Adaptive Sparse Recovery with Limited Adaptivity

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Outline

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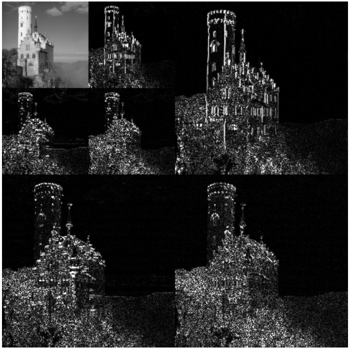
- Images sparse in wavelet basis



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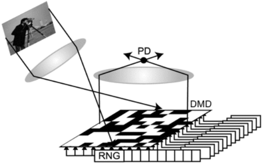

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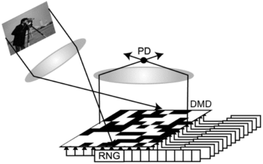

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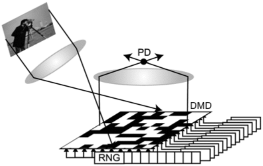

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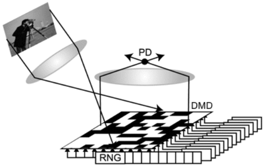
- Extremely well studied: thousands of papers.



Standard Sparse Recovery Framework

- Specify distribution on $m \times n$ matrices A (independent of x).

- Given linear sketch Ax, recover $\widehat{x}$.
- Satisfying the recovery guarantee:

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- Solvable in $O(k \log \log \frac{n}{k})$ [Indyk-Price-Woodruff '11].

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- Lower bounds:
  - [Arias-Castro, Candès, Davenport '13]: $m \gtrsim \frac{1}{\varepsilon} k$
  - [Price, Woodruff '13]: $m \gtrsim \log \log n$.

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For $k < n^{o(1)}$, $m^* = \omega(k)$.

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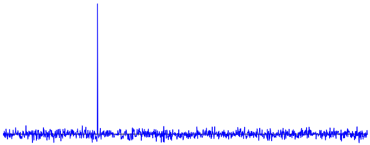

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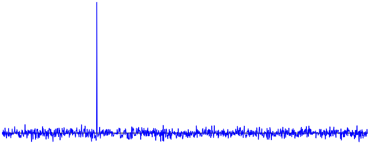

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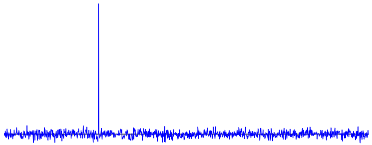

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- Observations $\langle v, x \rangle = v_z + \langle v, w \rangle = v_z + \frac{\|v\|_2}{\sqrt{n}} z$, for $z \sim N(0, 1)$.

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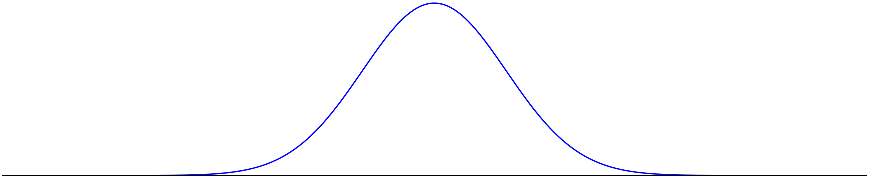

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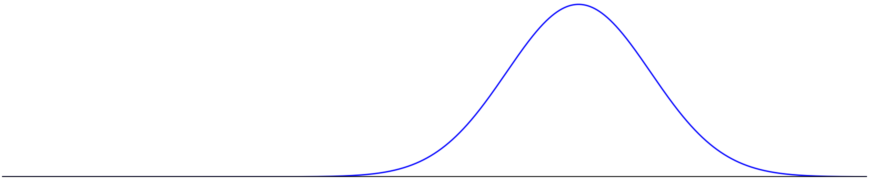

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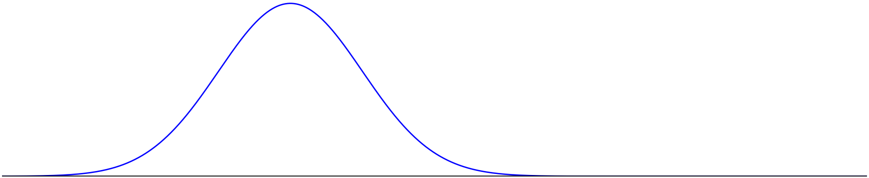

