Randomized Algorithms for Computing Full Matrix Factorizations

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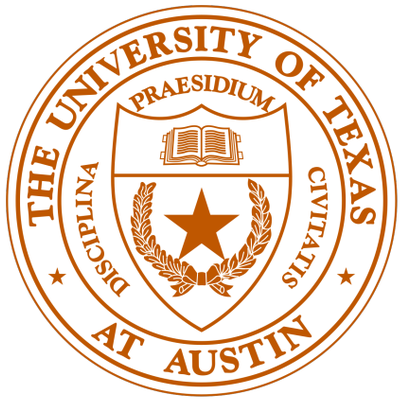

Background: Randomized methods such as the "randomized SVD (RSVD)" have proven effective at computing low rank approximations to matrices.

Topic of this talk: How to use methods based on randomized projections to efficiently compute *full* factorizations. Say you want *all* of the eigenvalues, or that the effective rank is not that much smaller than the matrix dimensions (say 10% or 50%).

Themes:

- Use randomization to reduce *communication* rather than flops.
- Quest for algorithms that spend most flops in matrix-matrix multiplications.
- Communication constrained environments:
  - Distributed memory computing.
  - Matrices stored "out-of-core" (say on a hard drive or an SSD).
  - GPU computing.

Outline:

- Review of randomized SVD.
- A randomized method for computing a UTV decomposition.
  (The UTV decomposition is a relaxation of a singular value decomposition.)
- A randomized method for computing a column pivoted QR decomposition.

Background/review: Randomized singular value decomposition (RSVD)

Problem: Given an $m \times n$ matrix $\mathbf{A}$, and a target rank k, where $k \ll \min(m, n)$, we seek to compute an approximate partial singular value decomposition:

$$\begin{array}{ccccc} \mathbf{A} & \approx & \mathbf{U} & \mathbf{D} & \mathbf{V}^*, \\ m \times n & & m \times k & k \times k & k \times n \end{array}$$

with $\mathbf{U}$ and $\mathbf{V}$ having orthonormal columns, and $\mathbf{D}$ diagonal.

Solution: Pick an over-sampling parameter p, say $p = 5$. Then proceed as follows:

1. Draw an $n \times (k + p)$ Gaussian random matrix $\mathbf{R}$. `R = randn(n,k+p)`
2. Form the $m \times (k + p)$ sample matrix $\mathbf{Y} = \mathbf{A}\,\mathbf{R}$. `Y = A * R`
3. Form an $m \times (k + p)$ orthonormal matrix $\mathbf{Q}$ s. t. $\text{col}(\mathbf{Y}) = \text{col}(\mathbf{Q})$. `[Q, ~] = qr(Y)`
4. Form the $(k + p) \times n$ matrix $\mathbf{B} = \mathbf{Q}^*\,\mathbf{A}$. `B = Q' * A`
5. Compute the SVD of $\mathbf{B}$ (small!): $\mathbf{B} = \hat{\mathbf{U}}\,\mathbf{D}\,\mathbf{V}^*$. `[Uhat, Sigma, V] = svd(B,'econ')`
6. Form the matrix $\mathbf{U} = \mathbf{Q}\,\hat{\mathbf{U}}$. `U = Q * Uhat`
7. Optional: Truncate the last p terms in the computed factors.

Background/review: Randomized singular value decomposition (RSVD)

Input: An $m \times n$ matrix $\mathbf{A}$, a target rank k, and an over-sampling parameter p (say $p = 5$).	
Output: Rank-$(k+p)$ factors $\mathbf{U}$, $\mathbf{D}$, and $\mathbf{V}$ in an approximate SVD $\mathbf{A} \approx \mathbf{UDV}^*$.	
(1) Draw an $n \times (k+p)$ random matrix $\mathbf{R}$.	(4) Form the small matrix $\mathbf{B} = \mathbf{Q}^* \mathbf{A}$.
(2) Form the $m \times (k+p)$ sample matrix $\mathbf{Y} = \mathbf{AR}$.	(5) Factor the small matrix $\mathbf{B} = \hat{\mathbf{U}}\mathbf{DV}^*$.
(3) Compute an ON matrix $\mathbf{Q}$ s.t. $\mathbf{Y} = \mathbf{QQ}^*\mathbf{Y}$.	(6) Form $\mathbf{U} = \mathbf{Q}\hat{\mathbf{U}}$.

Background/review: Randomized singular value decomposition (RSVD)

Input: An $m \times n$ matrix **A**, a target rank k, and an over-sampling parameter p (say $p = 5$).	
Output: Rank-$(k+p)$ factors **U**, **D**, and **V** in an approximate SVD $\mathbf{A} \approx \mathbf{UDV}^*$.	
(1) Draw an $n \times (k+p)$ random matrix **R**.	(4) Form the small matrix $\mathbf{B} = \mathbf{Q}^* \mathbf{A}$.
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(3) Compute an ON matrix **Q** s.t. $\mathbf{Y} = \mathbf{QQ}^*\mathbf{Y}$.	(6) Form $\mathbf{U} = \mathbf{Q}\hat{\mathbf{U}}$.

- Often faster than alternative algorithms for low-rank approximation (CPQR, Krylov, ...) on traditional CPU based platforms.
- Order of magnitude acceleration for data stored *out-of-core.*
- Single pass algorithms have been developed for *streaming environments.*
- Consider the problem of computing the dominant k eigenvectors/eigenvalues of a dense matrix of size $m \times n$. Reduction in complexity from $O(mnk)$ to $O(mn\log k)$.
- Highly efficient for GPU computing, or mobile computing (phones, etc).
- Well understood mathematically — detailed performance analysis.

Question: Can these ideas be applied for a *full* rank factorization?

Accelerate algorithms for FULL factorizations of matrices

Starting point (Demmel, Dumitriu, Holtz, 2007): Let $\mathbf{A}$ be an $n \times n$ matrix. We seek a rank-revealing UTV factorization $\mathbf{A} = \mathbf{UTV}^*$, with $\mathbf{U}, \mathbf{V}$ unitary, and $\mathbf{T}$ triangular.

Proceed as follows:

- Draw an $n \times n$ Gaussian matrix $\mathbf{G}$ and orthonormalize its columns $[\mathbf{V}, \sim] = \mathtt{qr}(\mathbf{G})$.
- Form a QR factorization of $\mathbf{AV}$ so that $\mathbf{AV} = \mathbf{UT}$.

Then $\mathbf{A} = \mathbf{UTV}^*$ is provably "rank-revealing." But in a *very* weak sense.

Improved Demmel UTV (with power iteration): Same set-up.

- Draw an $n \times n$ Gaussian matrix $\mathbf{G}$ and compute $\mathbf{Y} = \left(\mathbf{A}^*\mathbf{A}\right)^q \mathbf{G}$ for $q = 1$ or 2.
- Orthonormalize the columns of $\mathbf{Y}$ so that $[\mathbf{V}, \sim] = \mathtt{qr}(\mathbf{Y})$.
- Form a QR factorization of $\mathbf{AV}$ so that $\mathbf{AV} = \mathbf{UT}$.

Then $\mathbf{A} = \mathbf{UTV}^*$ is "rank-revealing." Very good for $q = 1$. Excellent for $q = 2$.

These algorithms require a huge number of flops!
But much faster in practice than, say, CPQR.

Key fact: The matrix-matrix multiply can be done *very* rapidly in many environments . . . GPU, distributed memory, fast algorithms, Strassen, etc.

Numerical results for the "Demmel URV" factorization

There are many different ways to measure the quality of a rank-revealing factorization. Let us describe one common measure: Let $\mathbf{A}$ be an $n \times n$ matrix factored as

$$\mathbf{A} = \mathbf{UTV}^*$$

where $\mathbf{U}$ and $\mathbf{V}$ are unitary, and where $\mathbf{T}$ is upper triangular. Define for $k \in \{1, 2, \ldots, n-1\}$ the quantities

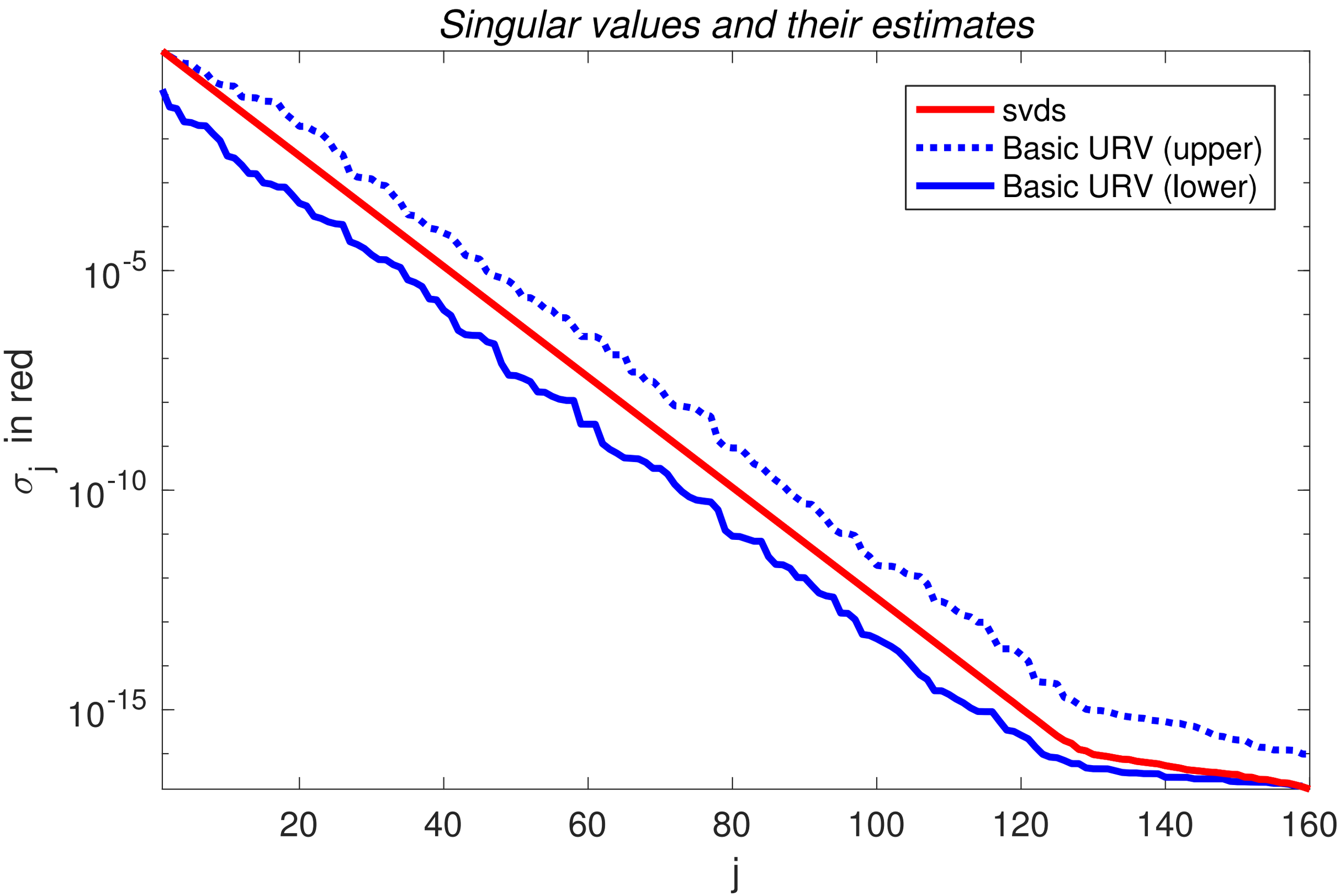
$$\nu_k = \sigma_k(\mathbf{T}(1:k, 1:k)),$$
$$\tau_{k+1} = \sigma_1(\mathbf{T}((k+1):n, (k+1):n)),$$

where $\sigma_j(\mathbf{X})$ denotes the j'th singular value of $\mathbf{X}$. One can easily prove that

$$\nu_k \leq \sigma_k(\mathbf{A}) \leq \tau_k.$$

The more tightly that (ν_k, τ_k) constrains the k'th singular value, the better.

Numerical experiments illustrating the errors in the URV factorization



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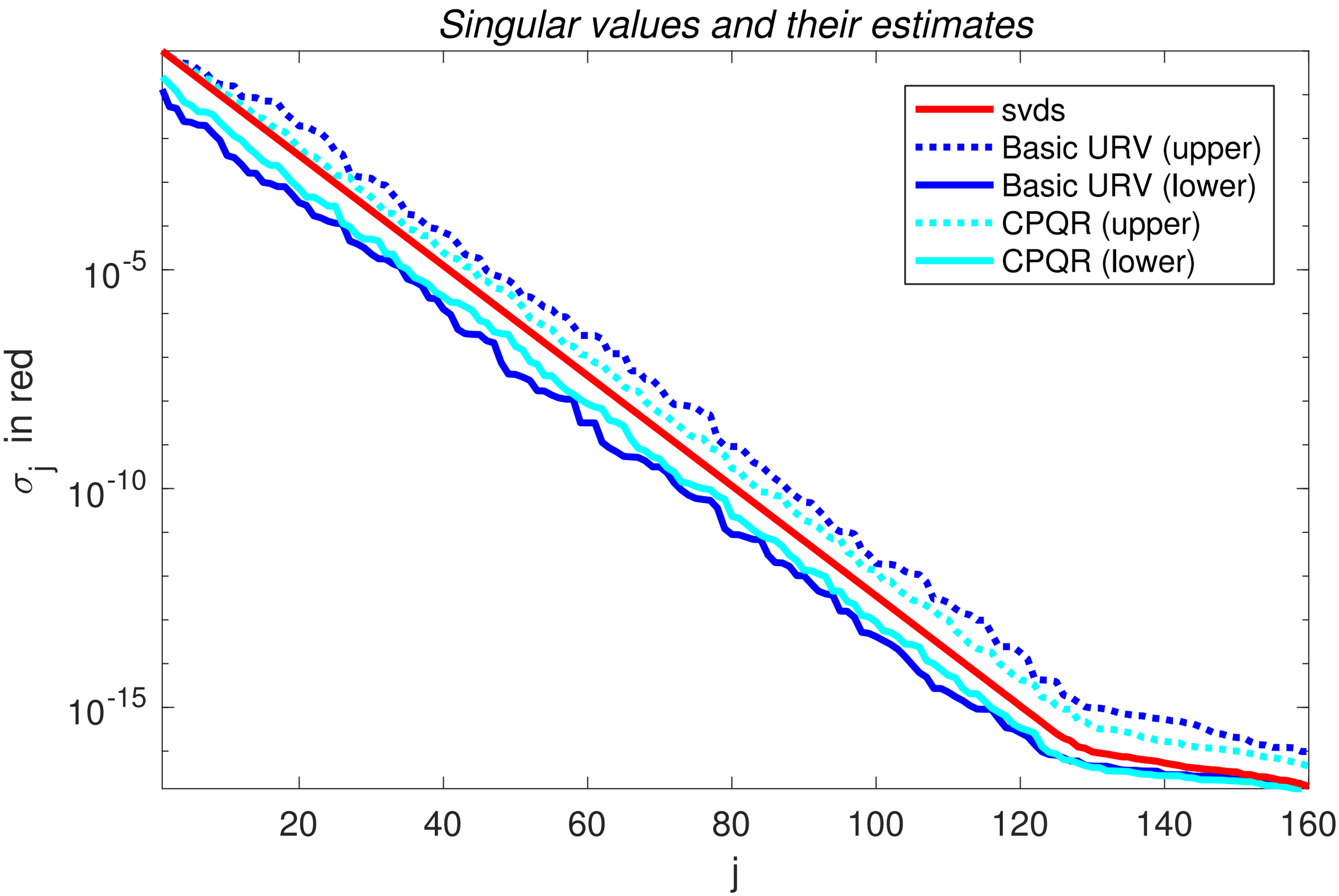

