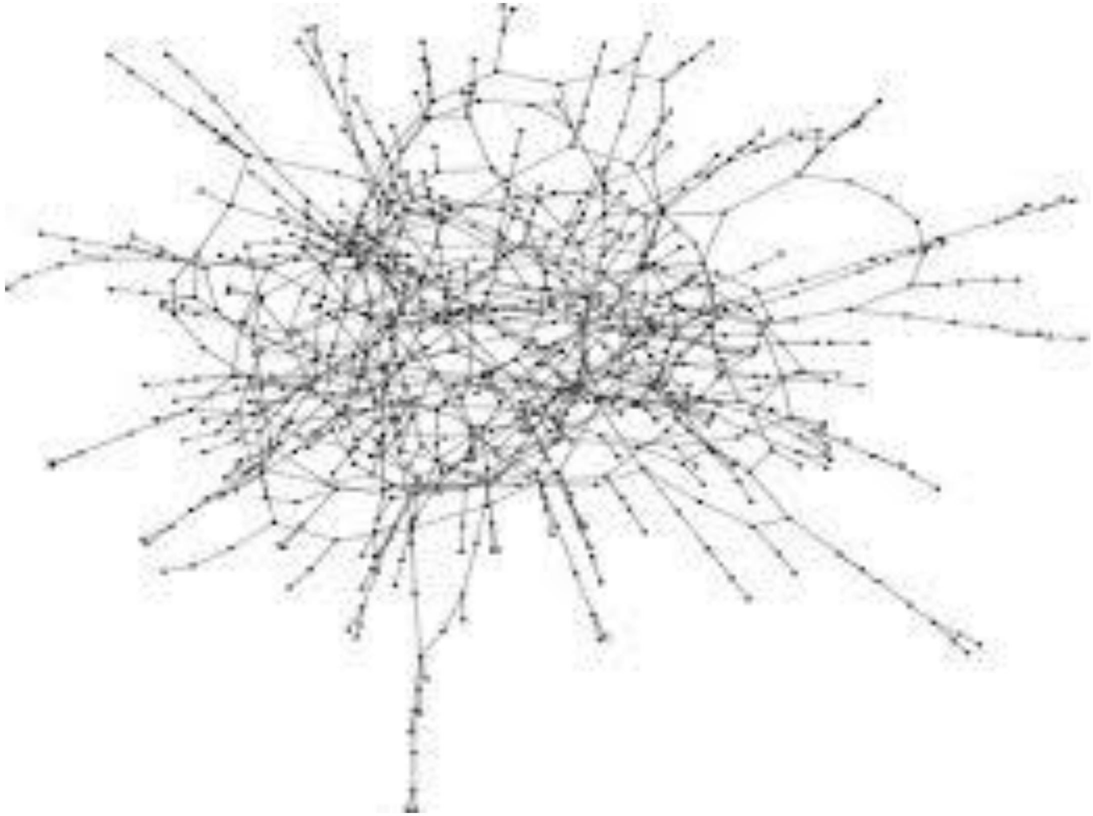
- Graphs generated:
  - Degree at most polylog
  - Indistiguishable from uniform distribution for few queries
- Queries:
  - Adjacency, all-neighbor
  - Up to polylog queries



Implementation II of sparse G(n,p) graph

[Naor-Nussboim 2007]

- Graphs generated:
  - Degree at most polylog
- Queries:
  - Adjacency, all-neighbor
  - Bound on number stated in paper, but not necessary in some settings



Implementations of Barabasi-Albert Preferential Attachment Graphs [Even-Levi-Medina-Rosen 2017]

- Graphs generated:
  - essentially a rooted tree/forest structure
  - Highly sequential random process
  - Sparse, but degree not bounded
- Queries:
  - Adjacency
  - Introduce next-neighbor query (implement polylog(n) resources)
  - No bound on number

Give local implementation without reconstructing full history!!

Models

Two models for random generation of graphs

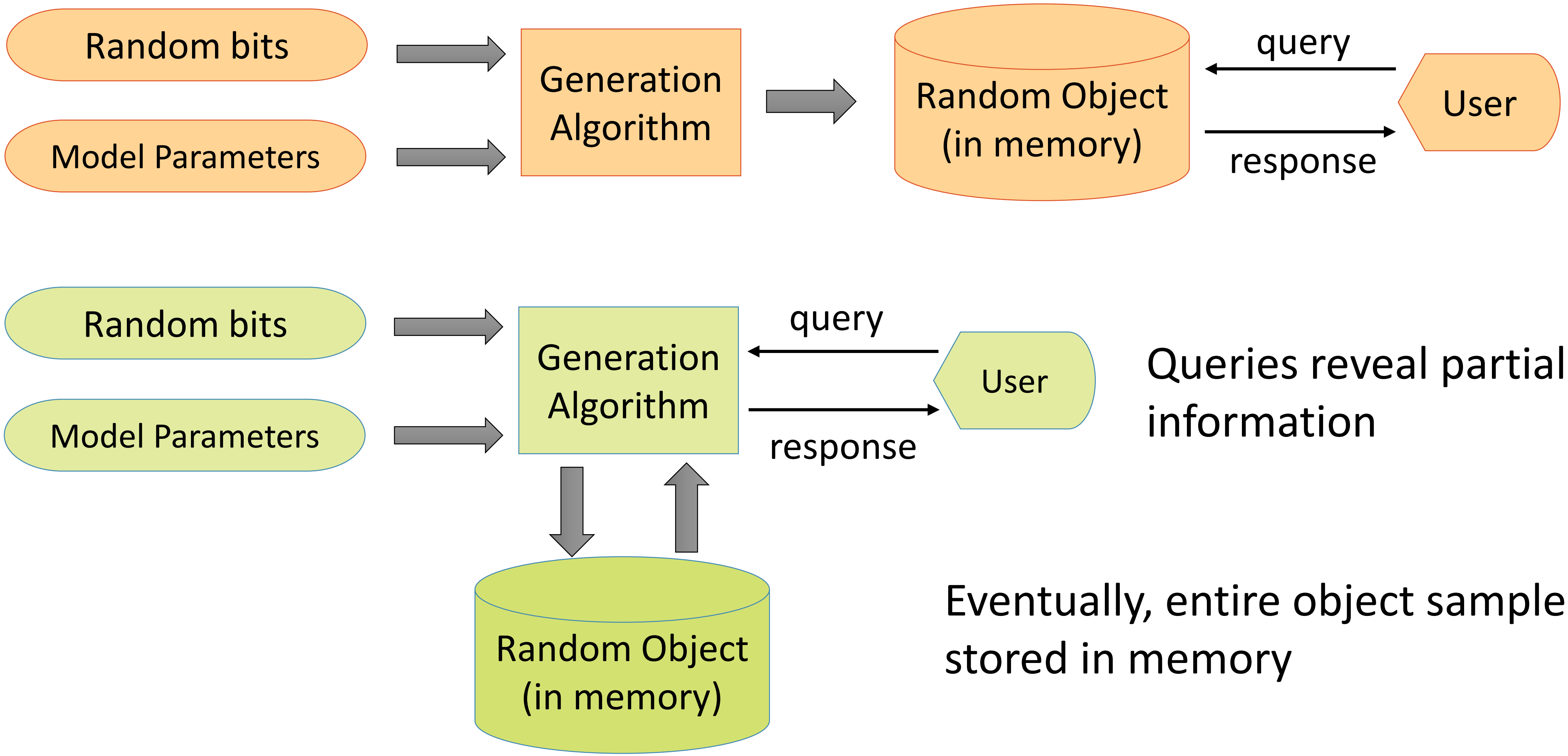
Huge random graphs/objects [Goldreich Goldwasser Nussboim]

