building monotone expanders

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work with Jean Bourgain, IAS

expanders

expanders are constant degree “highly connected” graphs

motivation

several ways to define

(bipartite) vertex expansion

a bipartite graph $H = (A \cup B, E)$ with $A = B = [n]$ is an **expander** if there exist $c, d > 0$ independent of n

- degree of each vertex is at most d
- for every $A' \subset A$ of size $|A'| \leq n/2$

$$|\Gamma(A')| \geq (1 + c)|A'|$$

where

$$\Gamma(A') = \left\{ b \in B : \exists\ a \in A' \ \ \{a, b\} \in E \right\}$$

interested in infinite families

background

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existence:

constructions:

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- most graphs are expanders [Pinsker]

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- zig-zag: if G_1, G_2 are expanders, then $zigzag(G_1, G_2)$ is too [Reingold-Vadhan-Wigderson]

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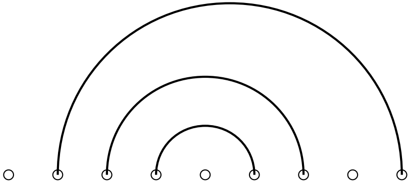
- are there d-page expanders?
- are there d-monotone expanders?

d-page graphs

vertices are on a spine of a book with *d*-pages and edges do not cross each other

1-page



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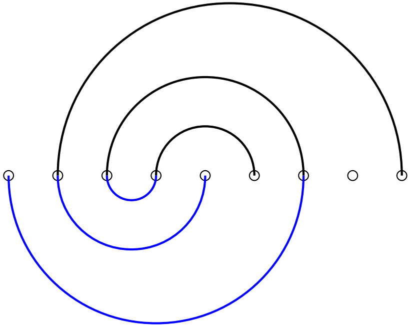

2-page

d-page graphs

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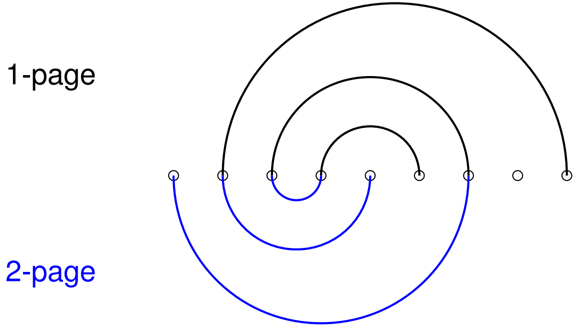

comment. related to Turing machines simulations [Galil-Kannan-Szemeredi, Dvir-Wigderson]

d-monotone graphs

the bipartite graph $H = (A \cup B, E)$ with $A = B = [n]$ is d-**monotone** if its edges are a union of d partial monotone maps:

there are partial[1] monotone[2] maps $\psi_1, \ldots, \psi_d$ so that edges are of the form

$$e = \{a, \psi_i(a)\}$$

---

[1] $\psi_i : A_i \to B$ with $A_i \subset A$

[2] think of x, y as integers: $\psi_i(x) < \psi_i(y)$ for $x < y$ in A_i

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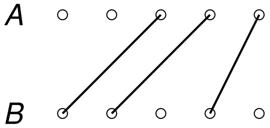

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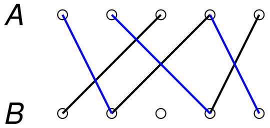

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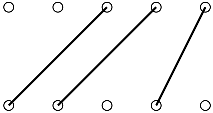

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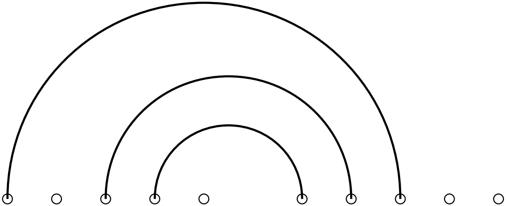

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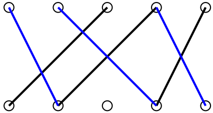