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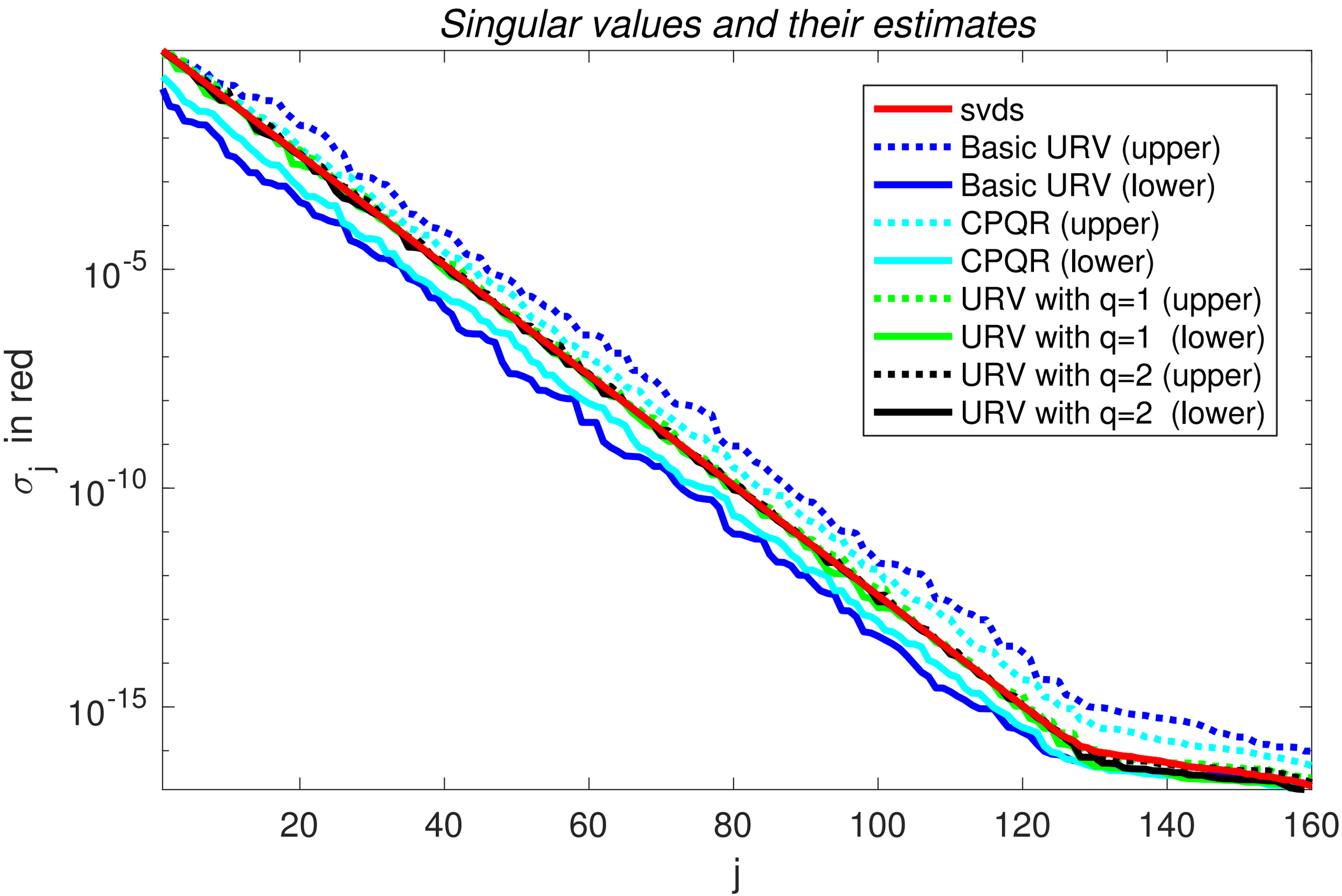

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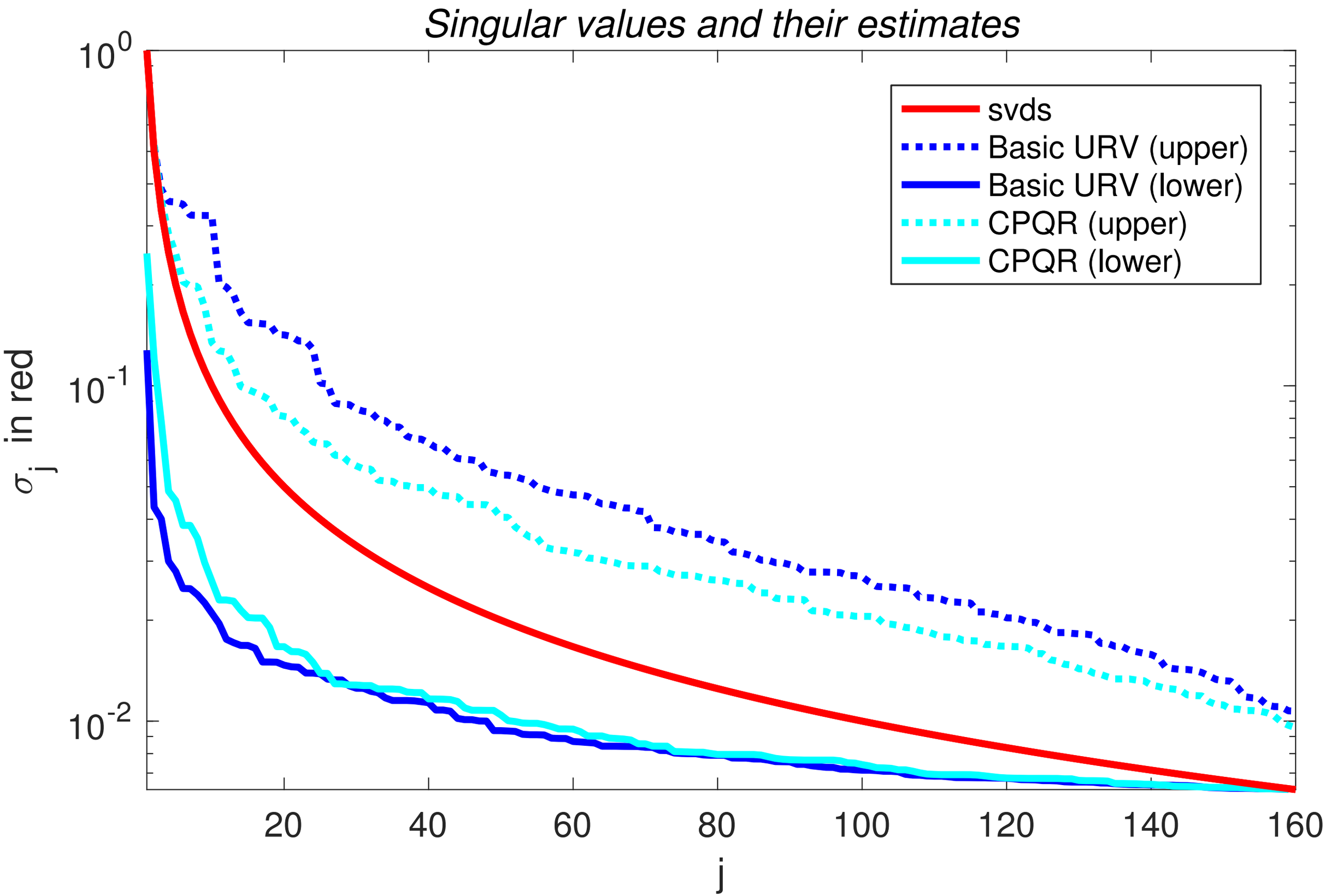

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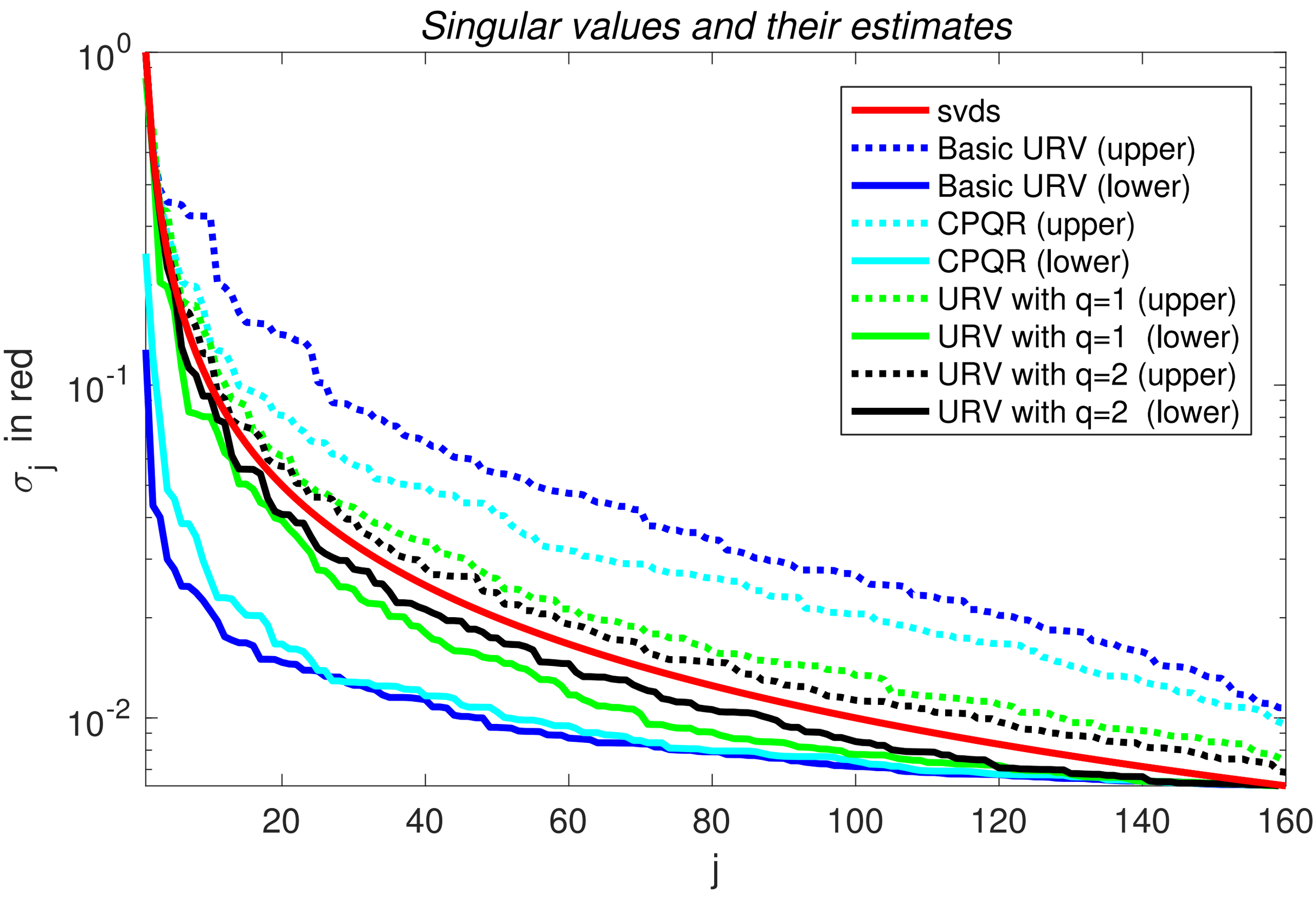

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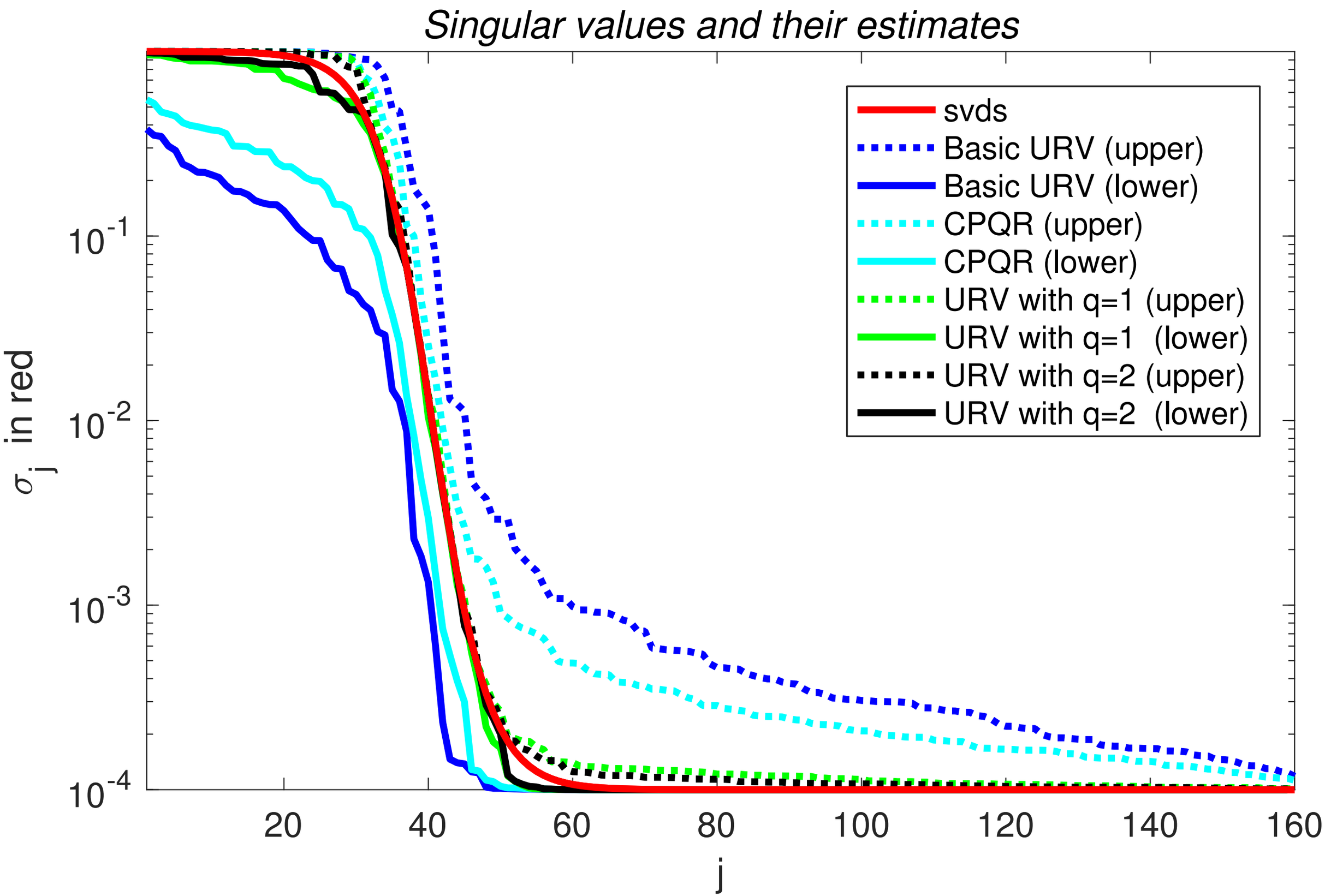

The UTV decomposition: A rank-revealing factorization

Given a dense $m \times n$ matrix $\mathbf{A}$, with $m \geq n$, compute a factorization

$$\begin{array}{cccccc} \mathbf{A} & = & \mathbf{U} & \mathbf{T} & \mathbf{V}^*, \\ m \times n & & m \times n & n \times n & n \times n \end{array} \tag{1}$$

where $\mathbf{T}$ is upper triangular, and $\mathbf{U}$ and $\mathbf{V}$ are unitary. We want a factorization that is "rank-revealing", in the sense its truncation to a rank-k approximation should be of close to optimal accuracy. We also would like for the diagonal entries of $\mathbf{T}$ to approximate the singular values of $\mathbf{A}$.

A rank-revealing factorization has many uses:

- Finding a low-rank approximation to a matrix. (Obviously!)
- Solving ill-conditioned linear systems, or linear regression problems.
- Finding bases for fundamental subspaces.

Basically, when (1) is rank-revealing, it can be used for almost anything that the SVD is recommended for.

The UTV decomposition: Overview of proposed algorithm randUTV

Given a dense $m \times n$ matrix $\mathbf{A}$, with $m \geq n$, compute a factorization

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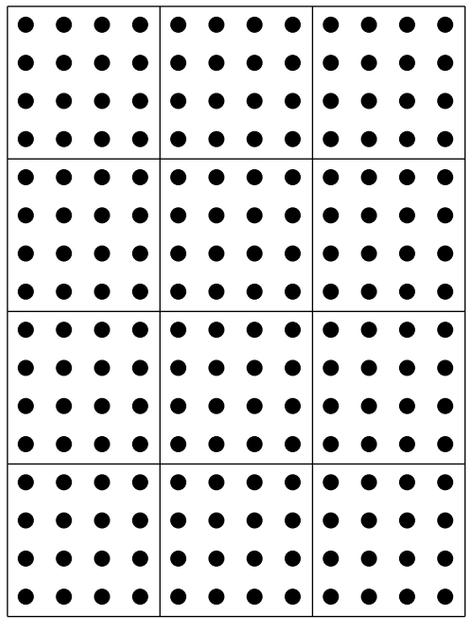
The technique proposed drives $\mathbf{A}$ to upper triangular form via unitary transformations:



$$\mathbf{A}_0 = \mathbf{A}$$

The UTV decomposition: Overview of proposed algorithm randUTV

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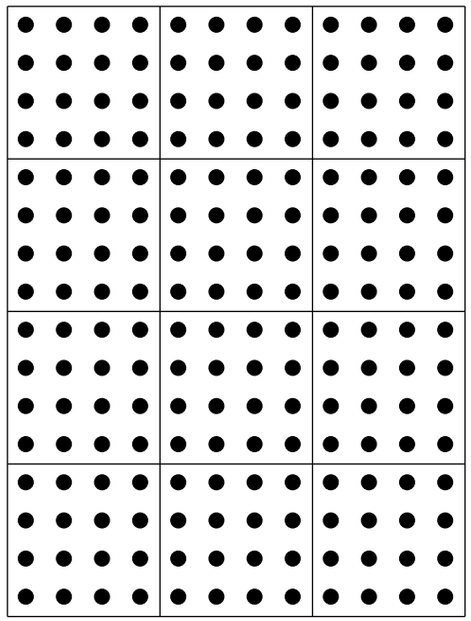

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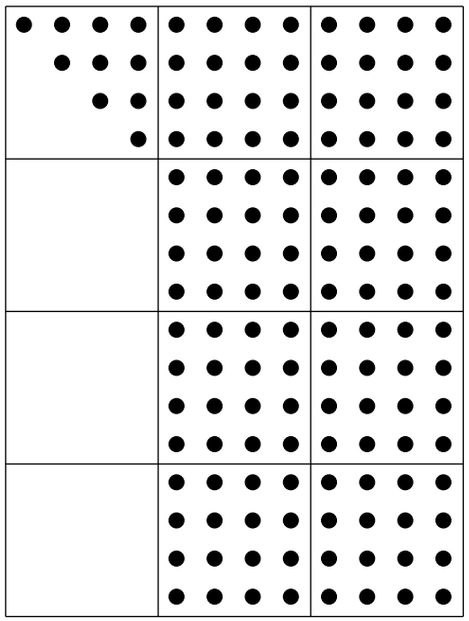