- Huge = exponential size
- User will not query more than poly locations
- In some versions, sufficient to generate graph that “looks” random to poly time algorithm

Big random graphs/objects [Even Levi Medina Rosen]

- Big = poly size
- Might eventually write down the whole graph, but don’t want to pay cost up-front
- End result should be random according to the claimed process

“On-the-fly” Sampler
(Adapted from [Even-Levi-Medina-Rosen 2017])



Standard paradigm

“on-the-fly” sampler

Queries reveal partial information

Eventually, entire object sampled, stored in memory

Desiderata:

- Efficiency:
  - Answer queries in polylogarithmic time
- Succinct Representation
- Consistency over future queries:
  - Can store past decisions
  - eventually give complete valid sample
- Distribution equivalence:
  - Output distribution is ϵ-close (in ℓ_1-distance) to goal distribution

Error from implementation issues

- Not considered today:
  - pseudo-random distributions (indistinguishable from goal distribution, or preserving properties)
  - bounds on number of queries
  - Very succinct representation

Possible queries:

- Vertex-pair (adjacency): Is edge *(u,v)* present?
- All-Neighbors: What are all neighbors of *u*?
- Degree: What is degree(*u*)?
- *i*th neighbor: What is *i*th neighbor of u?
- Next-neighbor: What is next neighbor of u?
- Random-neighbor: Output random neighbor of u?

considered by [GGN] [NN]

considered by [Even Levi Medina Rosen 2017]

today

can take random walk in large degree graph!

New Generators

Today’s Goal: Graph models supporting typical graph queries

G(n,p)

Community structure: Stochastic Block Model

Small world graphs

$G(n,p)$ graphs

Vertex-pair query:
Is there an edge from u to v?

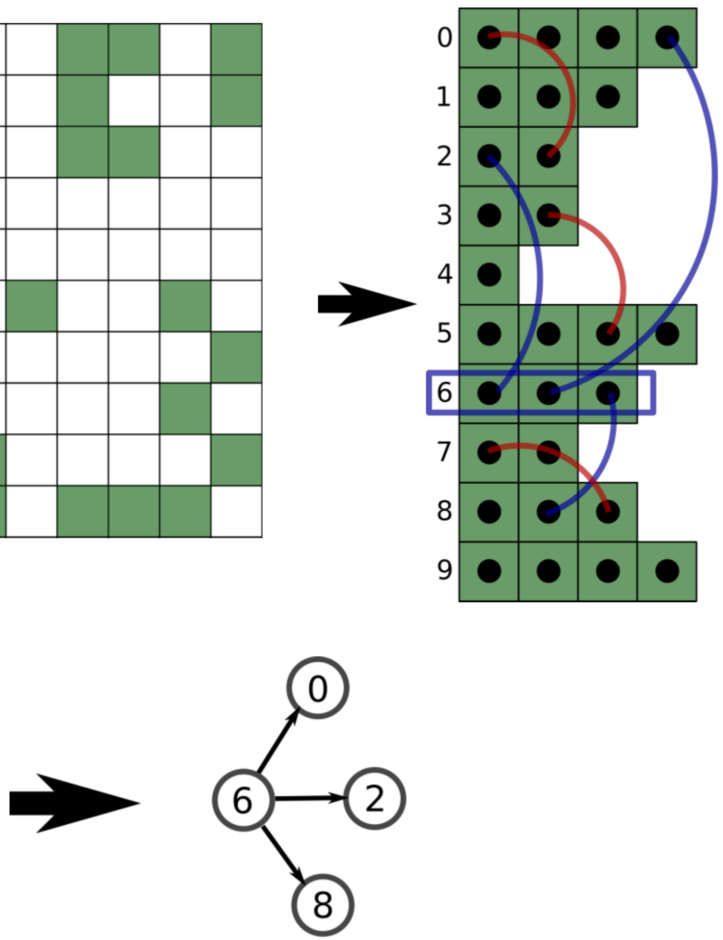
	1	2	3	4	5	6	7	8	9	10
1			0	1	0		0			
2			0	1	0		0	1		1
3	0	0		0	0	0	0	0	1	
4	1	1	0		0		0	1		
5	0	0	0	0		1	1			
6			0		1				0	
7	0	0	0	0	1					
8		1	0	1						0
9			1			0				
10		1						0		