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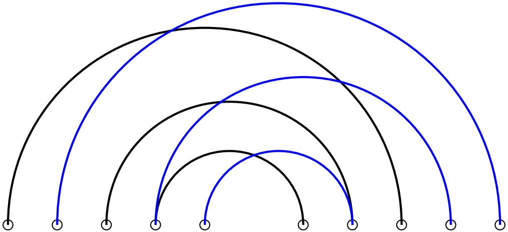

monotone expanders

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corollary [Dvir-Shpilka, Bourgain, Dvir-Wigderson]. there are dimension expanders

dimension expanders

a d-**dimension expander** over $\mathbb{F}^n$ is a collection of linear maps $L_1, \ldots, L_d$ so that for every subspace V of dimension $k \leq n/2$,

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lemma. if there is a d-monotone expander then there is a d-dimension expander over any field with L_i defined by zero-one matrices

a monotone expander

presentation will have 4 parts

(a) Schreier diagrams
(b) continuous monotone expanders
(c) choices
(d) overview of proof

(a) Schreier diagrams

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a **Schreier diagram:** a graph $H = Sch(G, S, X)$ defined by

a group G

a finite subset S of G

an action: $G \curvearrowright X$

- every g in G defines a map $g : X \to X$
- $g(h(x)) = (gh)(x)$ for all g, h in G

vertex set: $A = B = X$
edge set: $\{(x, g(x)) : x \in A,\ g \in S \cup S^{-1}\}$

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Cayley graphs: action of G on itself

(a) an example

1 group G

$$G = SL_2(\mathbb{F}_p) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{F}_p,\ ad - bc = 1 \right\}$$

2 subset S of G

$$S = \left\{ \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\}$$

3 $G \curvearrowright X$: the Möbius action of G on $X = \mathbb{F}_p \cup \{\infty\}$

$$g(x) = \frac{ax + b}{cx + d}$$

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$|G| \sim p^3$, $|A| = |B| = p + 1$, 4-regular

(b) continuous monotone expanders

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a **continuous monotone expander** is an (infinite) bipartite graph defined by $\psi_1, \ldots, \psi_d$ as follows

- vertices: $A = B = [0, 1]$
- monotone: edges of the form $(x, \psi_i(x))$
  - $\psi_i : A_i \to B$ is smooth with $A_i \subset A$ an interval
  - $\psi_i(x) < \psi_i(y)$ for $x < y$ in A_i
- expansion: for every $A' \subset A$ of measure $|A'| \leq 1/2$

$$|\Gamma(A')| \geq (1 + c)|A'|$$

where $\Gamma(A') = \bigcup_{i \in [d]} \psi_i(A')$

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how? if A is partitioned to $a_1, \ldots, a_n$ and B to $b_1, \ldots, b_n$, connect intervals a_j, b_k when $\psi_i(a_j) \cap b_k \neq \emptyset$ for some ψ_i

(c) choices

an explicit continuous Schreier diagram

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3 $G \curvearrowright X$:

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[3]

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3 $G \curvearrowright X := [0,1]$: the Möbius action[3] $g(x) = \frac{ax+b}{cx+d}$ restricted so that $x, g(x)$ in $[0,1]$ for all x, g

[3] no longer an action due to restriction

(c) monotone expanders: concluding

theorem. the (restricted) Möbius action of $SL_2(\mathbb{R})$ on $[0, 1]$ with a constant number of simple matrices as generators yields a continuous monotone expander

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- monotone since action is monotone...

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thus

$$g'(x) = \frac{a(cx + d) - c(ax + b)}{(cx + d)^2} = \frac{1}{(cx + d)^2} > 0$$

except at pole $x = -d/c$

(d) a three-step proof: chess game [Sarnak]

Bourgain-Gamburd, Helfgott, ... :

opening.

middle-game.