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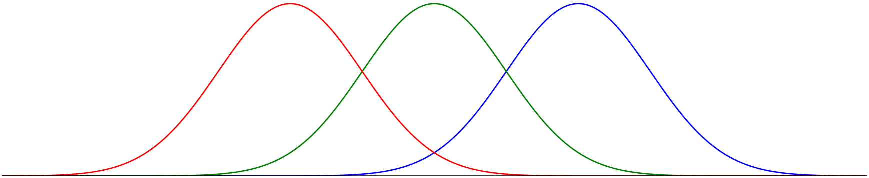

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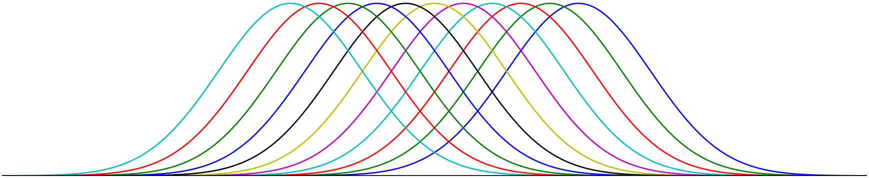

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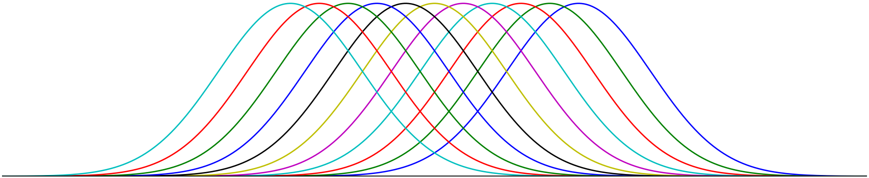

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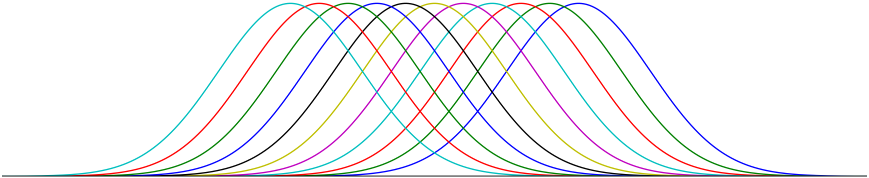

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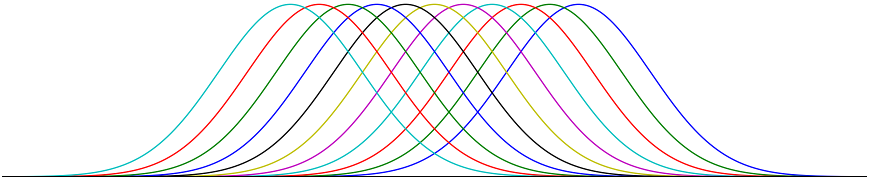

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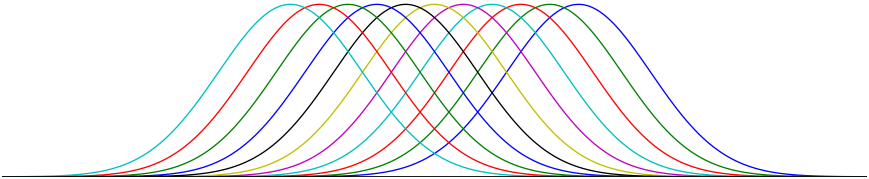

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- Finding z needs $\Omega(\log n)$ non-adaptive measurements.

1-sparse recovery: changes in adaptive setting

- Information capacity

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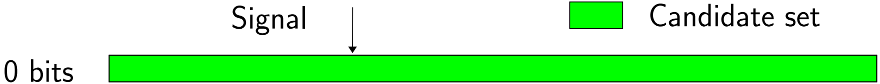
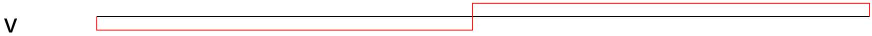
- As we learn about z, we can increase the SNR.