Generate “on the fly”?

toss coins as needed

All-neighbor queries for sparse G(n,p): Implementation adapted from [NN07]

0	🟩					🟩	🟩		🟩
1		🟩				🟩			🟩
2						🟩	🟩		
3		🟩	🟩						
4			🟩						
5		🟩	🟩		🟩			🟩	
6		🟩	🟩						🟩
7		🟩						🟩	
8	🟩			🟩					🟩
9				🟩		🟩	🟩	🟩	



- Edges defined via “Ports”:
  - For each node, pick “ports”: "1" (green)
  - Ports matched to others on the fly: indicated via edge (red)
- Two equivalent processes:
  - Pick number of edges for each u and sum to get total edges
  - Picking total number of edges and dividing among u’s

  → Compute u’s locations using locally computable interval-summable functions [GGIKMS 02] [GGN03][NN07]

- Given an “all neighbor” query vertex (6), match its ports to other **unmatched** ports
  - Match each port to random open position in degree sequence

Next-Neighbor Query: what is u's next neighbor?

Dense case: $p \geq 1/poly(\log n)$

- Algorithm:
  - Start at last found neighbor
  - Go down row, flipping coins to fill empty entries, until find neighbor.
- Time $O(1/p)$.

Can we do $o(1/p)$ for $p = o(1)$?

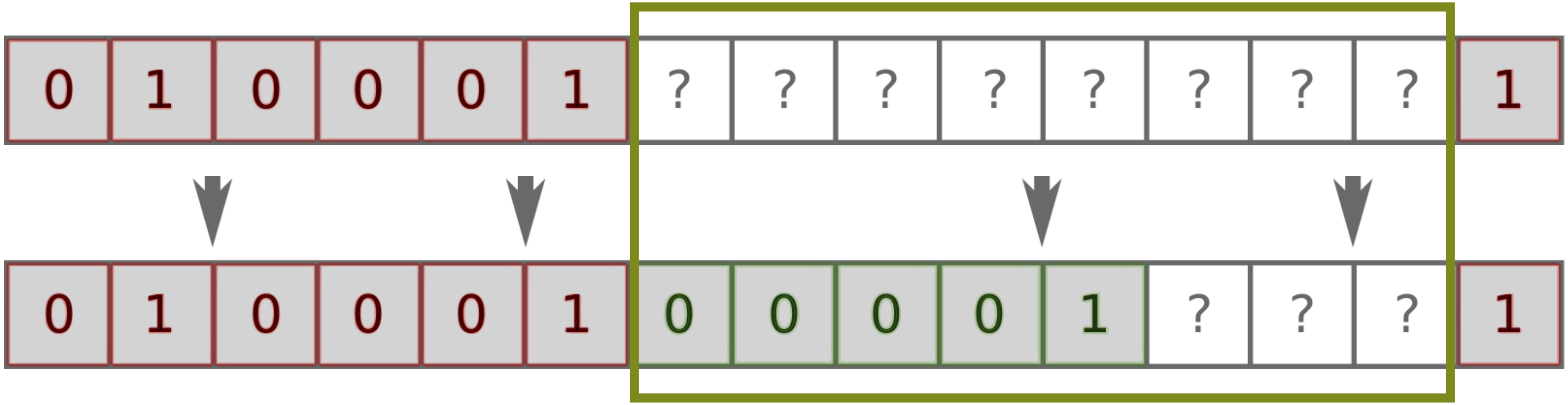
Sparse case: $p \leq poly(\log n)/n$

- Algorithm: Use "all neighbor" query [Naor Nussboim 07]
- Time $O(E[degree]) = O(polylog\ n)$

Intermediate case: (e.g. $p = \frac{1}{\sqrt{n}}$)

- Idea: Sample "length of 0's run" according to hypergeometric distribution $p(1-p)^i$
- Challenge: some entries already filled in!

Skip-sampling for next-neighbor queries: The case of directed graphs



Algorithm idea:

Pick length of *0*-run according to hypergeometric distribution (via binary search on CDF):

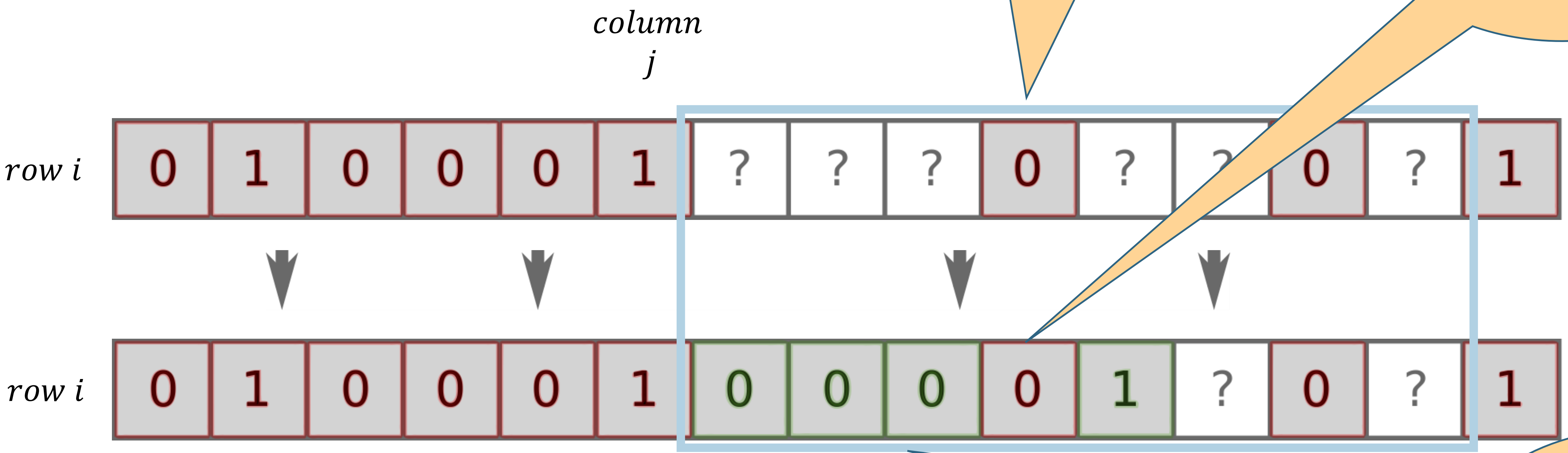
$$\sum_{k=0}^{b-a-1} p\,(1-p)^k \;= 1\;-(1-p)^{b-a}$$