$\mathbf{A}_1 = \mathbf{U}_1^* \mathbf{A}_0 \mathbf{V}_1$

The UTV decomposition: Overview of proposed algorithm randUTV

Given a dense $m \times n$ matrix $\mathbf{A}$, with $m \geq n$, compute a factorization

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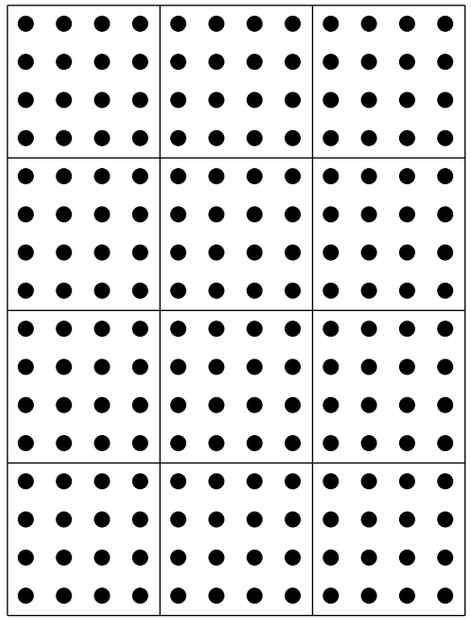

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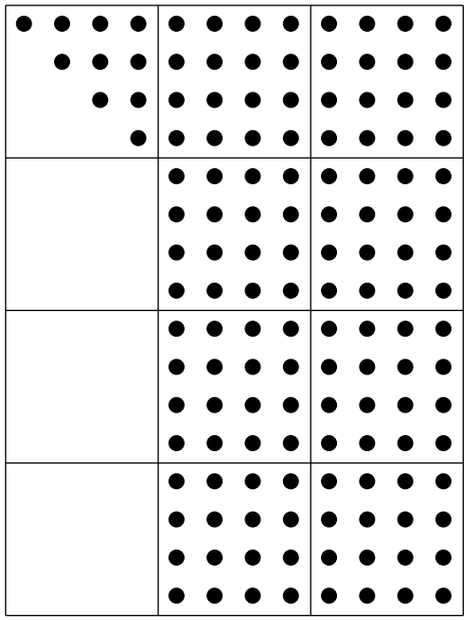

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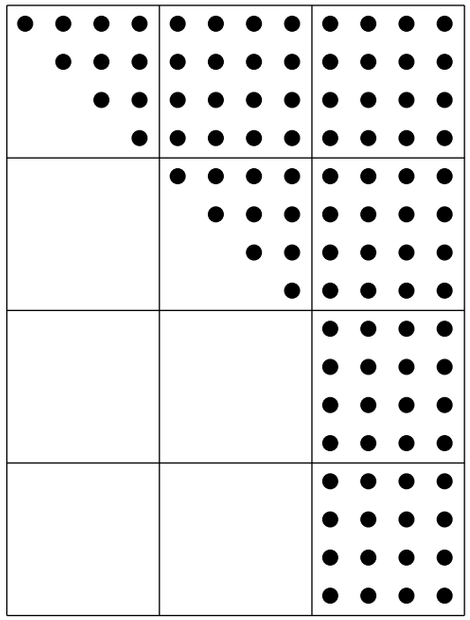

$\mathbf{A}_2 = \mathbf{U}_2^* \mathbf{A}_1 \mathbf{V}_2$

The UTV decomposition: Overview of proposed algorithm randUTV

Given a dense $m \times n$ matrix $\mathbf{A}$, with $m \geq n$, compute a factorization

$$\underset{m \times n}{\mathbf{A}} = \underset{m \times n}{\mathbf{U}} \; \underset{n \times n}{\mathbf{T}} \; \underset{n \times n}{\mathbf{V}^*},$$

where $\mathbf{T}$ is upper triangular, and $\mathbf{U}$ and $\mathbf{V}$ are unitary.

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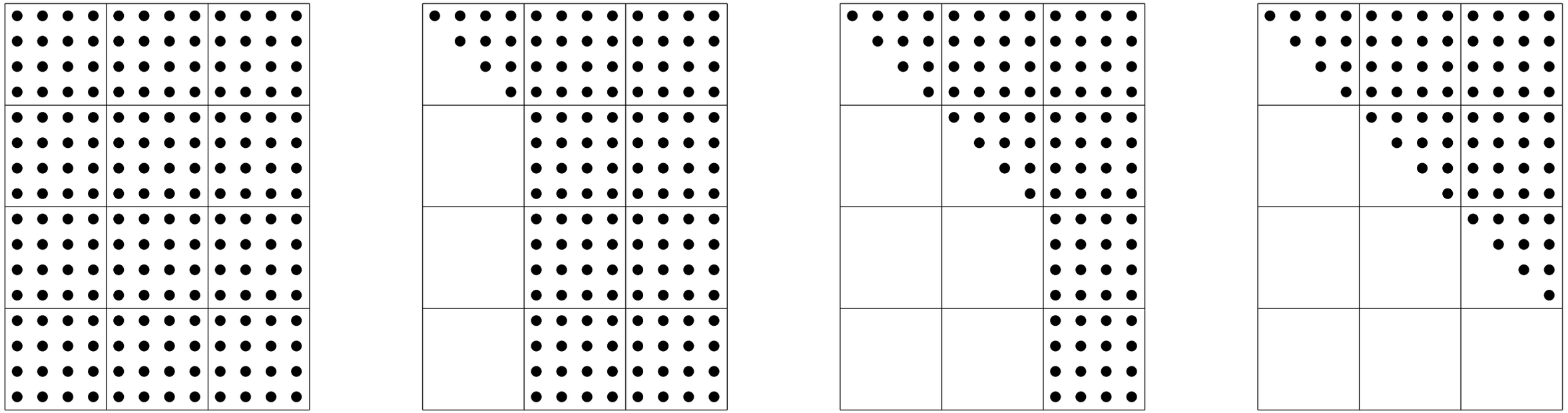

$\mathbf{A}_0 = \mathbf{A}$ $\quad$ $\mathbf{A}_1 = \mathbf{U}_1^* \mathbf{A}_0 \mathbf{V}_1$ $\quad$ $\mathbf{A}_2 = \mathbf{U}_2^* \mathbf{A}_1 \mathbf{V}_2$ $\quad$ $\mathbf{A}_3 = \mathbf{U}_3^* \mathbf{A}_2 \mathbf{V}_3$

Both $\mathbf{U}_j$ and $\mathbf{V}_j$ are (mostly...) products of b Householder reflectors.

Blocking enables high performance. Most flops are spent in matrix-matrix multiplication.

The UTV decomposition: A single blocked step

Consider a single blocked step: We apply unitary matrices $\mathbf{U}$ and $\mathbf{V}$ to get

$$\mathbf{T} = \mathbf{U}^*\mathbf{A}\mathbf{V}.$$

Let b be a block size, and separate out the first b rows and columns so that

$$\mathbf{T} = \begin{bmatrix} \mathbf{U}_1^* \\ \mathbf{U}_2^* \end{bmatrix} \mathbf{A} \begin{bmatrix} \mathbf{V}_1 & \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{0} & \mathbf{T}_{22} \end{bmatrix}.$$

We want the following properties in the transformed matrix $\mathbf{T}$:

- $\mathbf{T}_{11}$ should hold as *much* mass as possible.
- $\mathbf{T}_{12}$ should be tiny.

A *perfect* choice of $\mathbf{U}$ and $\mathbf{V}$ would be:

- The columns of $\mathbf{U}_1$ span the space spanned by the first k left singular vectors.
- The columns of $\mathbf{V}_1$ span the space spanned by the first k right singular vectors.

We use randomization to cheaply find a *close to optimal* choice:

$$\mathbf{Y} = (\mathbf{A}^*\mathbf{A})^q \mathbf{A}^*\mathbf{G}, \qquad [\mathbf{V}, \sim] = \mathtt{qr}(\mathbf{Y}),$$

where $\mathbf{G}$ is an $m \times b$ Gaussian random matrix, and where $q \in \{0, 1, 2\}$.

(Over-sampling can be used as well.)

The UTV decomposition: Overview of proposed algorithm randUTV

Given a dense $m \times n$ matrix $\mathbf{A}$, with $m \geq n$, compute a factorization

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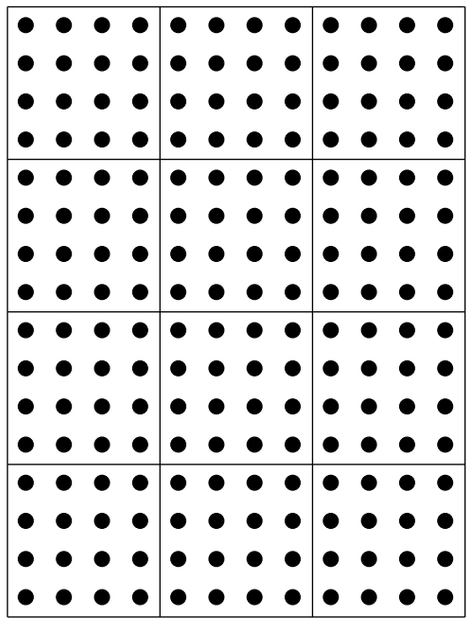
The technique proposed drives $\mathbf{A}$ to upper triangular form via unitary transformations:



$$\mathbf{A}_0 = \mathbf{A}$$

The UTV decomposition: Overview of proposed algorithm randUTV

Given a dense $m \times n$ matrix $\mathbf{A}$, with $m \geq n$, compute a factorization

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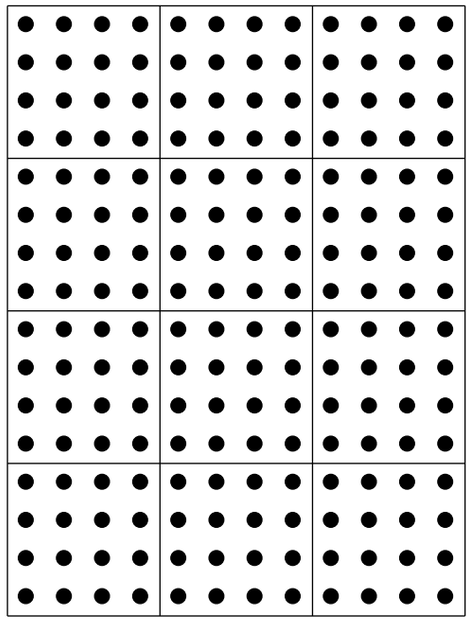
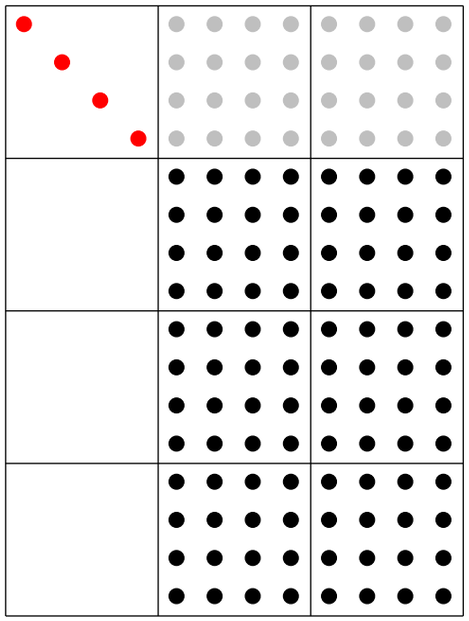


$\mathbf{A}_0 = \mathbf{A}$



$\mathbf{A}_1 = \mathbf{U}_1^* \mathbf{A}_0 \mathbf{V}_1$

The UTV decomposition: Overview of proposed algorithm randUTV

Given a dense $m \times n$ matrix $\mathbf{A}$, with $m \geq n$, compute a factorization

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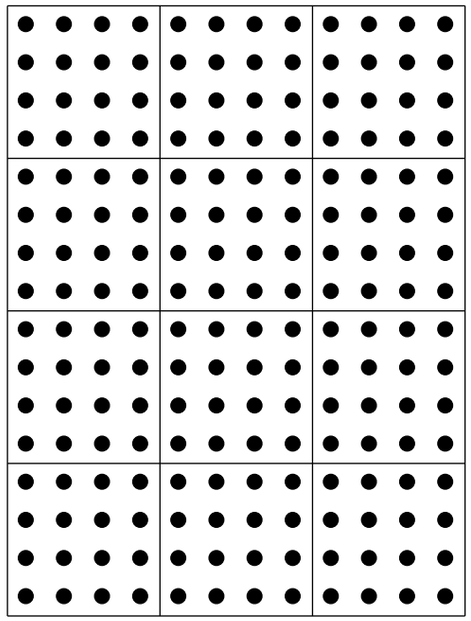

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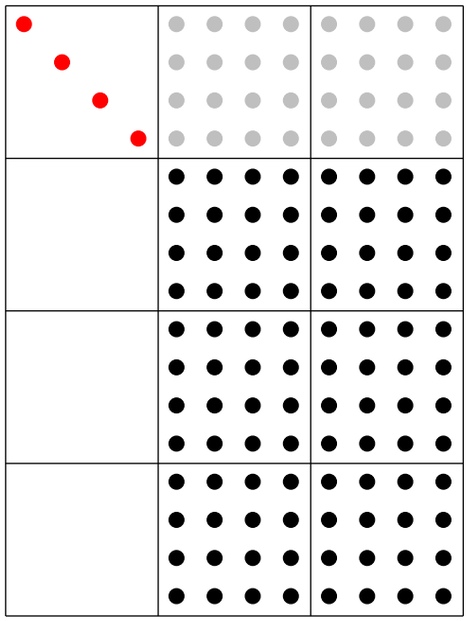
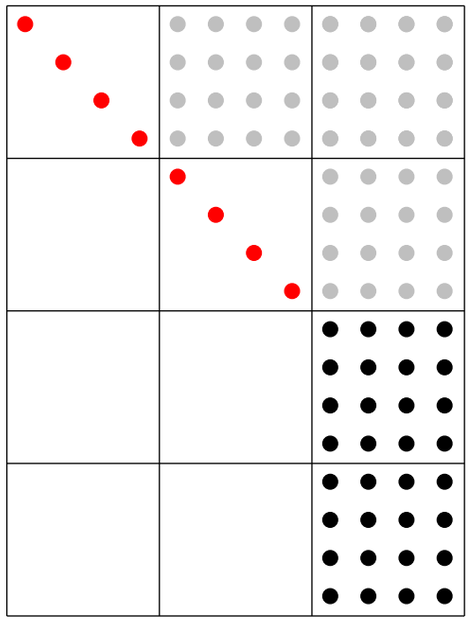


$\mathbf{A}_1 = \mathbf{U}_1^* \mathbf{A}_0 \mathbf{V}_1$



$\mathbf{A}_2 = \mathbf{U}_2^* \mathbf{A}_1 \mathbf{V}_2$

The UTV decomposition: Overview of proposed algorithm randUTV

Given a dense $m \times n$ matrix $\mathbf{A}$, with $m \geq n$, compute a factorization

$$\underset{m \times n}{\mathbf{A}} = \underset{m \times n}{\mathbf{U}} \quad \underset{n \times n}{\mathbf{T}} \quad \underset{n \times n}{\mathbf{V}^*},$$

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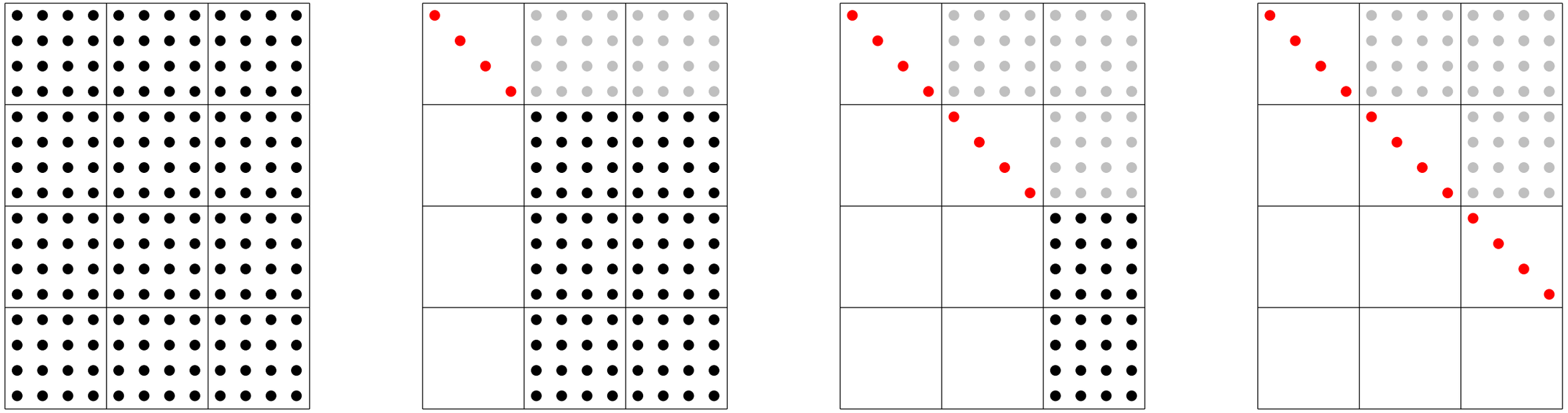


$\mathbf{A}_0 = \mathbf{A}$ $\quad$ $\mathbf{A}_1 = \mathbf{U}_1^* \mathbf{A}_0 \mathbf{V}_1$ $\quad$ $\mathbf{A}_2 = \mathbf{U}_2^* \mathbf{A}_1 \mathbf{V}_2$ $\quad$ $\mathbf{A}_3 = \mathbf{U}_3^* \mathbf{A}_2 \mathbf{V}_3$

The $\mathbf{V}$ matrices are found using the randomized projections. (Basically RSVD.)