1-sparse recovery: adaptive upper bound

$$x = e_z + w$$

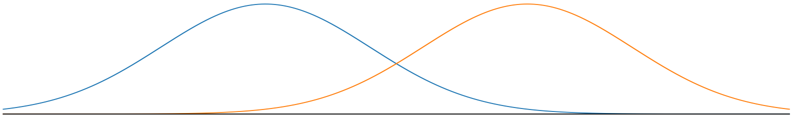




$SNR = 2$

$I(z, \langle v, x\rangle) \leqslant \log SNR = 1$

$$\langle v, x\rangle = v_z + \langle v, w\rangle$$



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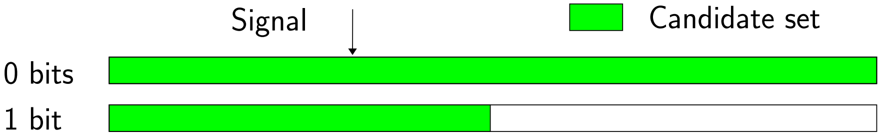

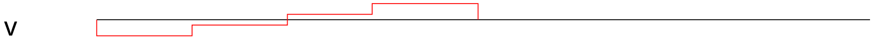

$SNR = 2^2$

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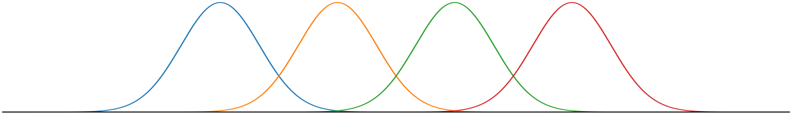

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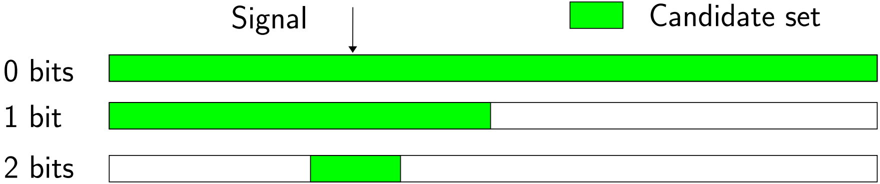
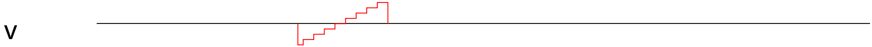
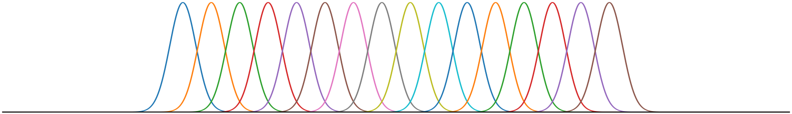




$SNR = 2^4$

$I(z, \langle v, x \rangle) \leqslant \log SNR = 4$

$\langle v, x \rangle = v_z + \langle v, w \rangle$



1-sparse recovery: adaptive upper bound

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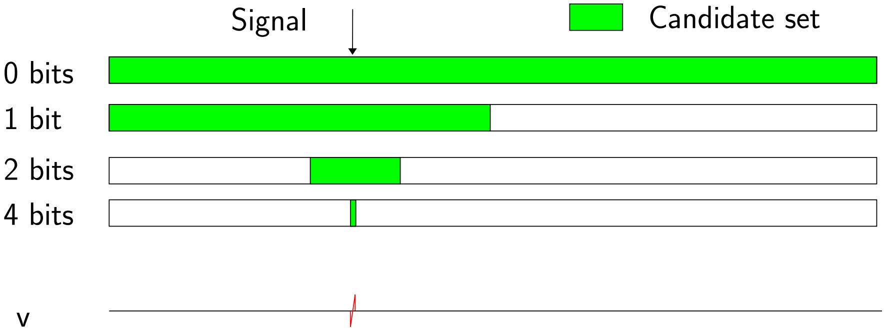

$SNR = 2^8$

$I(z, \langle v, x \rangle) \leqslant \log SNR = 8$

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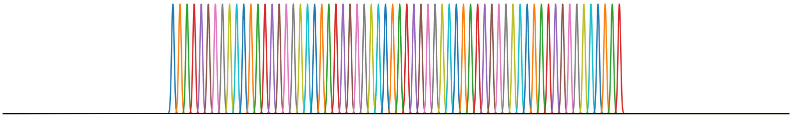

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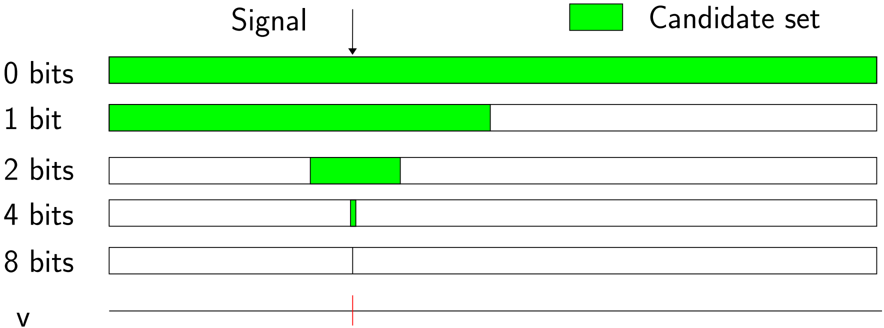