Fill in next entry *(i, j+k)* with a *1*

Skip-sampling for next-ne ries: Undirected graphs

some are determined by other neighbor?

yields correct distribution?



						column j									
row i	0	1	0	0	0	1	?	?	?	0	?	?	0	?	1
row i	0	1	0	0	0	1	0	0	0	0	1	?	0	?	1

need to write all 0s?

Algorithm idea:

Pick length of *0*-run according to hypergeometric distribution:

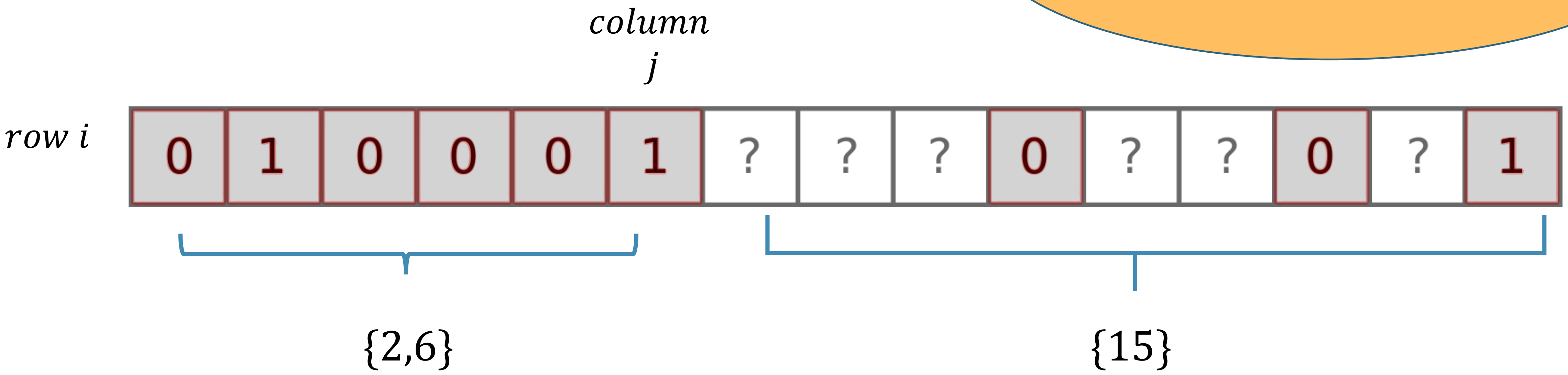
$$\sum_{k=0}^{b-a-1} p\,(1-p)^k = 1 - (1-p)^{b-a}$$

Fill in next entry *(i, j+k)* with a *1*

Implementation of next neighbor queries: (assume no adjacency queries)

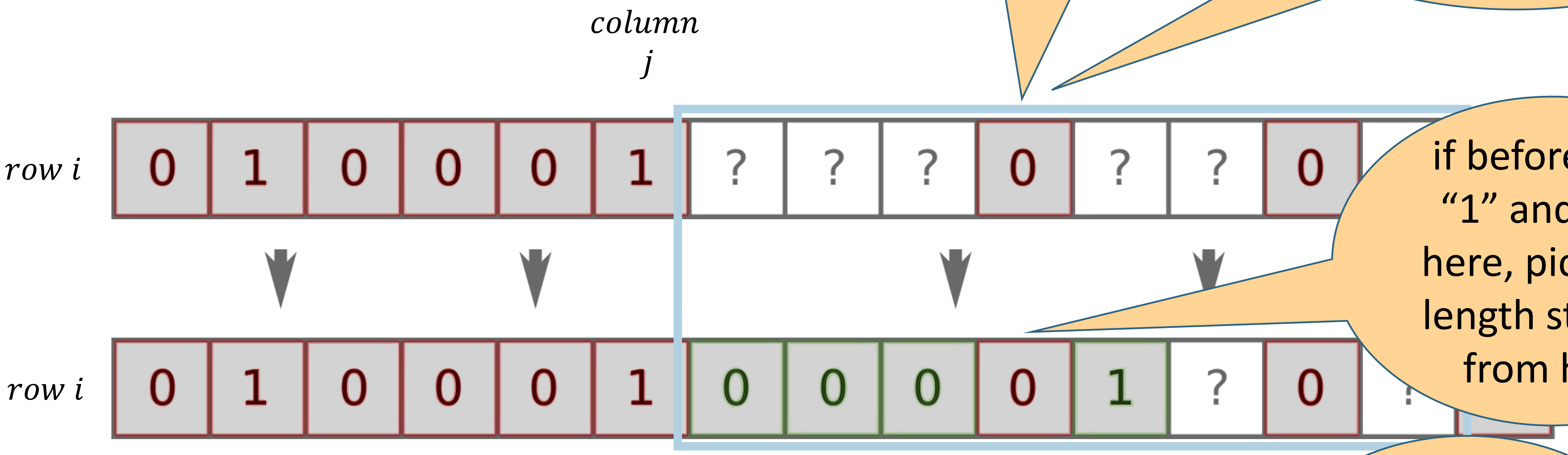
- For each node *i* maintain:
  1. last seen neighbor *j* (row entries *1..j* are determined, and *j* is a “1”)
  2. list of “1”s coming before *j (everything else is “0”)*
  3. remaining“1”s via min-heap
  4. Keep track of “0”s on row implicitly