The $\mathbf{U}$ matrices zero out the sub-diagonal elements.

Both $\mathbf{U}$ and $\mathbf{V}$ must be represented efficiently as products of Householder reflectors.

A full, but small (of size $b \times b$) SVD is used to diagonalize the diagonal blocks.

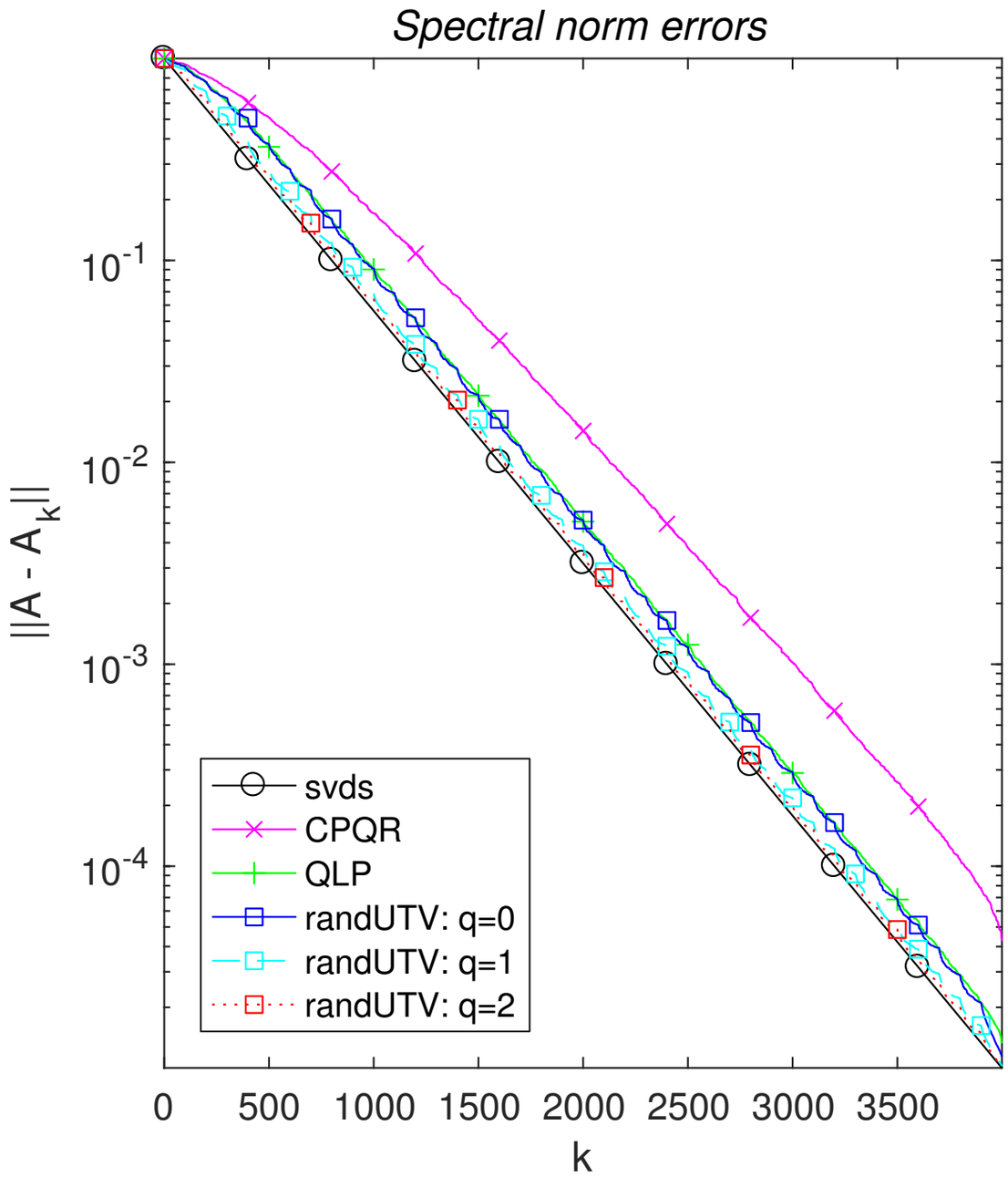
The super-diagonal elements are very small — often of relative size 10^{-5} *or so!*

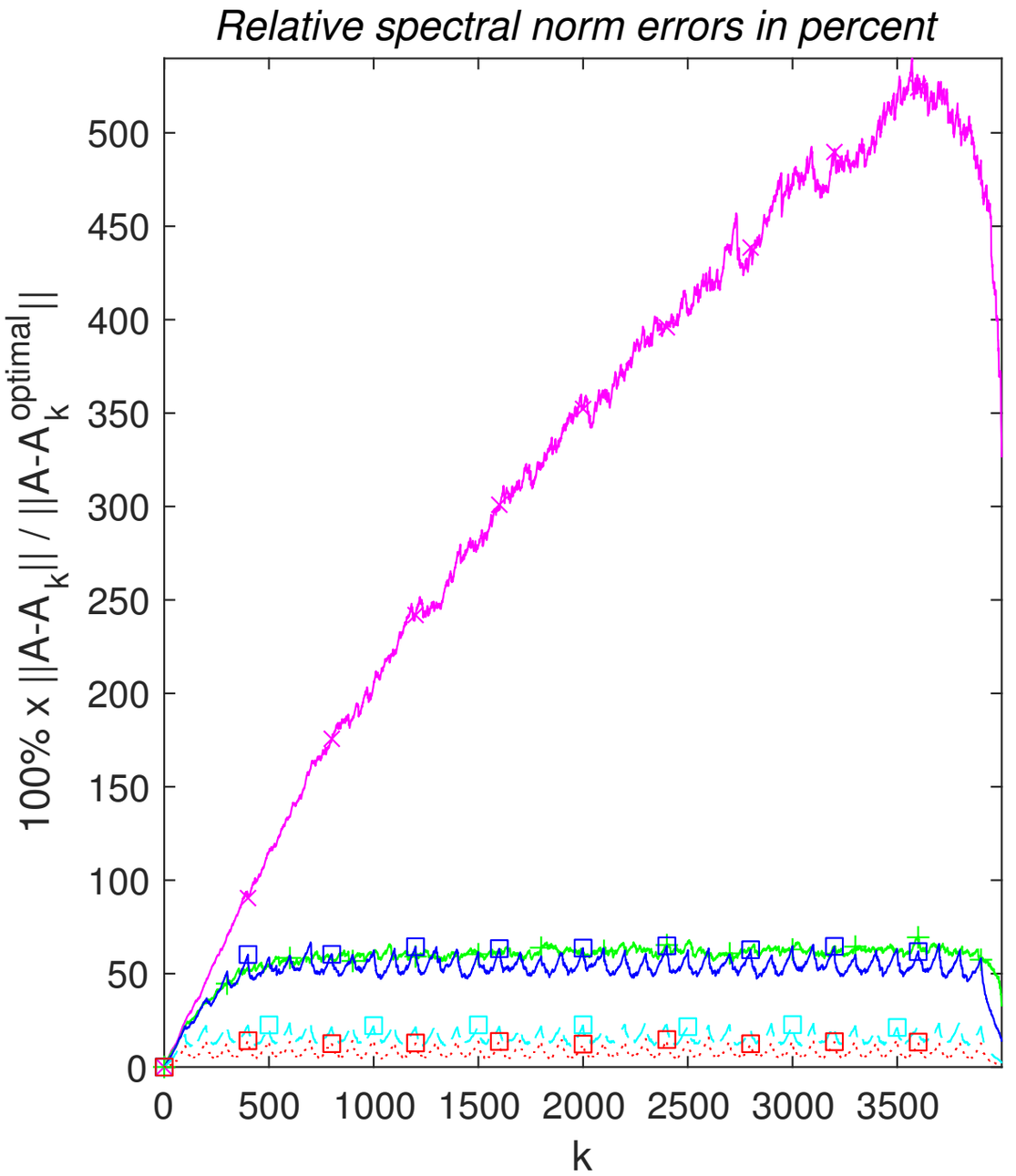
```
function [U,T,V] = randUTV(A,b,q)
  T = A;
  U = eye(size(A,1));
  V = eye(size(A,2));
  for i = 1:ceil(size(A,2)/b)
    I1 = 1:(b*(i-1));
    I2 = (b*(i-1)+1):size(A,1);
    J2 = (b*(i-1)+1):size(A,2);
    if (length(J2) > b)
      [UU,TT,VV] = stepUTV(T(I2,J2),b,q);
    else
      [UU,TT,VV] = svd(T(I2,J2));
    end
    U(:,I2)  = U(:,I2)*UU;
    V(:,J2)  = V(:,J2)*VV;
    T(I2,J2) = TT;
    T(I1,J2) = T(I1,J2)*VV;
  end
return
```

```
function [U,T,V] = stepUTV(A,b,q)
  G = randn(size(A,1),b);
  Y = A'*G;
  for i = 1:q
    Y = A'*(A*Y);
  end
  [V,~]    = qr(Y);
  [U,D,W]  = svd(A*V(:,1:b));
  T        = [D,U'*A*...
                V(:,(b+1):end)];
  V(:,1:b) = V(:,1:b)*W;
return
```

Matlab code for the algorithm randUTV that given an $m \times n$ matrix $\mathbf{A}$ *computes its UTV factorization* $\mathbf{A} = \mathbf{UTV}^*$*. The input parameters b and q reflect the block size and the number of steps of power iteration, respectively. In actual implementations, all unitary matrices are stored as products of Householder reflectors.*

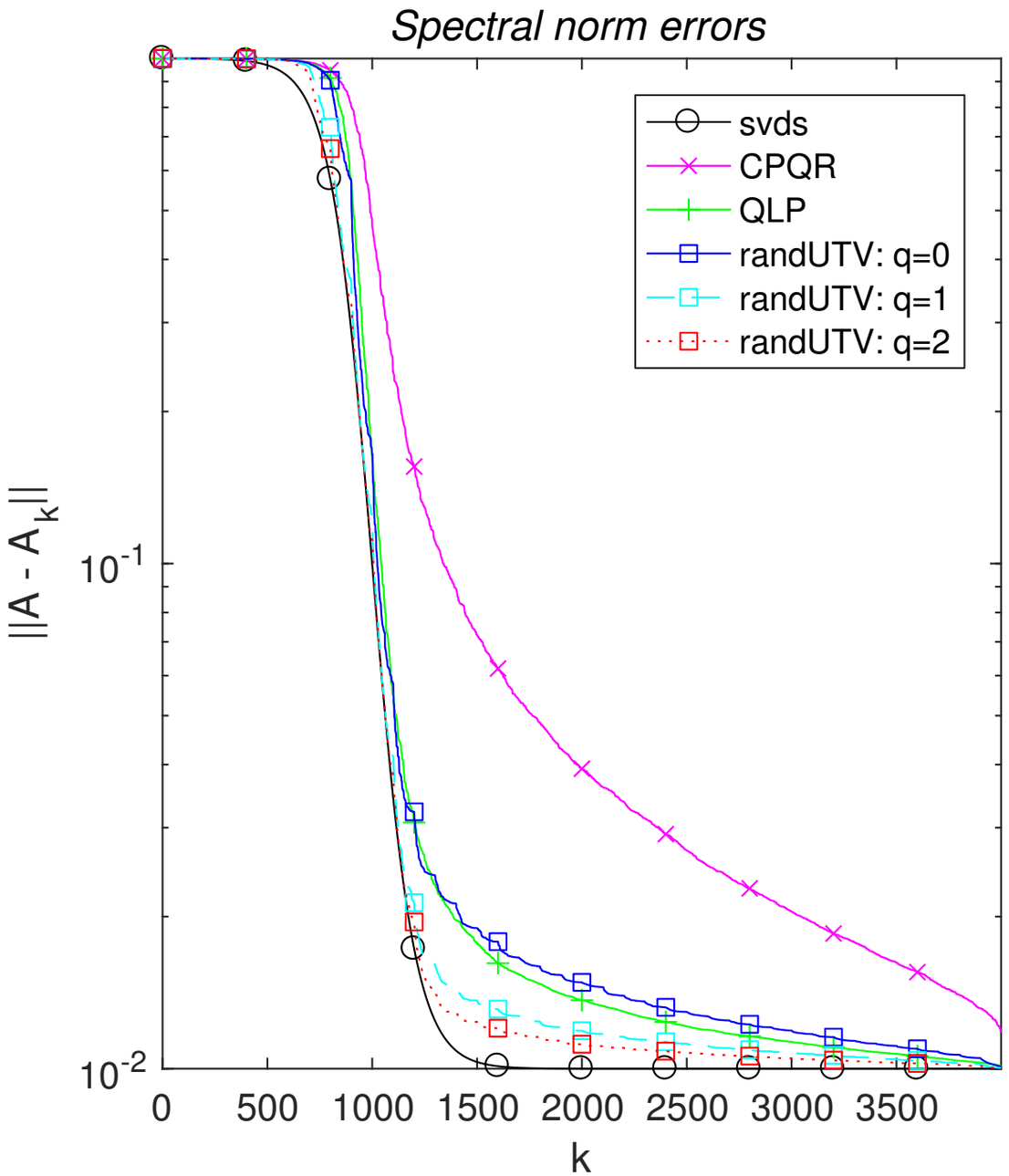
Numerical experiments illustrating the errors in the UTV factorization

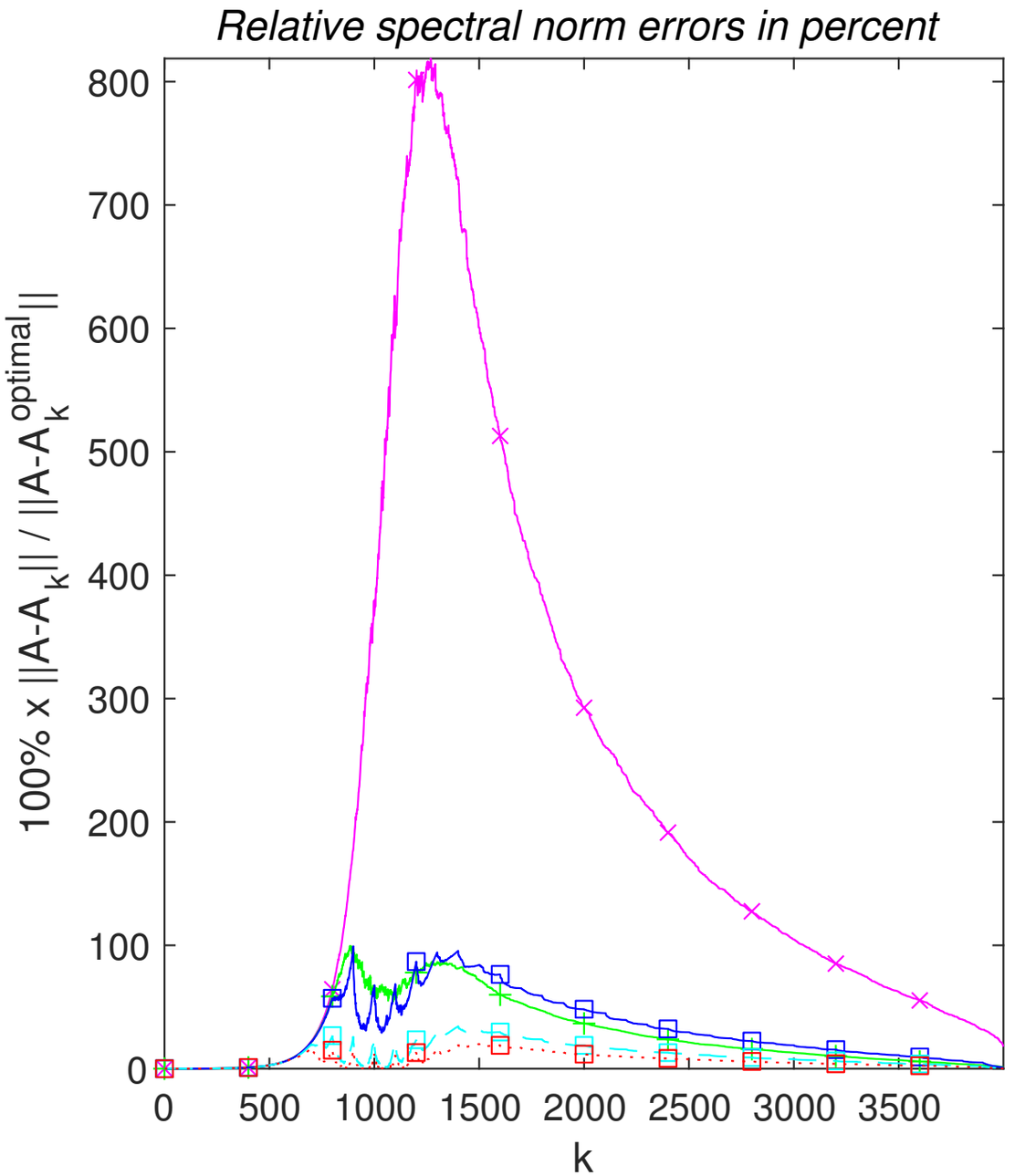




Rank-k approximation errors for the matrix "Fast Decay" of size 4000×4000. *The block size was* $b = 100$. *Left: Absolute errors in spectral norm. The black line (circles) marks the theoretically minimal errors. Right: Relative errors* $e_k^{\text{relative}} = 100\% \times \dfrac{\|\mathbf{A} - \mathbf{A}_k\|}{\|\mathbf{A} - \mathbf{A}_k^{\text{optimal}}\|}$.