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effective Tits' alternative [Eskin-Mozes-Oh, Breuillard, Gelander]: there is a constant r so that if $S \subset SL_2(\mathbb{R})$ generates a group containing $SL_2(\mathbb{Z})$ then in words of length r in S there are two elements that generate a free group F_2

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corollary: there is a constant r so that for every k, if $S \subset SL_2(\mathbb{R})$ generates a group containing $SL_2(\mathbb{Z})$ then in words of length k^r in S there are k elements that generate a free group F_k

(d) middle-game: product-growth

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product growth: under some conditions, if A is a subset of $SL_2(\mathbb{R})$ then the metric entropy of $A \cdot A \cdot A$ is much larger than that of A

background:
discretized ring conjecture [Bourgain]
spectral gaps in $SU(2)$ [Bourgain-Gamburd]

sum-product theorem [Bourgain-Katz-Tao]
growth in $SL_2(\mathbb{F}_p)$ [Helfgott]
expansion for $SL_2(\mathbb{F}_p)$ [Bourgain-Gamburd]

(d) endgame: mixing property (an example)

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assume $G \curvearrowright X$ (both finite)

mixing property: for every $\mu : G \to \mathbb{R}$ and $f : X \to \mathbb{R}$ so that $\sum_{x \in X} f(x) = 0$, we have

$$\|\mu * f\|_2^2 \leq \frac{|G|}{N} \|\mu\|_2^2 \|f\|_2^2$$

where $(\mu * f)(x) = \sum_{g \in G} \mu(g) f(g^{-1}(x))$

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useful: when N is large

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well known: Möbius action is 3-transitive

concluding

* there are “simple” expanders: monotone and constant-page

* proof has 3 parts:

- Tits’ alternative (groups, geometry)
- product growth (additive combinatorics)
- 3-transitivity (replaces representation theory)

even simpler?

a natural way to construct monotone graphs is using affine maps: given a_i, b_i for $i \in [d]$ define edges via

$$\{0, 1, 2, \ldots, n-1\} \ni x \mapsto \lceil a_i x + b_i \rceil \mod n$$

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question. what about $a_i, b_i \in \mathbb{R}$?

comments.

- can slightly generalise $\mathbb{Q}$: diophanite approximation
- no expanders using $\mathbb{R}$ for groups of polynomial growth

thank you

(d) combining parts (roughly)

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goal (spectral expansion): for every $f : [0,1] \to \mathbb{R}$ with $\mathbb{E}f = 0$,

$$\|T_\nu * f\|_2 \leq c\|f\|_2, \quad c < 1$$

with T_ν the Hecke operator that corresponds to the uniform distribution ν on (free) generators S

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lemma: $\|\mu^{*t}\|_\infty \leq \delta^{-0.01}$ for $t \lesssim \log(1/\delta)$

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with ν the uniform distribution on generators S

first work over G: let $\mu = \nu * p$ with p the density of uniform measure on δ-ball around 1 in $SL_2(\mathbb{R})$ $(\|\mu\|_\infty \sim \delta^{-3})$

lemma: $\|\mu^{*t}\|_\infty \leq \delta^{-0.01}$ for $t \lesssim \log(1/\delta)$

using lemma: lemma + endgame (3-transitivity): can non-trivially bound $\|T_\nu^t * f\|_2$

(d) combining parts (roughly)

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$\mu = \nu * p$ with ν uniform on generators and p on δ-ball

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proof idea:

opening:

middle-game:

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proof idea:
opening: $\|\mu^{*s}\|_2^2 \leq \delta^{-2.99}$ for small s

(μ grows along a tree so iterations smoothen it)

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proof idea:
opening: $\|\mu^{*s}\|_2^2 \leq \delta^{-2.99}$ for small s

(μ grows along a tree so iterations smoothen it)

middle-game: as long as $\|\mu^{*r}\|_2$ is not too small,

$$\|\mu^{*3r}\|_2 \leq \delta^{0.01}\|\mu^{*r}\|_2$$

(think of $A = supp(\mu^{*r})$ [Balog-Szemeredi-Gowers])