$SNR = 2^{16}$

$I(z, \langle v, x \rangle) \leqslant \log SNR = 16$

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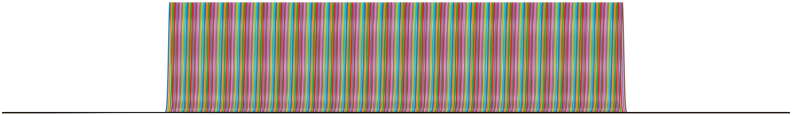

1-sparse recovery: adaptive lower bound

- Review of upper bound:
  - Given b bits of information about z.
  - Identifies z to set of size $n/2^b$.
  - Increases SNR, $\mathbb{E}[v_z^2]$, by 2^b.
  - Recover b bits of information in one measurement.
  - $1 \to 2 \to \cdots \to \log n$ in $\log \log n$ measurements.
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Lemma (Key lemma for $k = 1$)

For any measurement vector v,

$$I(z; \langle v, x \rangle) \lesssim b + 1$$

1-sparse recovery: adaptive lower bound

- Lower bound outline:
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- Can be terrible in general: $b = 1$ but $SNR = n/\log n$.

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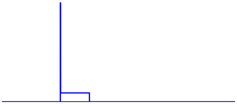

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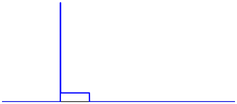

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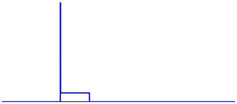

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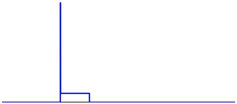

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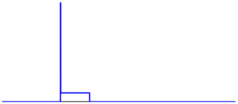

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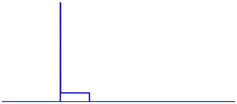

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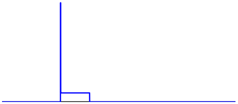

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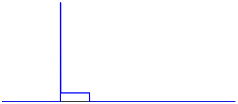

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1-sparse recovery: adaptive lower bound: finishing up

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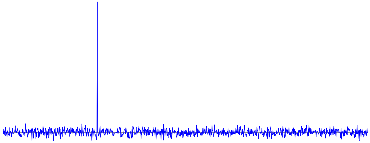
  - $\Omega(\log \log n)$ for unlimited R.

Outline

Recall the $k = 1$ proof outline



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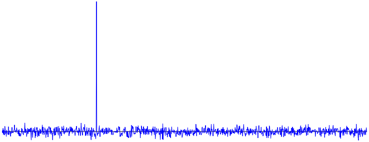

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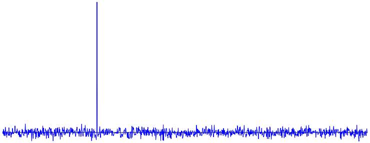

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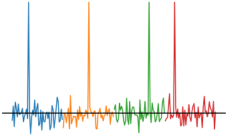
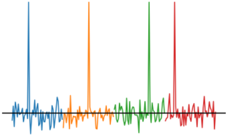
- **Question**: How to extend this to $k > 1$?

Extending to general k



- Create k independent copies over domain $N = nk$.

Extending to general k



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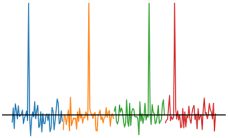

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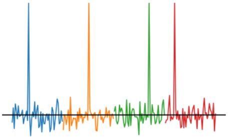

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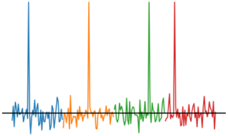

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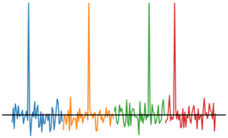
Lemma (Key lemma for general k)

$$\sout{I(Z; \langle v, x \rangle) \lesssim b + 1} \text{ ????}$$

What lemma do we want for general k?