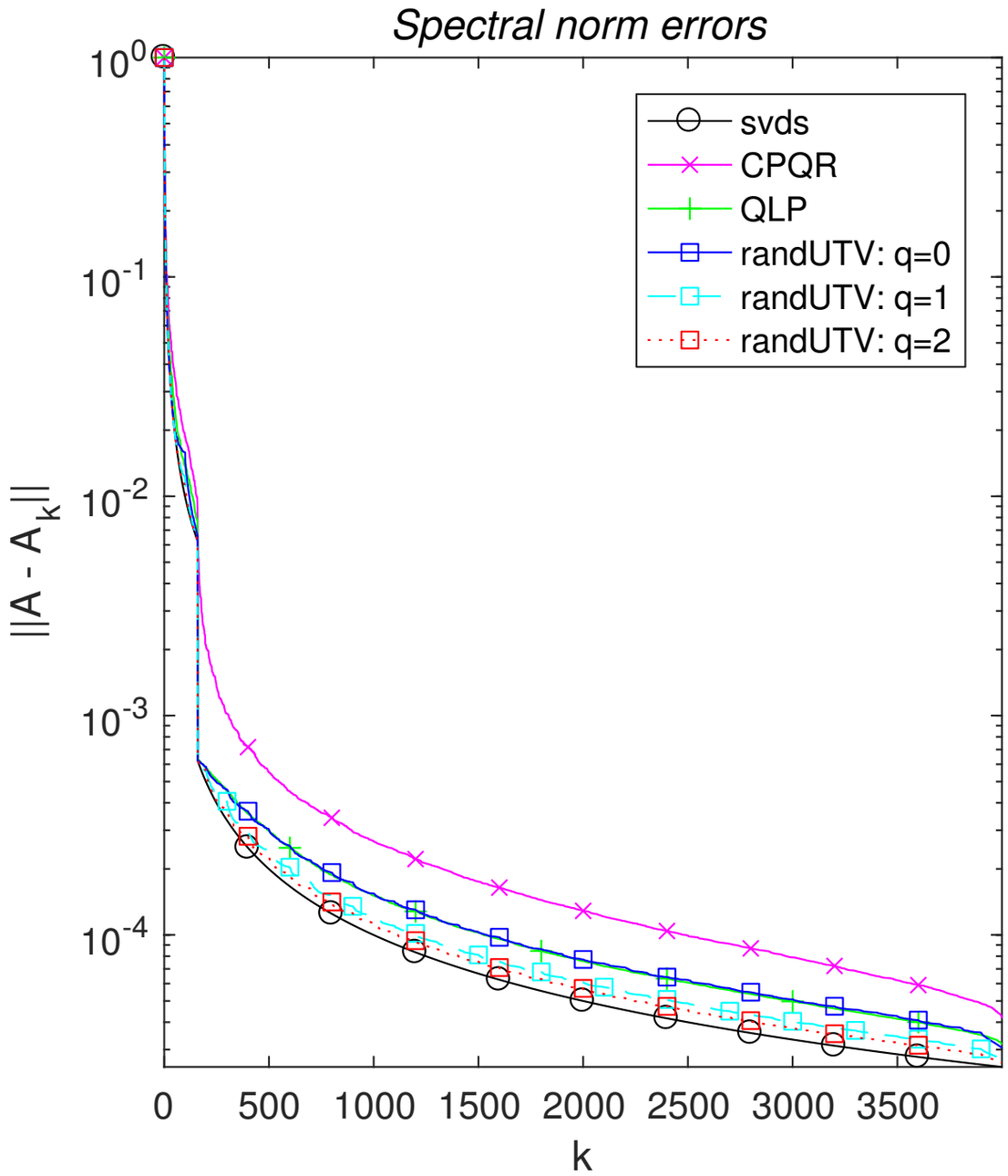
Numerical experiments illustrating the errors in the UTV factorization

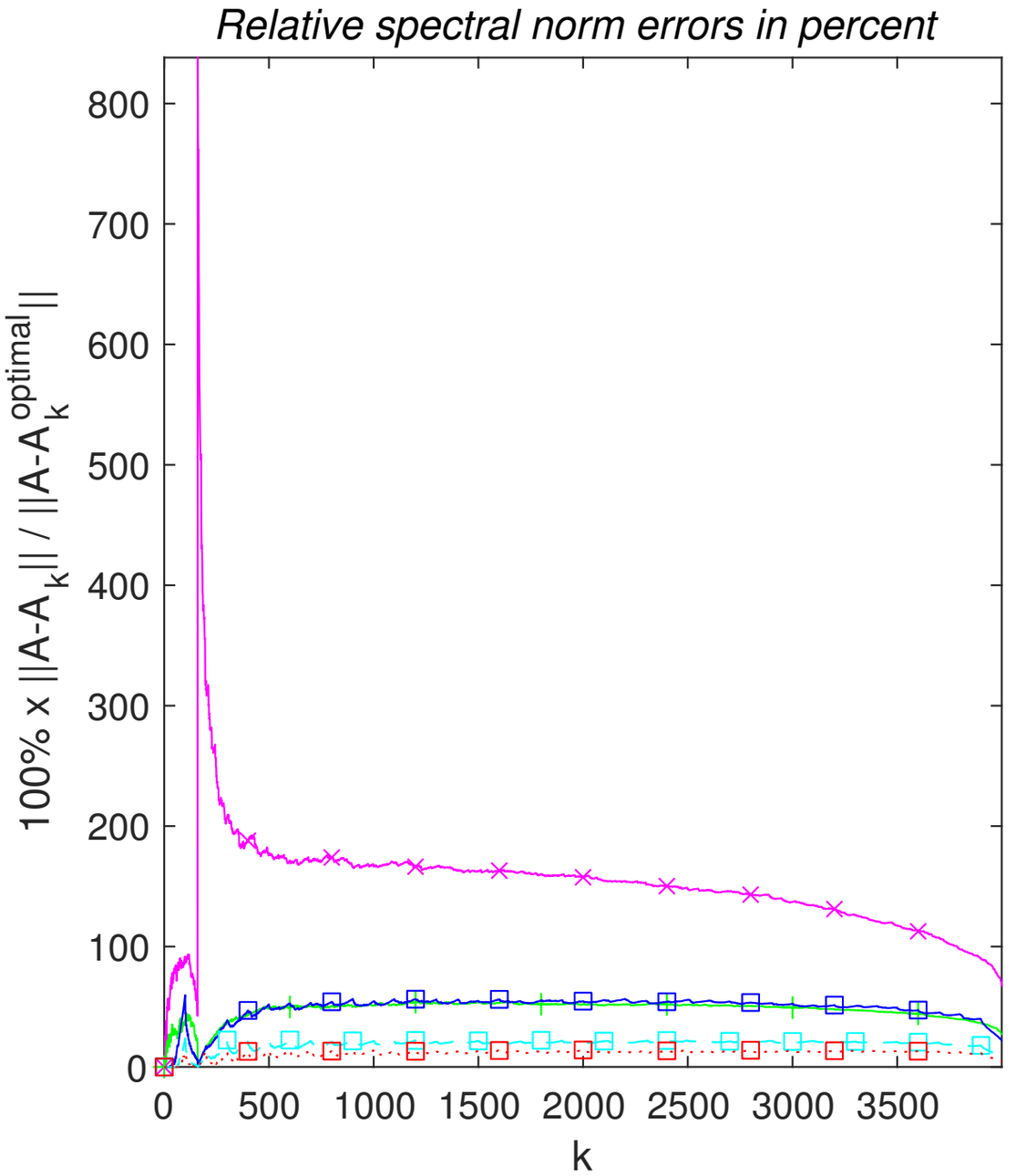




Rank-k approximation errors for the matrix "S-shape" of size 4000×4000. *The block size was* $b = 100$. *Left: Absolute errors in spectral norm. The black line (circles) marks the theoretically minimal errors. Right: Relative errors* $e_k^{\text{relative}} = 100\% \times \dfrac{\|\mathbf{A} - \mathbf{A}_k\|}{\|\mathbf{A} - \mathbf{A}_k^{\text{optimal}}\|}$.

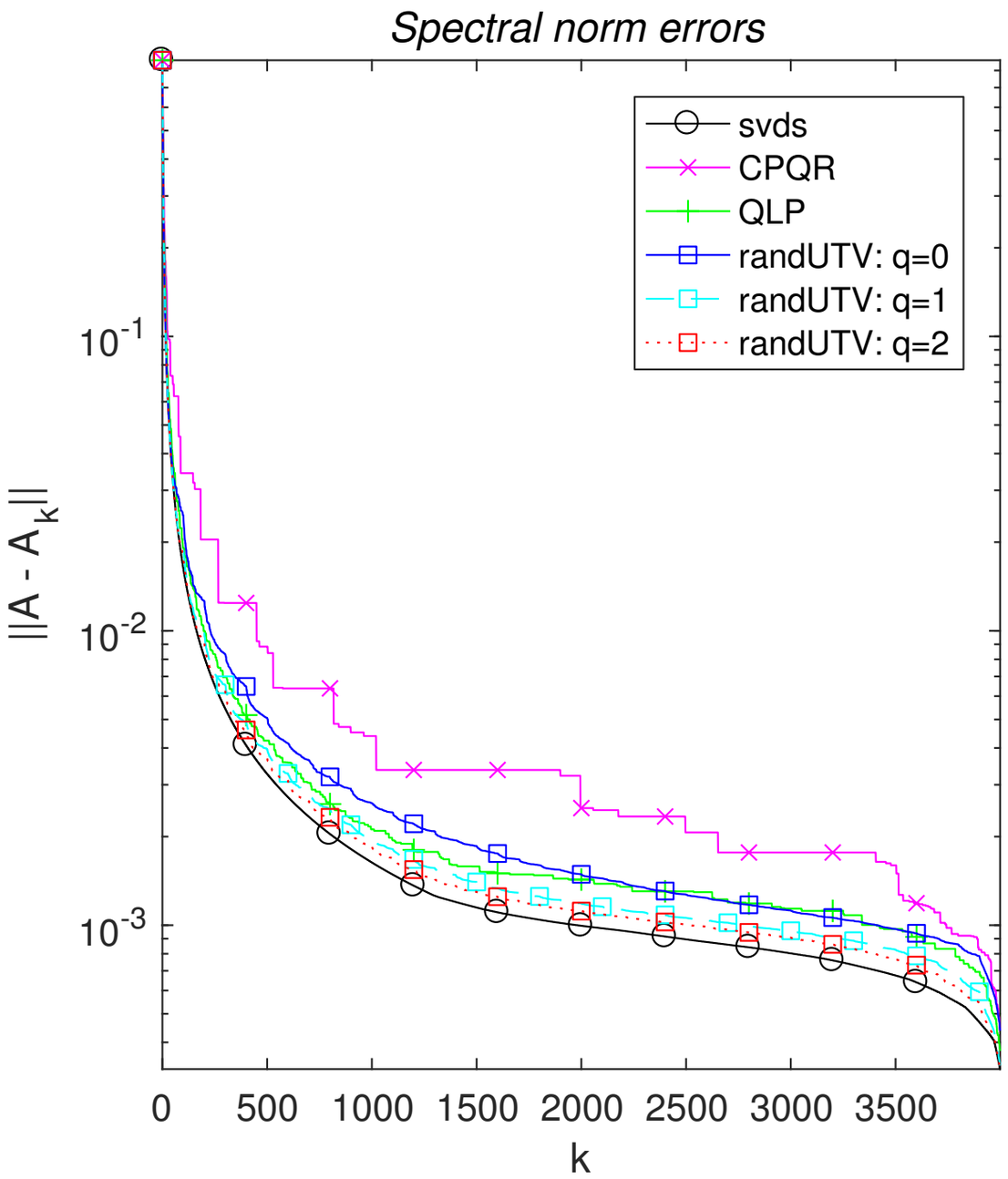
Numerical experiments illustrating the errors in the UTV factorization

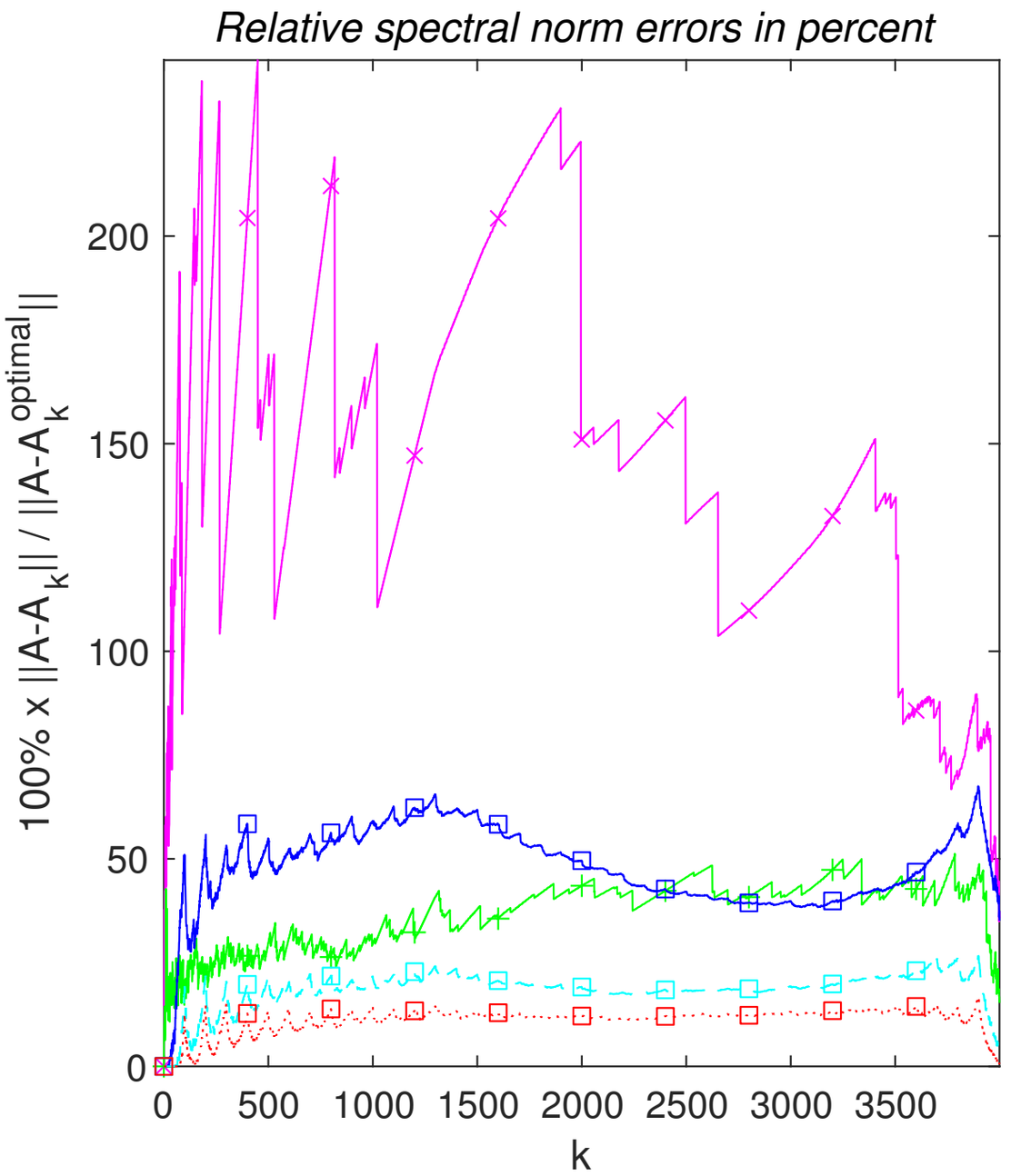




Rank-k approximation errors for the matrix "Gap" of size 4000×4000. *The block size was* $b = 100$. *Left: Absolute errors in spectral norm. The black line (circles) marks the theoretically minimal errors. Right: Relative errors* $e_k^{\text{relative}} = 100\% \times \dfrac{\|\mathbf{A} - \mathbf{A}_k\|}{\|\mathbf{A} - \mathbf{A}_k^{\text{optimal}}\|}$.

Numerical experiments illustrating the errors in the UTV factorization





Rank-k approximation errors for the matrix "BIE" of size 4000×4000. *The block size was* $b = 100$. *Left: Absolute errors in spectral norm. The black line (circles) marks the theoretically minimal errors. Right: Relative errors* $e_k^{\text{relative}} = 100\% \times \dfrac{\|\mathbf{A} - \mathbf{A}_k\|}{\|\mathbf{A} - \mathbf{A}_k^{\text{optimal}}\|}$.

Numerical experiments illustrating how close the UTV is to the SVD

As a consequence of the fact that the super-diagonal elements of **T** are very small, the diagonal elements of **T** are excellent approximants to the singular values of **A**:

$$\mathbf{T}(j,j) \approx \sigma_j, \qquad j = 1,\, 2\, \ldots,\, \min(m,n).$$

Question: How good?