Only keep track of 1’s



						$column$ j									
$row\ i$	0	1	0	0	0	1	?	?	?	0	?	?	0	?	1

Skip-sampling for next- qu



						column *j*								
row i	0	1	0	0	0	1	?	?	?	0	?	?	0	
row i	0	1	0	0	0	1	0	0	0	0	1	?	0	?

some are determined by other neighbor?

if “1”, then neighbor has told i about it

if before next “1” and land here, pick new length starting from here

why correct distribution?

choose k according to geometric distribution

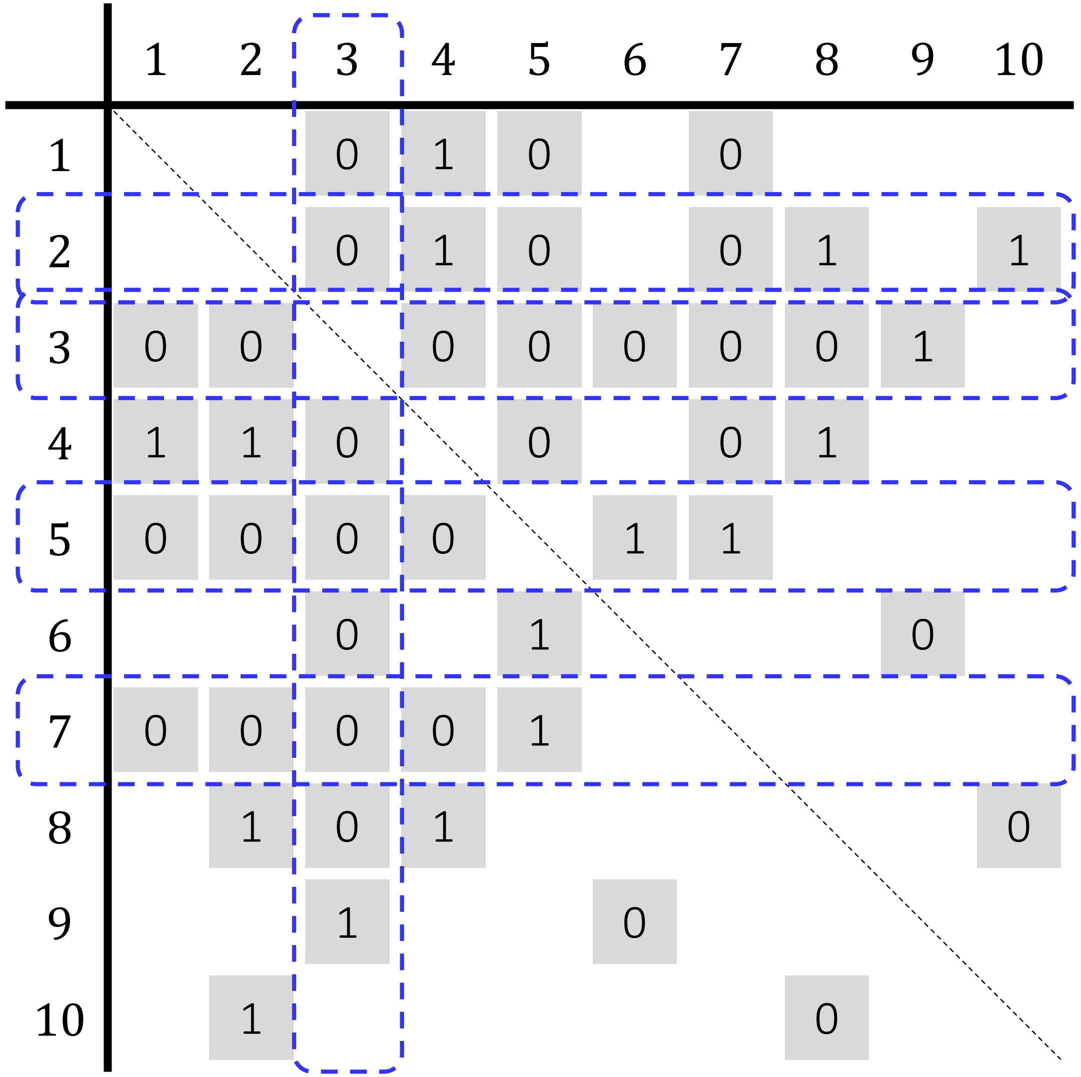
If j+k > next 1 in i’s heap, output next 1 in i’s heap

else check if (i,j+k) previously decided by j+k

if 0 then re-roll

else add (i, j+k) to heaps for i and j+k

Local-Access Generators – Difficulties



	1	2	3	4	5	6	7	8	9	10
1			0	1	0		0			
2			0	1	0		0	1		1
3	0	0		0	0	0	0	0	1	
4	1	1	0		0		0	1		
5	0	0	0	0		1	1			
6			0		1				0	
7	0	0	0	0	1					
8		1	0	1						0
9			1			0				
10		1						0		

next-neighbor

- how to sample for ***next-neighbor***?
- how to inform (non-)neighbors?
- how to find ***next-neighbor*** when some choices are already decided?

vertex-pair

- how to maintain information without obstructing ***next-neighbor***?

careful analysis can mitigate these .. but

random-neighbor

- how to sample without knowing/committing to a degree?

Random-Neighbor Query: output random neighbor of i

Dense case: $p \geq 1/poly(\log n)$

- Algorithm:
  - repeat until find neighbor:
    - pick random j
    - do vertex pair query on (i, j)
- Time $O(1/p)$.

Can we do $o(1/p)$ for $p = o(1)$?

Sparse case: $p \leq poly(\log n)/n$

- Algorithm: Use “all neighbor” query [Naor Nussboim 07]
- Time $O(E[degree]) = O(polylog\, n)$

Intermediate case: (e.g. $p = \frac{1}{\sqrt{n}}$)

???

we don’t even know degree?