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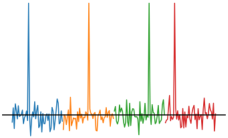

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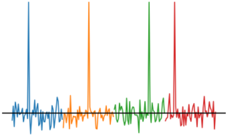

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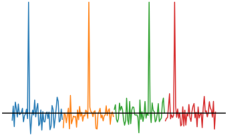

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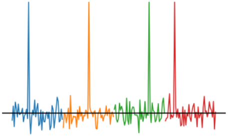

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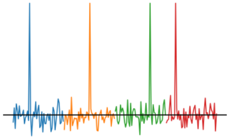

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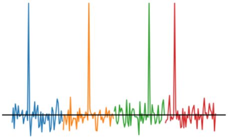

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- $I(Z_W; \langle v, x \rangle) \lesssim b/k + 1$ for $|W| > 0.99k$.
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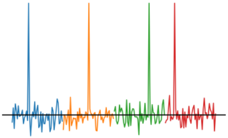

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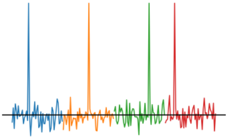

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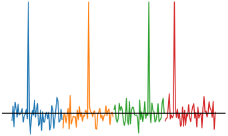

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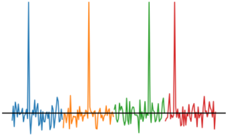

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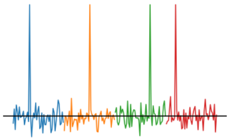

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- So we just need

$$\max_{i \in W} \log(1 + SNR(i)) \lesssim b/k.$$

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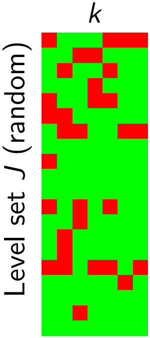

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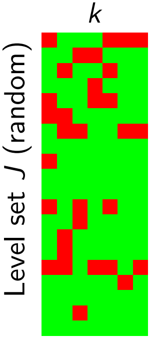

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- Find $W = W(J)$ so that

$$\max_{i \in W} \mathop{\mathbb{E}}_{J} \log(1 + (SNR(i)|J)) \lesssim b/k$$

and $|W| \geqslant 0.99k$ with 99% probability.

Goal for general k

Lemma (Key lemma for general k)

One can choose a set $W = W(J) \subset [k]$ of expected size $0.99k$ so that

$$I(Z_W; Ax) \lesssim m(b/k + \log k) + (b + k)$$

for any $A \in \mathbb{R}^{m \times N}$.

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- Recall $k = 1$ approach:

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Outline

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  - This is solvable nonadaptively in $O(k \log_C(n/k) \cdot \log^* k)$ measurements. [Price-Woodruff '12]

Basic approach: $R = 2$

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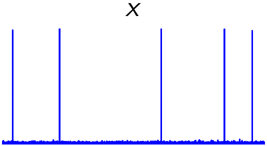
- Solution: *triple Gaussian hashing*.

Hashing



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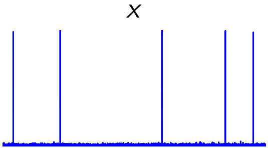

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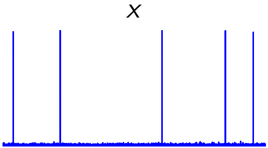

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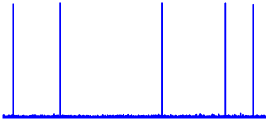

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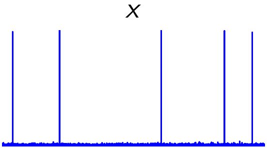

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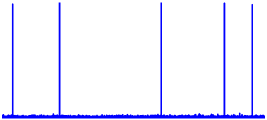

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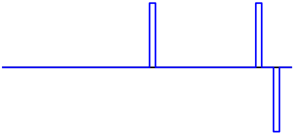
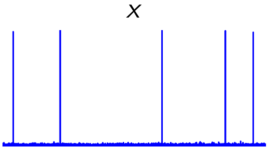


Without noise

Hashing



- For intuition, consider x being k-sparse binary + Gaussian with norm 1.
  - Successful recovery must find all but $O(1)$ binary entries of x.
- Given partition $h : [n] \to [B]$, how to condense $x \in \mathbb{R}^n$ into $y \in \mathbb{R}^B$?
  - Goal: preimage of k-sparse recovery on y includes large entries in x.
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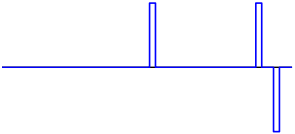
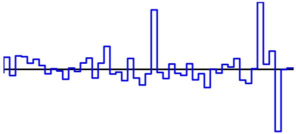
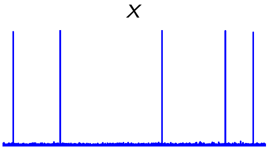