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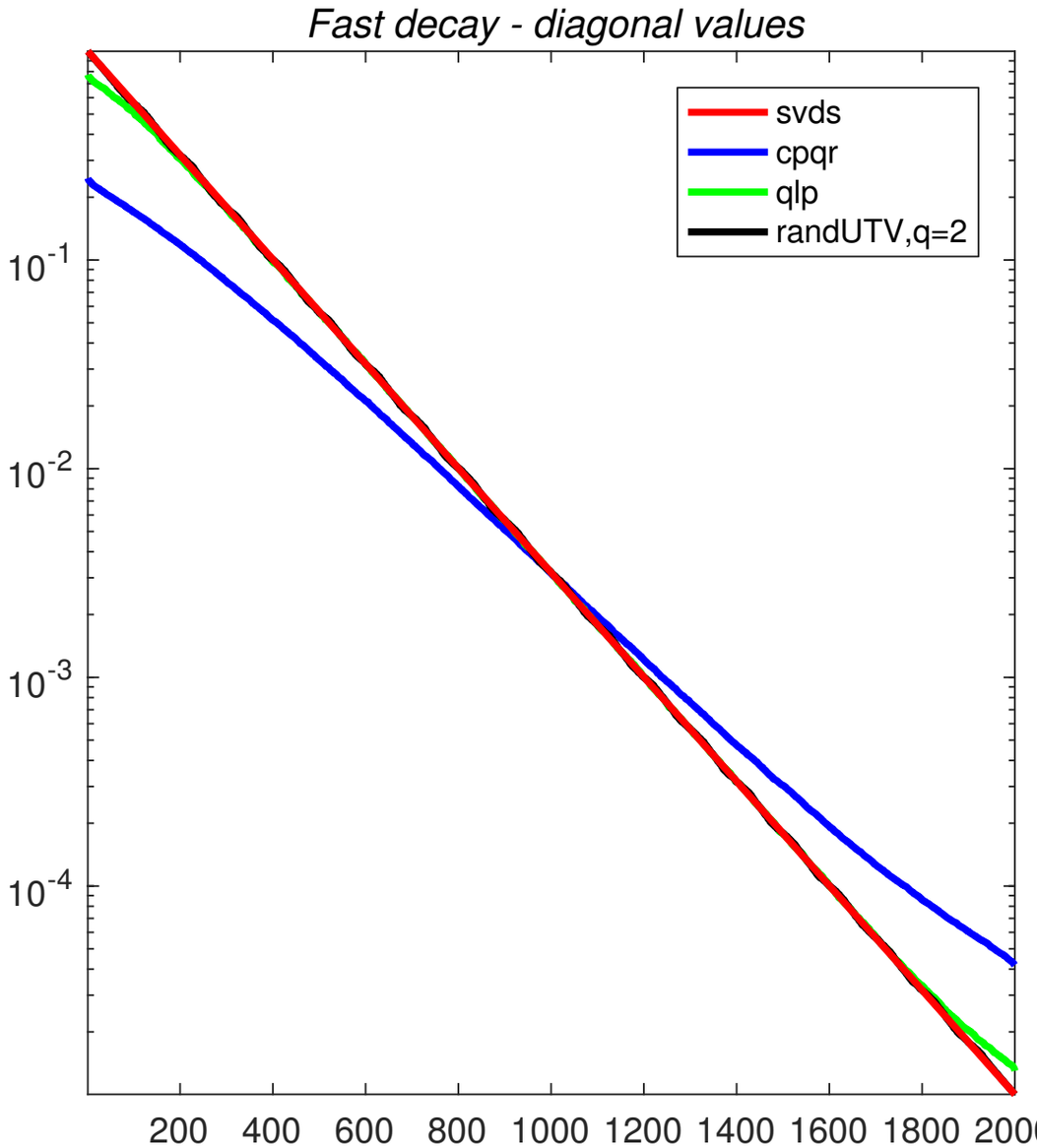

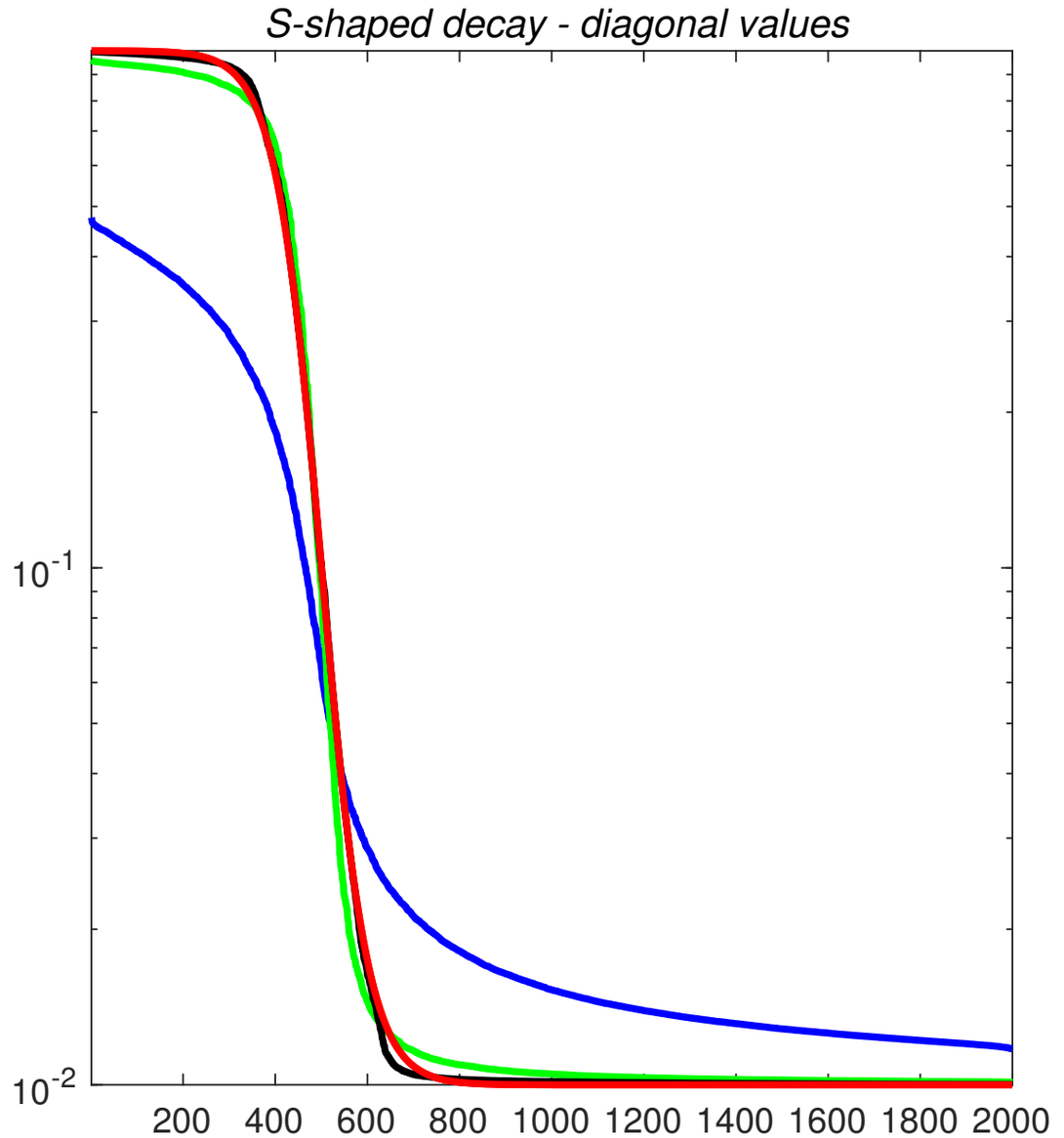

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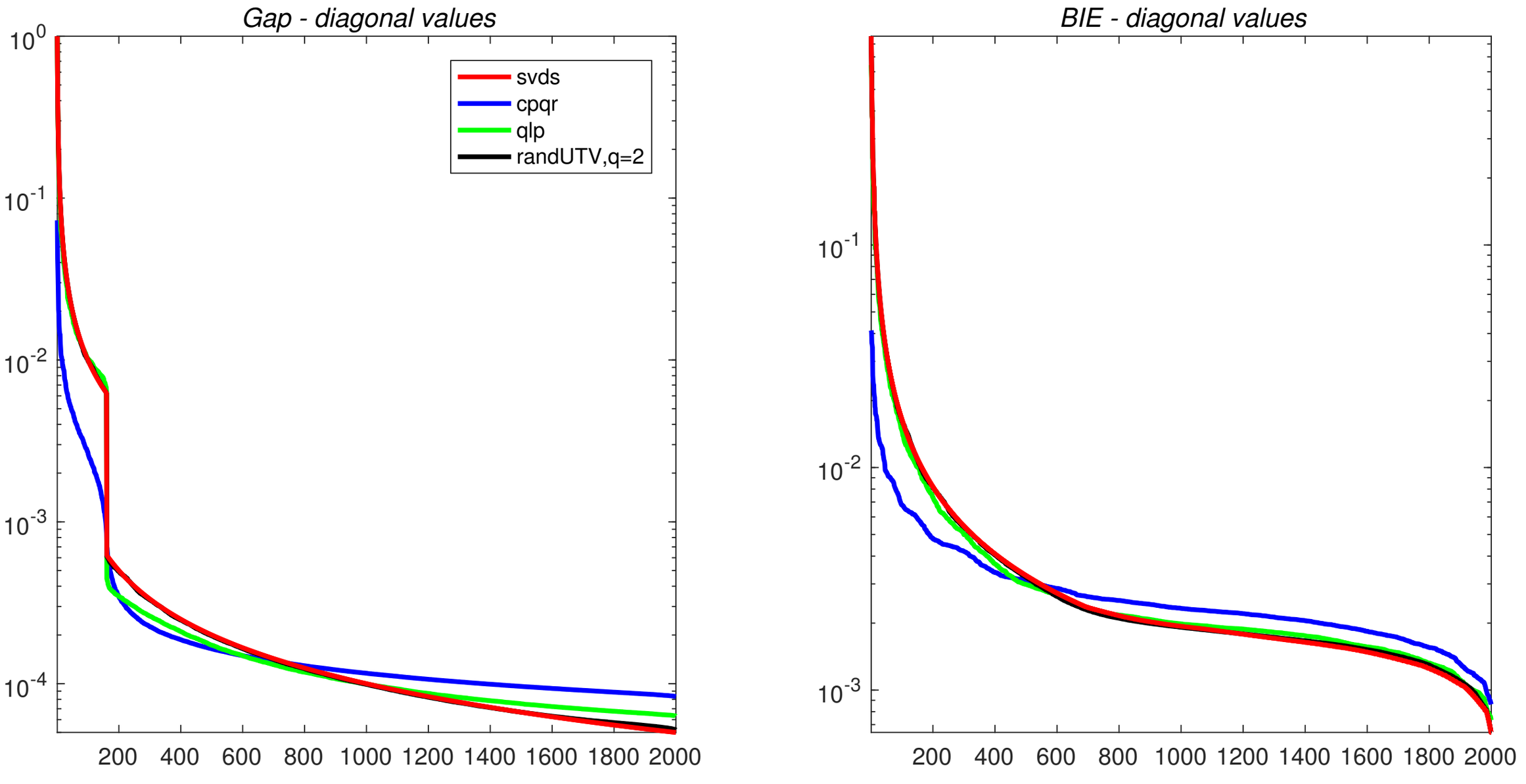


Orthonormal matrices (14 cores)

$10^{10} \times \text{Time} / n^3$

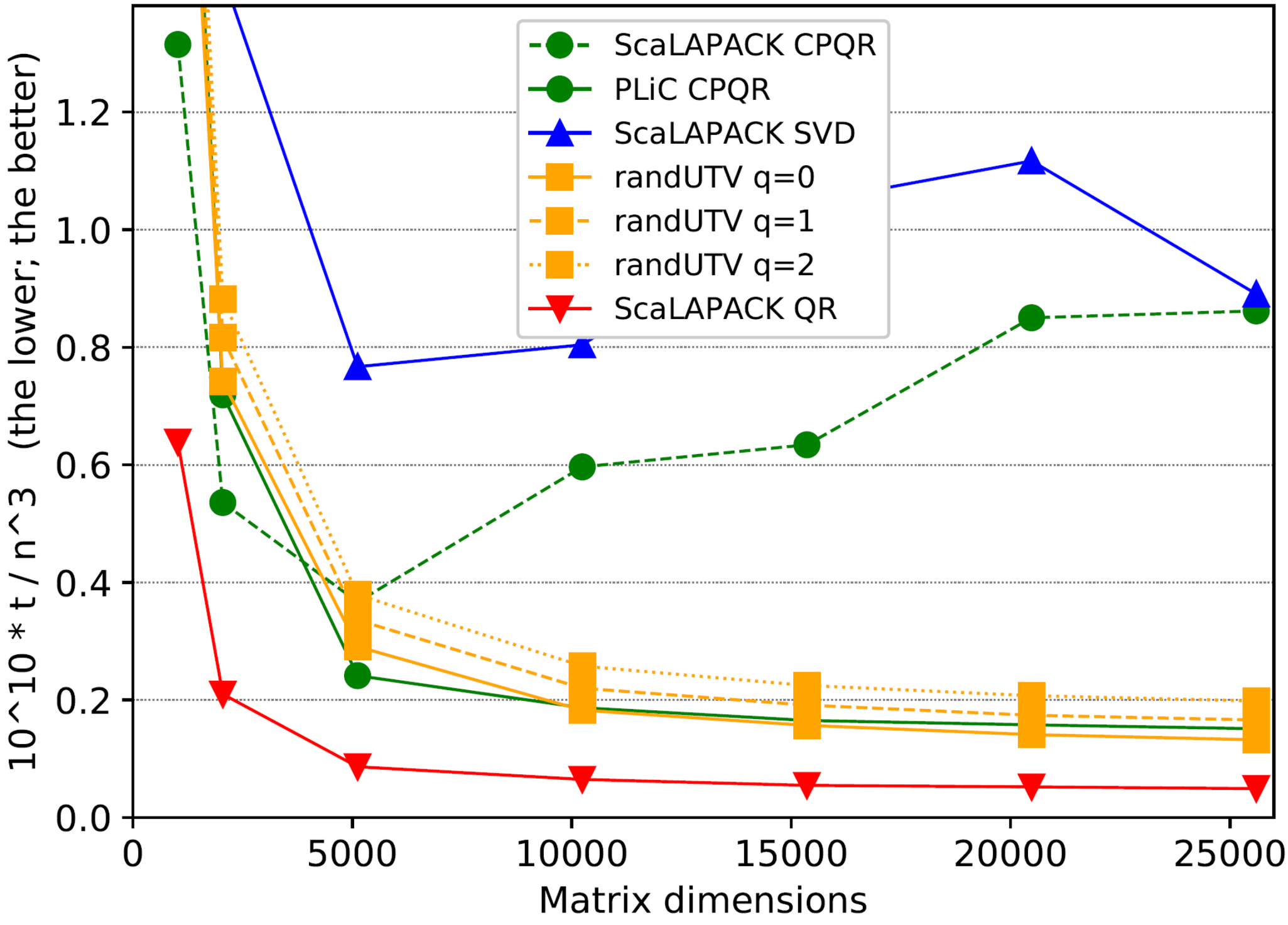
2
1.5
1
0.5
0

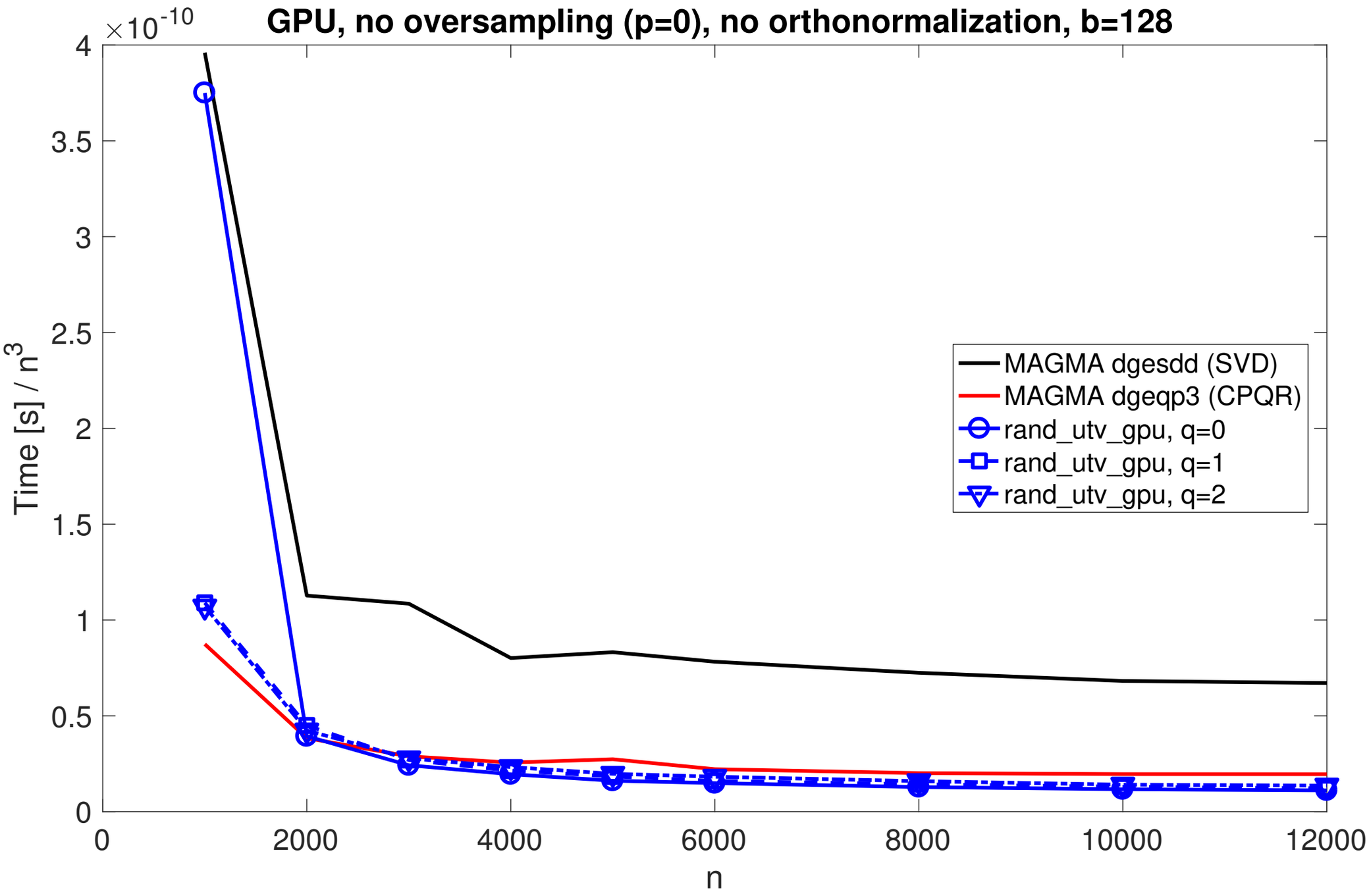
0
20480
40960
61440
81920
102400

n

In-core MKL SVD
In-core MKL CPQR
In-core randUTV pblas q=2
Out-of-core randUTV v21v q=2

Performances on a 6 x 16 mesh. Orthonormal matrices are built.