Implementation of Random-Neighbor queries via Bucketing

Plan: **Equipartition** each row into **contiguous buckets** such that:

Expected # of neighbors in a bucket is a constant
⇒ w.h.p. 1/3 of buckets are non-empty
⇒ w.h.p. no bucket has more than log n neighbors

(drumroll…)
⇒ can write down all $\log n$ neighbors for each bucket! (assuming you can figure them out)

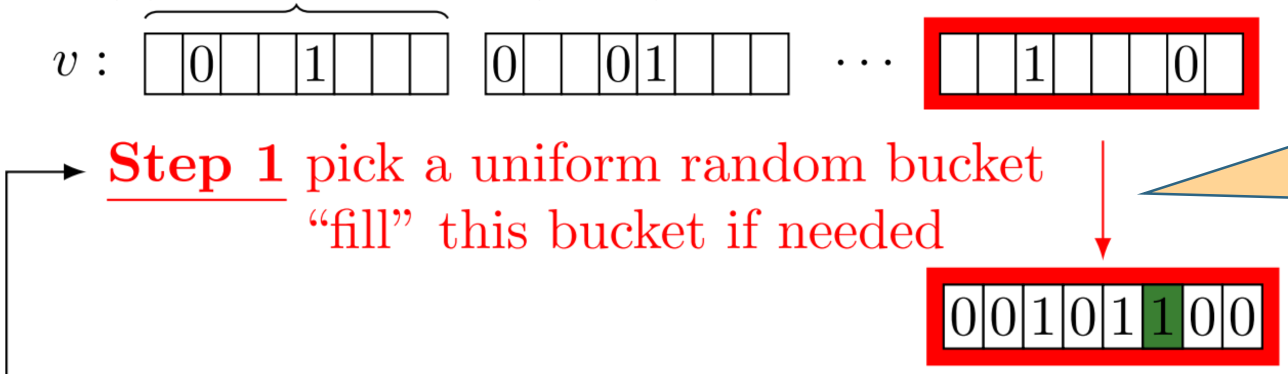
How many buckets?

pn, each of size $1/p$

Note that both size and number of buckets can be big

Random Neighbors with rejection sampling

Bucketing: expected #neighbors in a bucket $= \Theta(1)$ expected, $\leq \mathcal{O}(\log n)$ w.h.p. $\Rightarrow$ #neighbors $\approx$ #buckets



Keep list of 1's, then can pick nbr quickly

Step 1 pick a uniform random bucket
"fill" this bucket if needed

Step 2 pick a uniform random neighbor u

$\hookrightarrow$ **return** or **reject**

Step 3 return u with probability $\dfrac{\text{\#neighbors in the bucket}}{\mathcal{O}(\log n)}$
otherwise, try again

$$\mathbb{P}[\text{return } u] = \frac{1}{\text{\#buckets}} \times \frac{1}{\text{\#neighbors in bucket}} \times \frac{\text{\#neighbors in bucket}}{\mathcal{O}(\log n)} \approx \frac{\Omega(1/\log n)}{\text{\#neighbors of } v}$$

$$\mathbb{P}[\text{return any neighbor}] \approx \Omega(1/\log n) \Rightarrow \mathcal{O}(\log n) \text{ iterations suffice}$$

How to fill a bucket?

- Bucket may be ***indirectly*** filled in certain locations
  - "1" entries reported when created
  - "0" entries not reported but can query from complementary bucket

?	1	?	?	?	0	?	1	0	?	?

- First, skip-sample in the bucket ignoring the existing entries

0	0	1	0	0	1	1	0	0	1	0

- Re-insert all *indirectly filled* (red) "1" entries: {2,8}
- For each new (green) "1" entry: remove if coincides with indirectly filled "0" entries

0	1	1	0	0	~~1~~	1	1	0	1	0

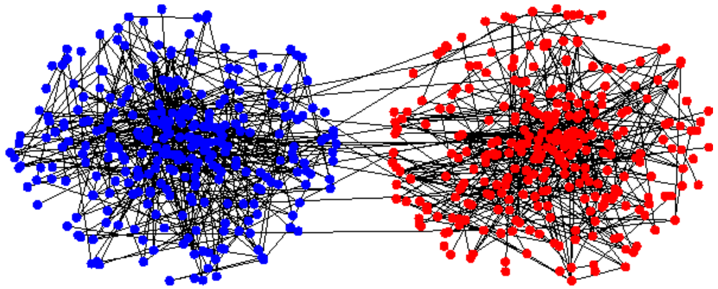
- Why fast? # of "1" entries is bounded by log n

Nice fact:

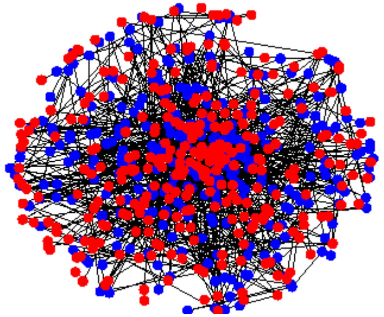
Bucketing improves next-neighbor queries too!