Without noise



With noise

Gaussian hashing



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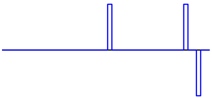
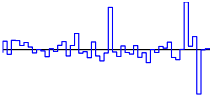


Without noise



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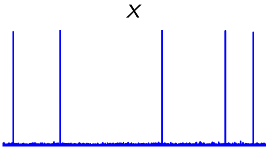

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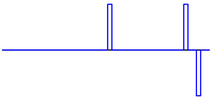

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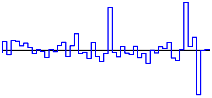

With noise

- Gaussian hashing: $g \sim N(I_n)$ and

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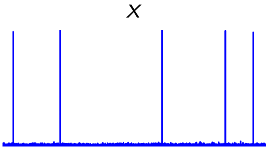

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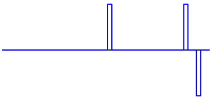

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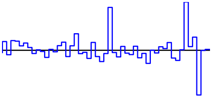

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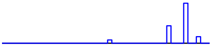

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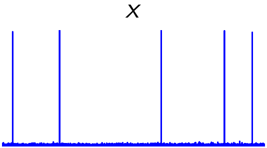

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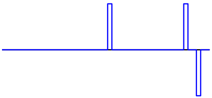

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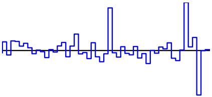

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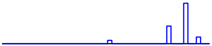
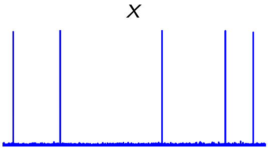


Without noise

With noise

Triple Gaussian hashing



- *Triple* Gaussian hashing: $g^1, g^2, g^3 \sim N(0, I_n)$;

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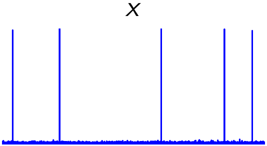

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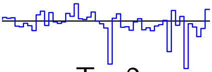

Try 2

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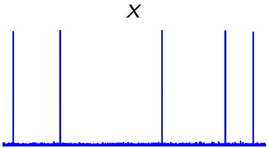

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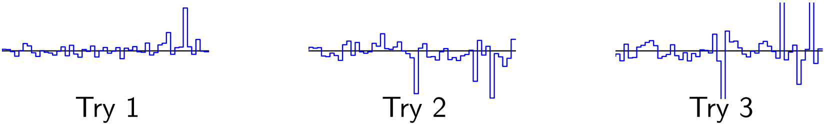

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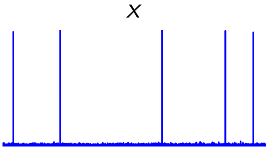

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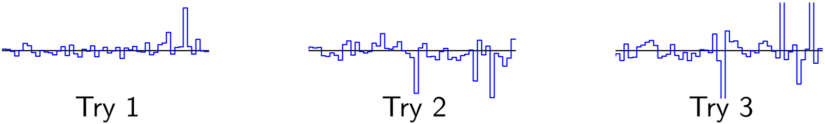

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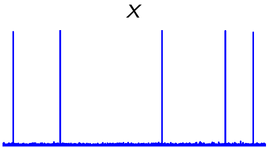

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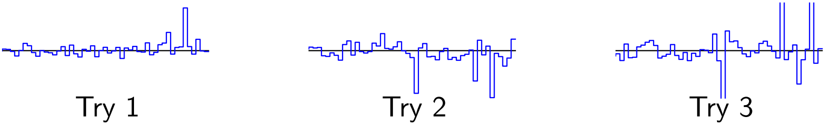

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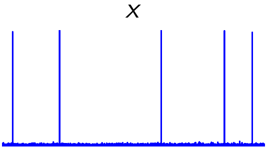

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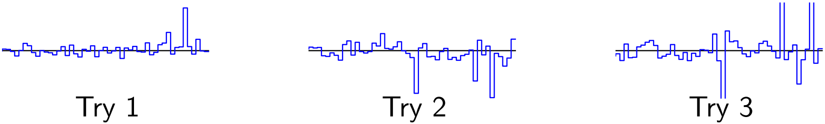

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- Avoids the cleanup rounds, getting

$$O(k \log^{1/R} n \cdot \log^* k)$$

measurements.

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Thank You