GPU, no oversampling (p=0), no orthonormalization, b=128
×10^-10
Time [s] / n^3
n
0
0.5
1
1.5
2
2.5
3
3.5
4
0
2000
4000
6000
8000
10000
12000
MAGMA dgesdd (SVD)
MAGMA dgeqp3 (CPQR)
rand_utv_gpu, q=0
rand_utv_gpu, q=1
rand_utv_gpu, q=2

Randomized Column Pivoted QR (randCPQR)

Given an $m \times n$ matrix $\mathbf{A}$ (with $m \geq n$), we seek a QR factorization

$$\begin{array}{cccc} \mathbf{A} & \mathbf{P} & \approx & \mathbf{Q} \quad \mathbf{R} \\ m \times n & n \times n & & m \times k \;\; k \times n \end{array}$$

for either $k = n$ (full factorization) or k comparable to $\min(m, n)$. As usual, $\mathbf{Q}$ is orthonormal, $\mathbf{P}$ is a permutation, and $\mathbf{R}$ is upper triangular.

Question: Is the CPQR "rank-revealing"? Does it satisfy:

- The truncated factorization is a close to optimal low-rank factorization, so that

$$\|\mathbf{A} - \mathbf{Q}(:, 1:k)\,\mathbf{R}(1:k, :)\mathbf{P}^*\| =\approx \inf\{\|\mathbf{A} - \mathbf{B}\| : \ \mathbf{B} \text{ has rank } k\}.$$

- $\sigma_j(\mathbf{T}(1:k, 1:k)) \approx \sigma_j(\mathbf{A})$ for $j \in \{1, 2, \ldots, k\}$.

In practice, it is pretty good; it is often used as a cheap substitute for SVD.

There are counter-examples, for which it performs very badly.

Note: There are sophisticated pivoting strategies that improve on how well CPQR reveals numerical rank — seminal work by Gu and Eisenstat (1996). Tricky to implement efficiently.

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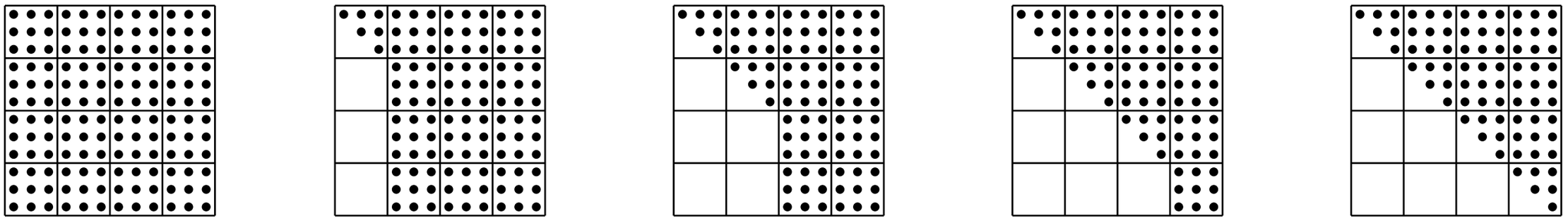
for either $k = n$ (full factorization) or k comparable to $\min(m, n)$. As usual, $\mathbf{Q}$ is orthonormal, $\mathbf{P}$ is a permutation, and $\mathbf{R}$ is upper triangular.

The technique proposed is based on a blocked version of classical Householder QR:

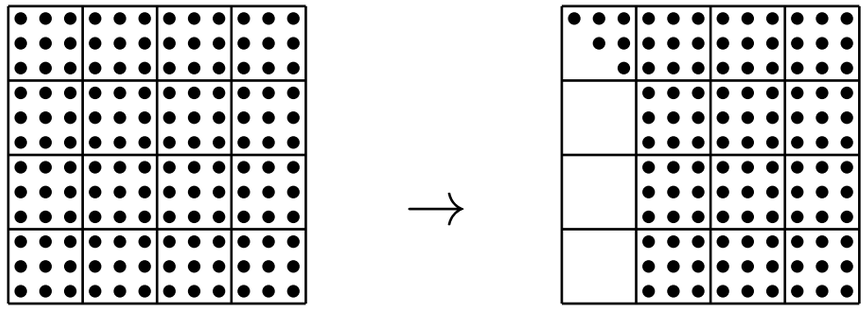


$$\mathbf{A}_0 = \mathbf{A} \qquad \mathbf{A}_1 = \mathbf{Q}_1^* \mathbf{A}_0 \mathbf{P}_1 \qquad \mathbf{A}_2 = \mathbf{Q}_2^* \mathbf{A}_1 \mathbf{P}_2 \qquad \mathbf{A}_3 = \mathbf{Q}_3^* \mathbf{A}_2 \mathbf{P}_3 \qquad \mathbf{A}_4 = \mathbf{Q}_4^* \mathbf{A}_3 \mathbf{P}_4$$

Each $\mathbf{Q}_j$ is a product of Householder reflectors. Each $\mathbf{P}_j$ is a permutation matrix computed via randomized sampling.

Randomized Column Pivoted QR. *How to do block pivoting using randomization:*

Let **A** be of size $m \times n$, and let b be a block size.



A $\qquad$ $\mathbf{Q}^* \mathbf{A} \mathbf{P}$

Q is a product of b Householder reflectors. **P** is a pivoting matrix that moves b "pivot" columns to the leftmost slots. We seek **P** so that the set of chosen columns *has maximal spanning volume.* Draw a Gaussian random matrix **G** of size $b \times m$ and form

$$\underset{b \times n}{\mathbf{Y}} = \underset{b \times m}{\mathbf{G}} \; \underset{m \times n}{\mathbf{A}}$$

The rows of **Y** are random linear combinations of the rows of **A**.

Then compute the pivot matrix **P** for the first block by executing traditional column pivoting on the small matrix **Y**:

$$\underset{b \times n}{\mathbf{Y}} \; \underset{n \times n}{\mathbf{P}} = \underset{b \times b}{\mathbf{Q}_{\text{trash}}} \; \underset{b \times n}{\mathbf{R}_{\text{trash}}}$$

References: *Martinsson, arxiv, 2015. Martinsson, Quintana-Orti, Heavner, van de Geijn, SISC, 2017. Duersch & Gu, arxiv, 2015. Duersch & Gu, SISC, 2017.*

Connection to randomized Interpolatory Decomposition (ID), CUR, etc.

Let **A** be an $m \times n$ matrix of numerical rank k. An *Interpolatory Decomposition (ID)* of **A** takes the form

$$\underset{m \times n}{\mathbf{A}} \approx \underset{m \times k}{\mathbf{C}} \; \underset{k \times n}{\mathbf{X}}$$

where **C** consists of k columns of **A**, and where **X** is a well-conditioned matrix.

Let J_{S} denote an index vector identifying the "skeleton" columns so that $\mathbf{C} = \mathbf{A}(:, J_{\mathrm{S}})$.

A randomized algorithm for computing the ID, given an over-sampling parameter p:

- Draw a $(k+p) \times m$ Gaussian matrix **G**.
- Form a $(k+p) \times n$ sampling matrix $\mathbf{Y} = \mathbf{GA}$.
- Perform a rank-k CPQR on **Y** so that $\mathbf{Y} \approx \mathbf{Y}(:, J_{\mathrm{S}})\mathbf{X}$.

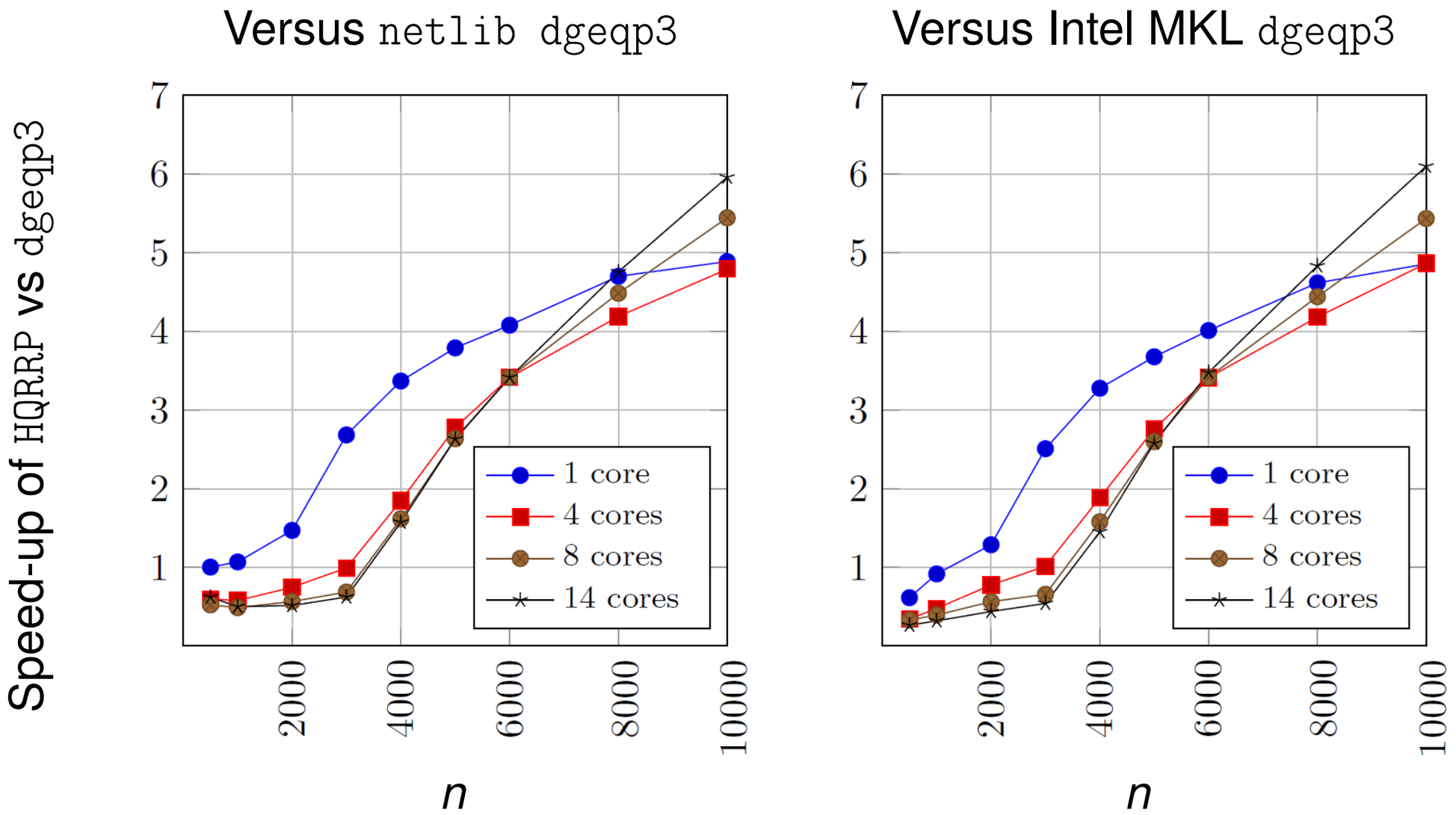
Then we automatically (and almost magically) get an ID of **A**:

$$\mathbf{A} \approx \mathbf{A}(:, J_{\mathrm{S}})\mathbf{X}.$$

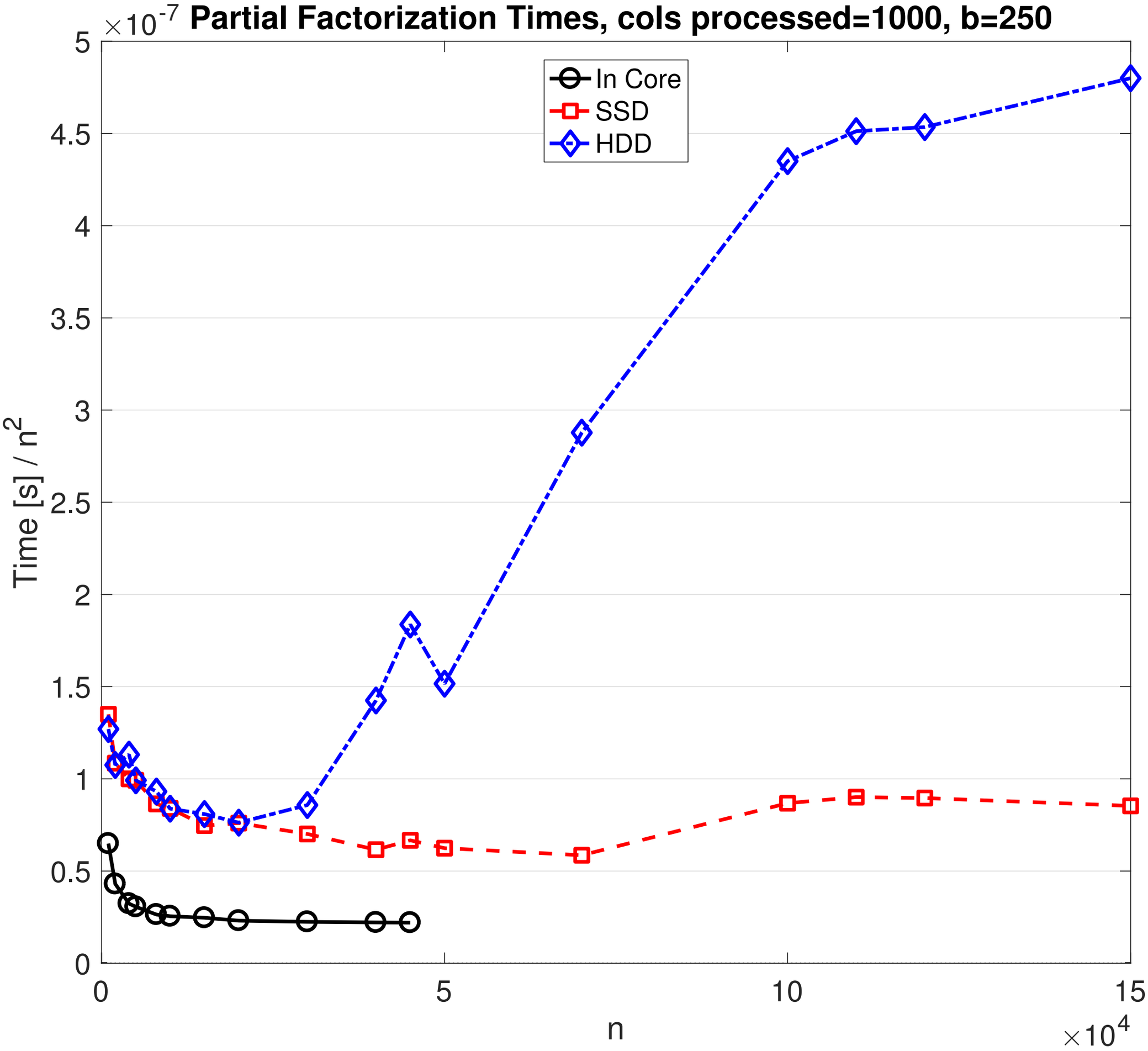
Can be used to compute a CUR decomposition as well.