Stochastic Block Model

Stochastic Block Model



- R communities each labelled via “color”
  - P_{ij} specifies probability of edges between community i and j
- how to assign colors to nodes?
  - contiguous blocks?
    - Algorithms for SBM are usually concerned with community *detection*
  - randomly?
    - assume given counts of members of each color

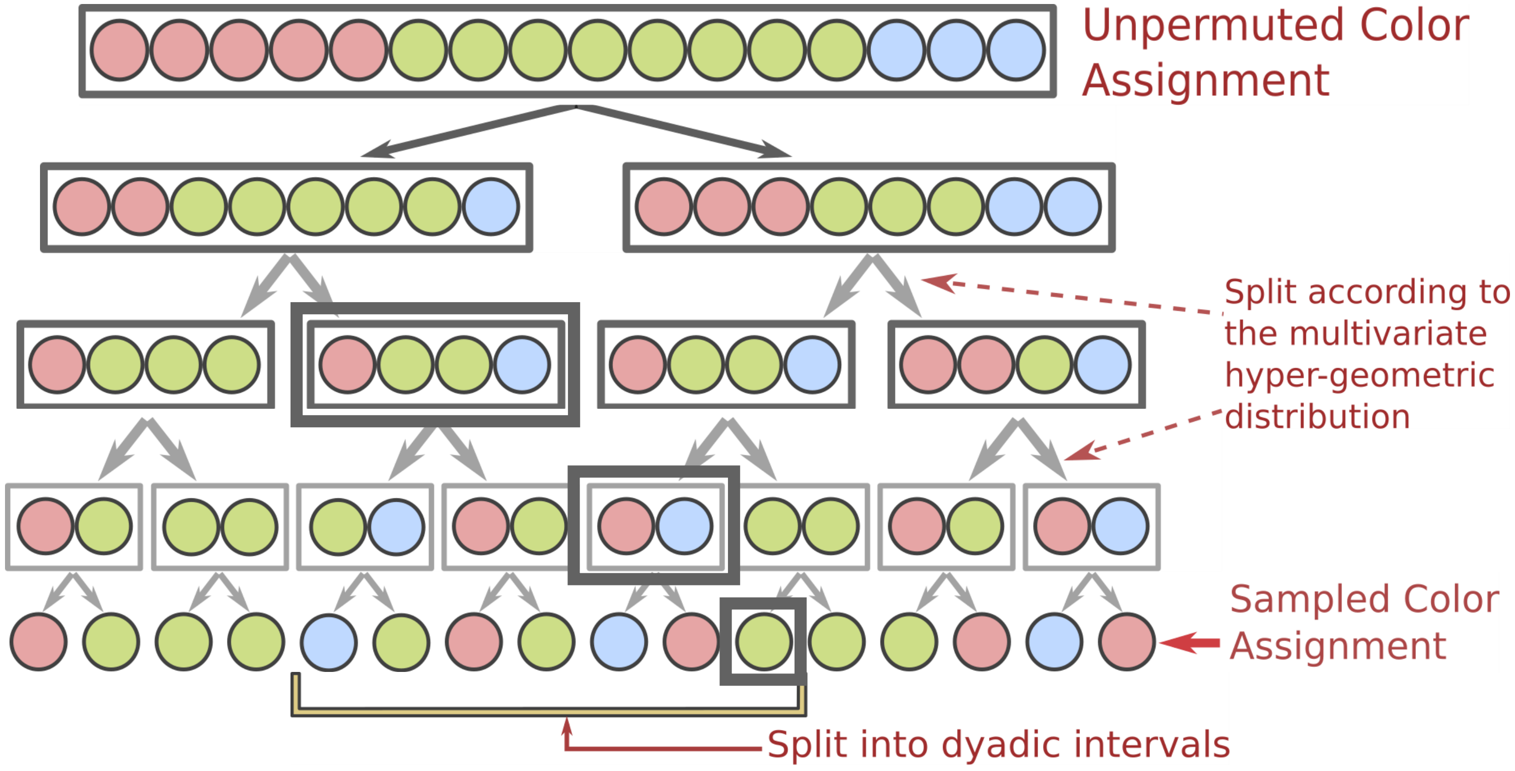


Skip-sampling probabilities

- New requirement
  - count # of members of each color within a specified interval [a,b]
    - E.g., Allows computing CDF of skip-sampling distributions
  - Equivalently: sample from the multivariate hypergeometric distribution

Count generator: Sample colors in an interval

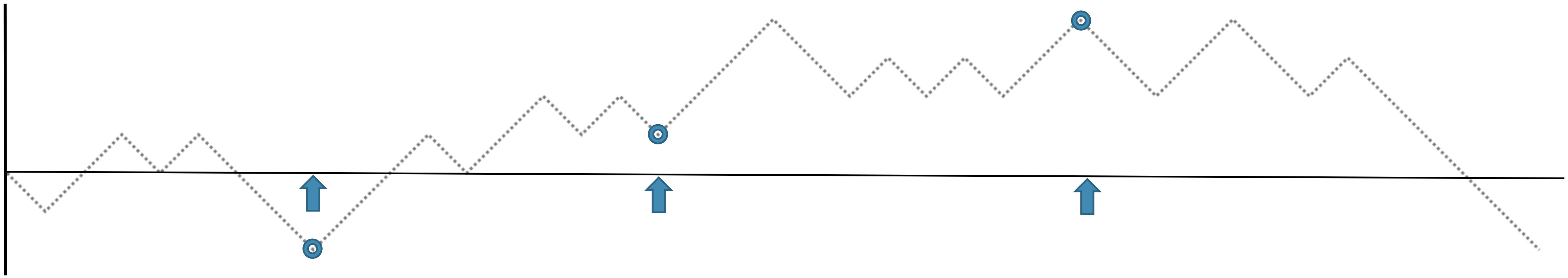
(see also GGIKMS, GGN, NN)


Unpermuted Color Assignment
Split according to the multivariate hyper-geometric distribution
Sampled Color Assignment
Split into dyadic intervals