Reference: "Randomized algorithms for the low-rank approximation of matrices." E. Liberty, F. Woolfe, P.G. Martinsson, V. Rokhlin, and M. Tygert; PNAS, 2007

Randomized Column Pivoted QR (randCPQR)

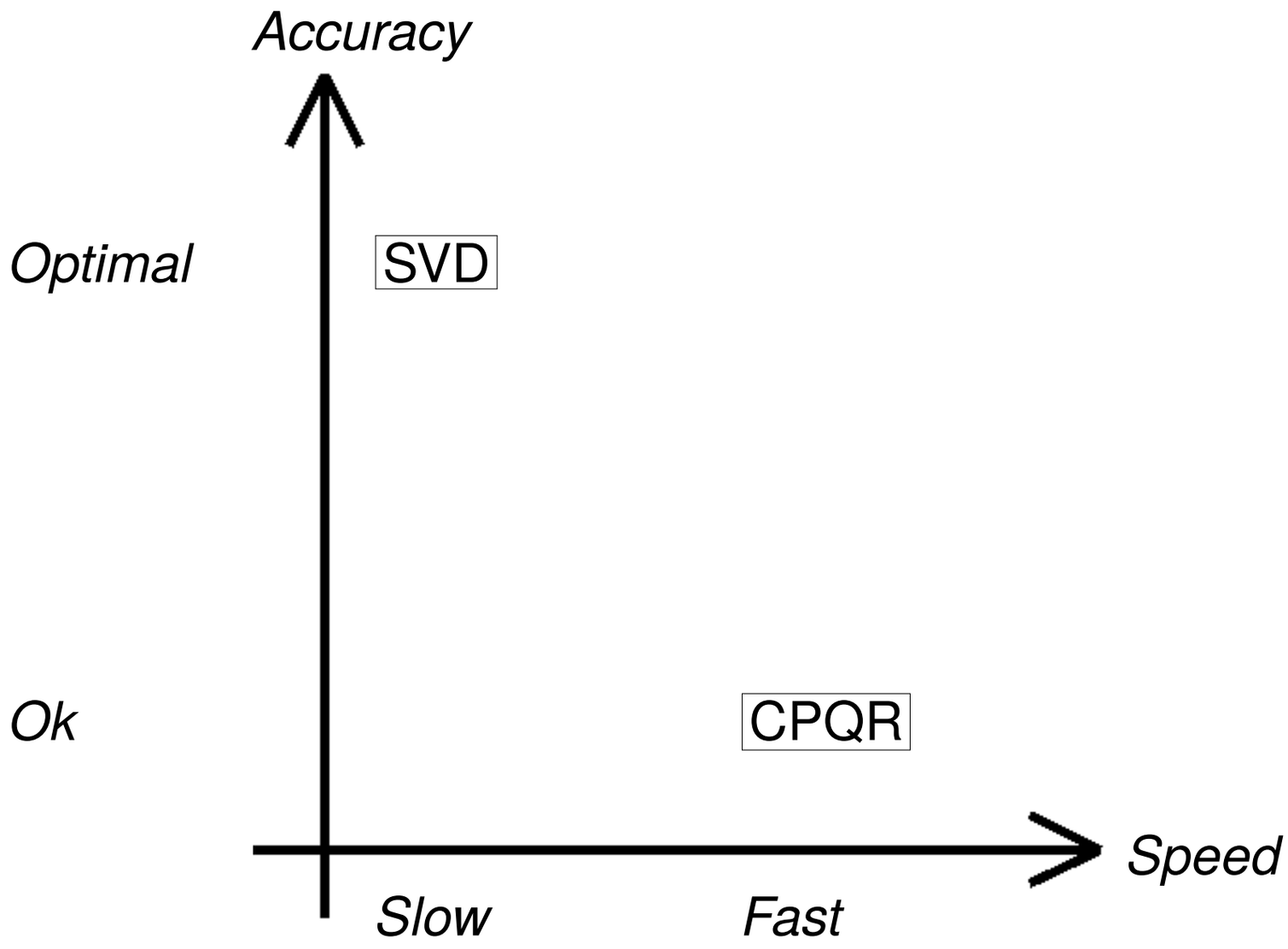


Speedup attained by our randomized algorithm `HQRRP` *for computing a full column pivoted QR factorization of an* $n \times n$ *matrix. The speed-up is measured versus LAPACK's faster routine* `dgeqp3` *as implemented in Netlib (left) and Intel's MKL (right). Our implementation was done in C, and was executed on an Intel Xeon E5-2695. Joint work with G. Quintana-Ortí, N. Heavner, and R. van de Geijn. Available at:* https://github.com/flame/hqrrp/


Partial Factorization Times, cols processed=1000, b=250
×10^-7
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
Time [s] / n^2
In Core
SSD
HDD
0
5
10
15
n
×10^4

Randomized Column Pivoted QR (randCPQR)

For the task of computing low-rank approximations to matrices, the classical choice is between SVD and column pivoted QR (CPQR). SVD is slow, and CPQR is inaccurate:



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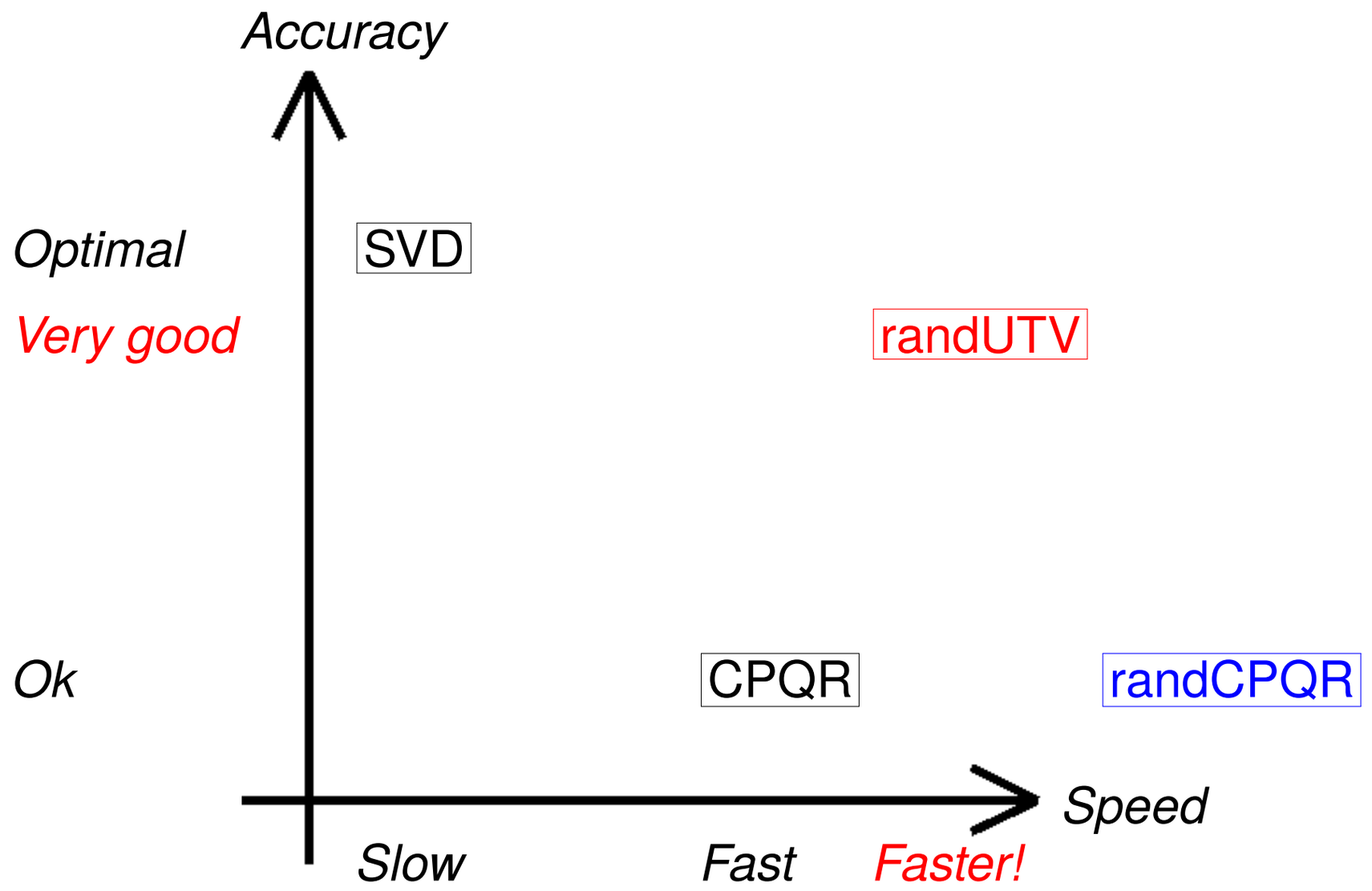

The randomized algorithm `randUTV` combines the best properties of both factorizations. Additionally, `randUTV` parallelizes better, and allows the computation of partial factorizations (like CPQR, but unlike SVD).

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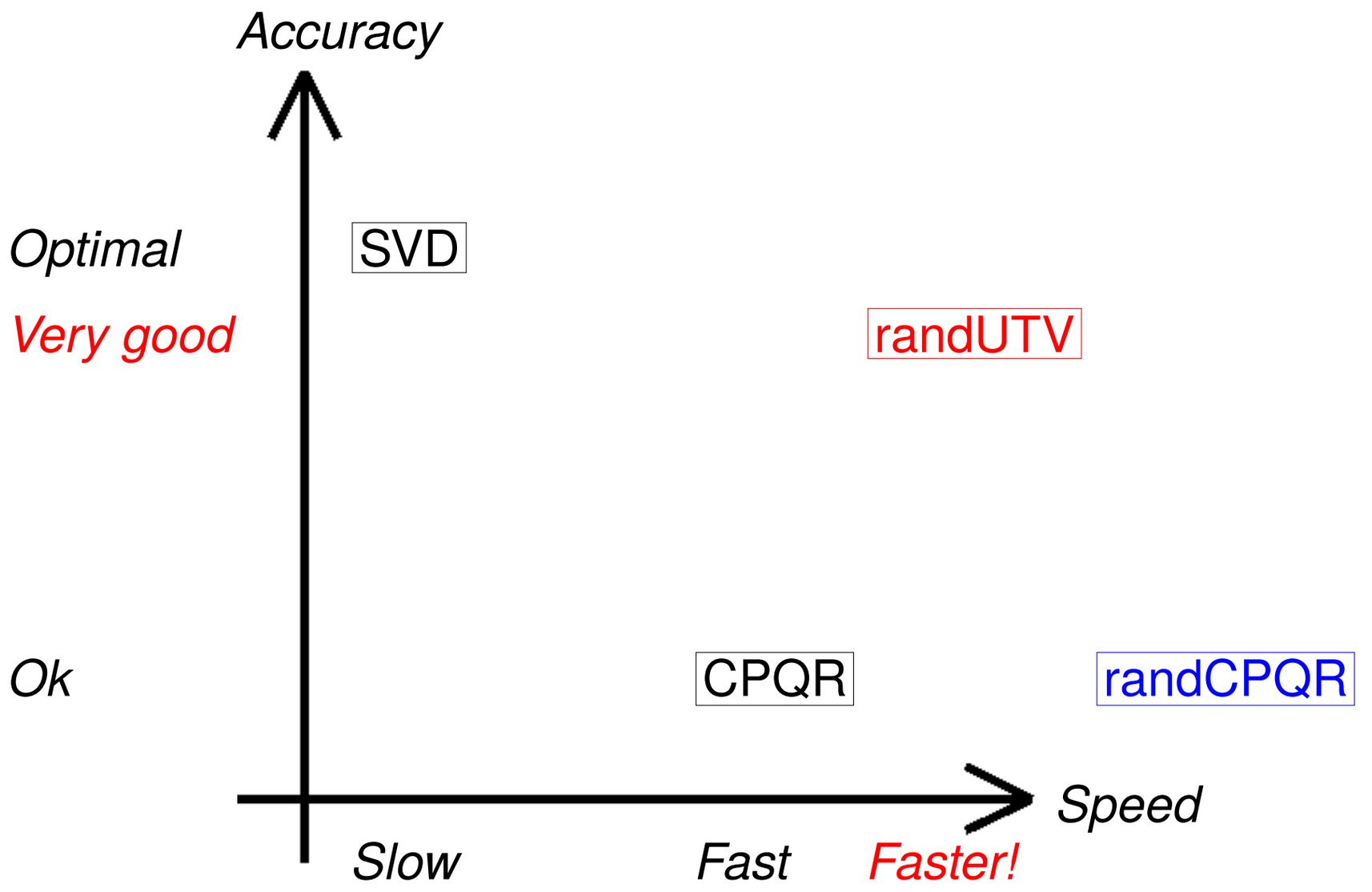

Future work: Continued development or randCPQR and randUTV. Adapt to different computing architectures (distributed memory, out-of-core, etc). Theory. Exploit information that is currently wasted. Multiple sweeps version. Algorithm-by-blocks.

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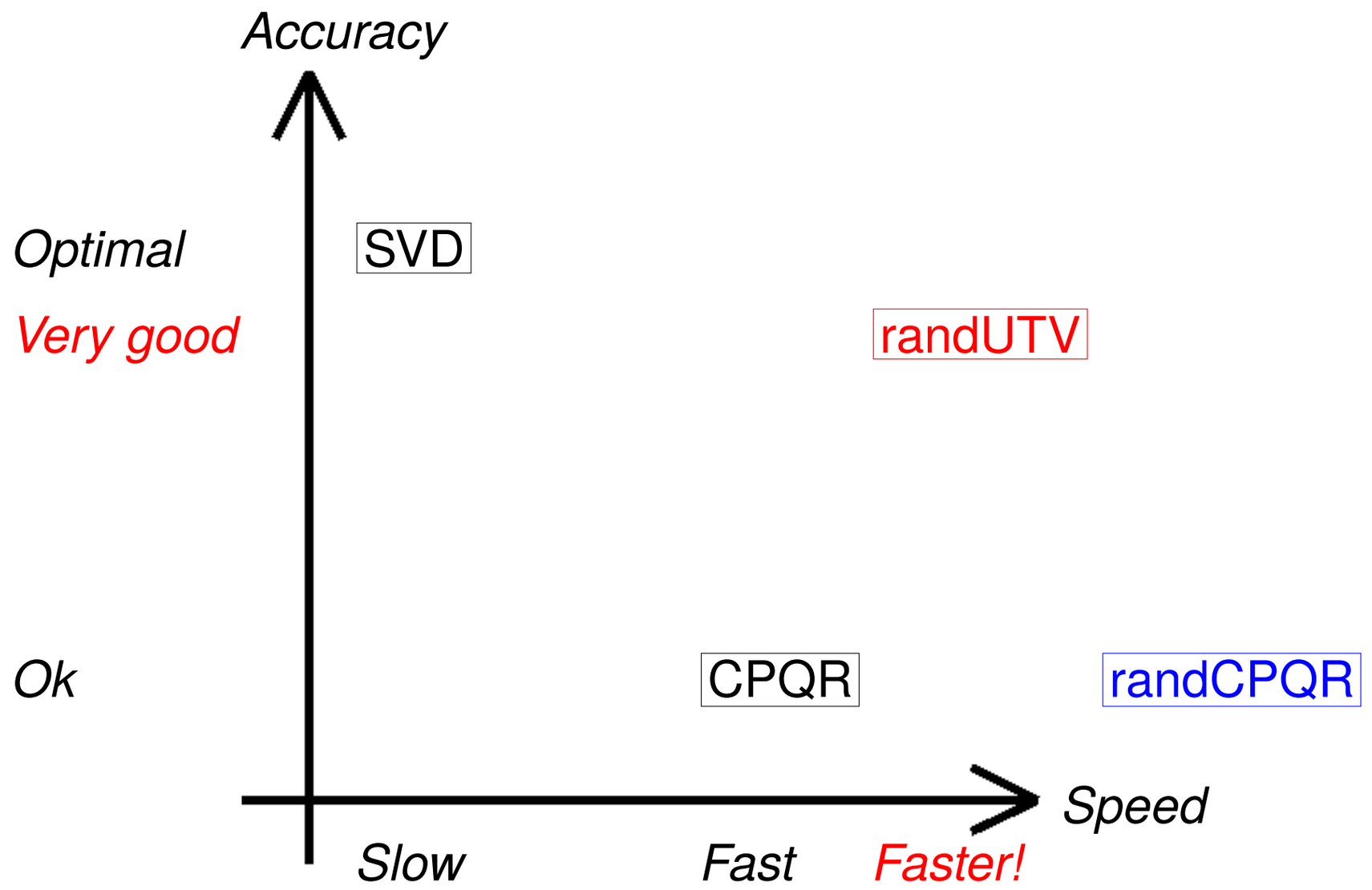

Block Krylov methods: For partial factorizations of sparse matrices, integrate ideas from Krylov methods. Explore design space between the basic RSVD and classical "single-vector" Krylov methods. Recent work by Musco & Musco; Tropp; Gu.

References:

- N. Halko, P.G. Martinsson, J. Tropp, “Finding structure with randomness: Probabilistic algorithms for constructing approximate matrix decompositions.” *SIAM Review*, **53**(2), pp. 217–288, 2011.

- P.G. Martinsson, “Randomized methods for matrix computations.” Arxiv.org #1607.01649. In proceedings book for 2016 PCMI summer school.

- P.G. Martinsson, G. Quintana-Ortí, N. Heavner, and R. van de Geijn, “Householder QR Factorization With Randomization for Column Pivoting (HQRRP).” *SIAM J. on Scientific Comp.*, **39**(2), pp. C96-C115, 2017.

- P.G. Martinsson, G. Quintana-Ortí, N. Heavner, and R. van de Geijn, “Householder QR Factorization With Randomization for Column Pivoting (HQRRP).” *SIAM J. on Scientific Comp.*, **39**(2), pp. C96-C115, 2017.

- P.G. Martinsson, G. Quintana-Ortí, N. Heavner, “randUTV: A blocked randomized algorithm for computing a rank-revealing UTV factorization.” Accepted for publication by *ACM TOMS*. arxiv.org #1703.00998.

- J. Duersch & M. Gu, “Randomized QR with Column Pivoting”, *SIAM Journal on Scientific Computing*, **39**(4), 2017.

Software for UTV: `https://github.com/flame/randutv`

Software for CPQR: `https://github.com/flame/hqrrp`