Tree contructed “lazily”: only as required

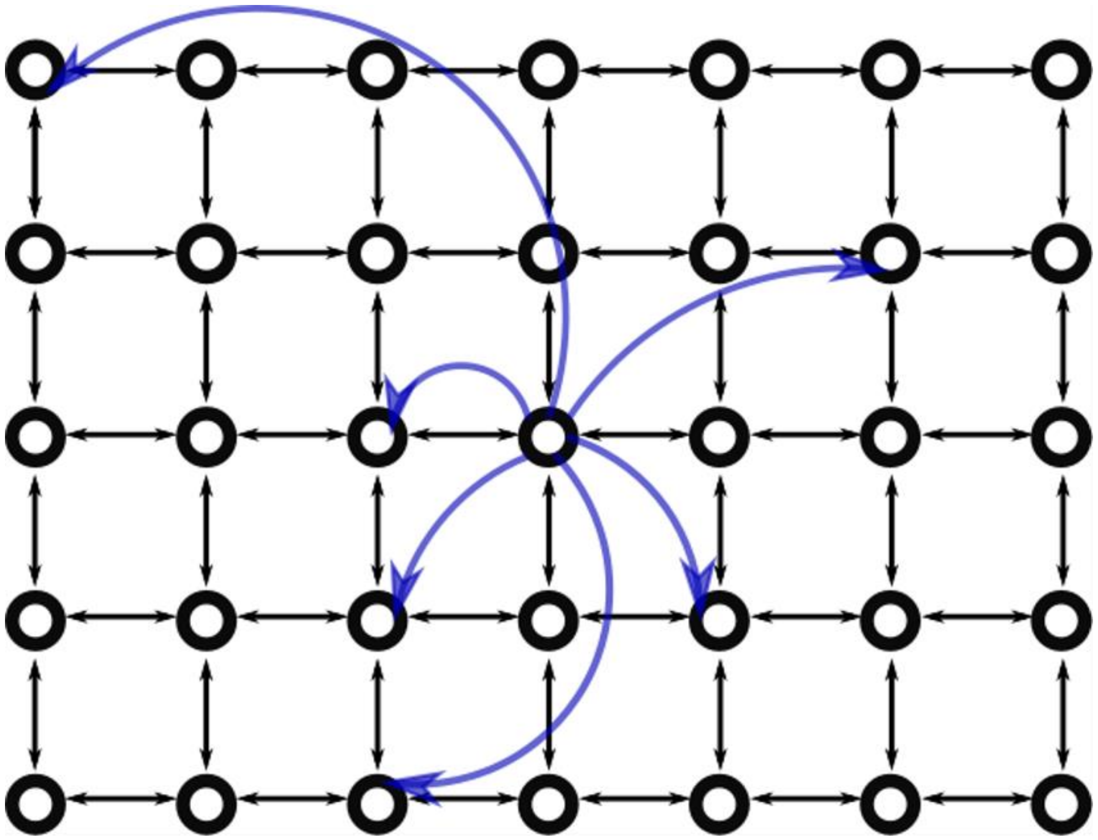
Another use:
Partially Sampling a Random Walk

Query Height(t) returns position of random walk at time t



Small world graphs

Small-World Model [Kleinberg]



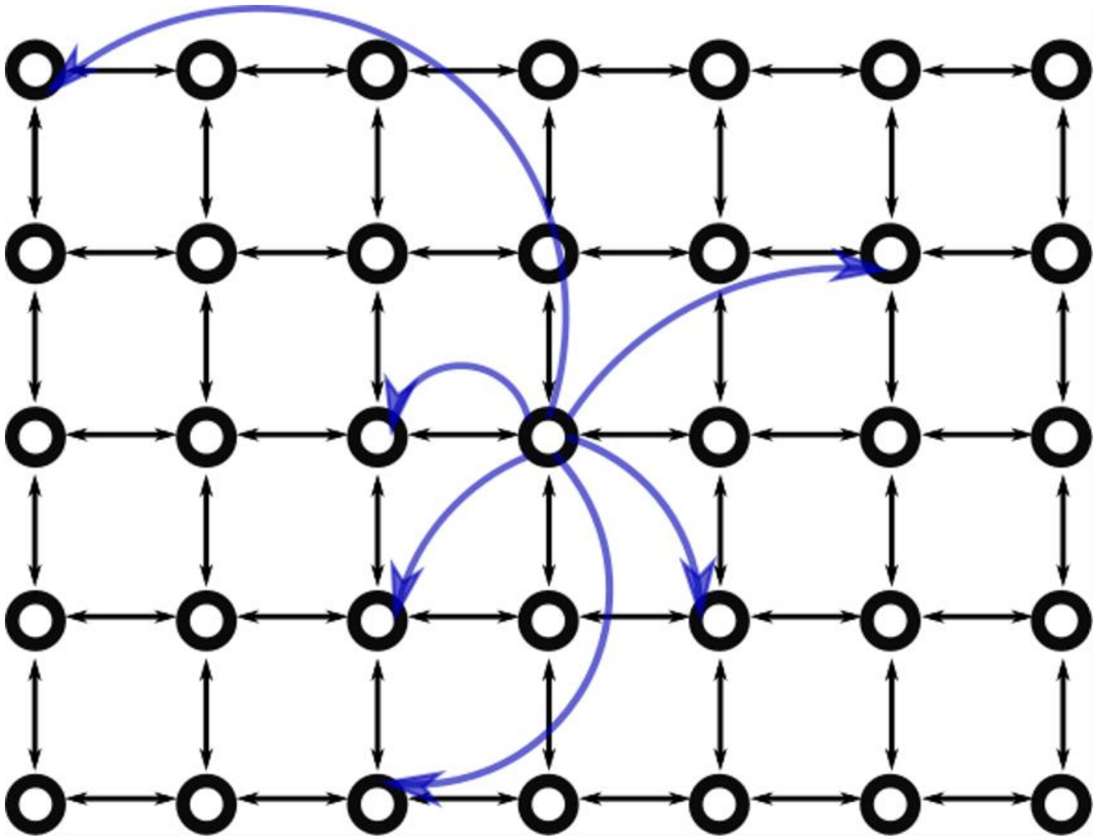
Edges:

- Uniform grid
- Directed long range edge (u, v) with probability $c/d(u, v)^2$

Will answer “All-neighbor queries” (implies implementation for other queries)

Small-World Model: All neighbor queries

- Model:
- Uniform grid
- (u, v) with probability $c/d(u,v)^2$



For increasing d:

(1) Sample next d which has nbrs of distance d

(2) skip sample among all O(d) nbrs at distance d

Future directions

Other random objects?

Support degree, ith neighbor queries?

Local generation without history?

Thank you!