Small Lifts of Expander Graphs are Expanding

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Overview

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Essentially expansion is 'good' and we seek ways of achieving high expansion efficiently

What are expander graphs ?

There are three main perspectives of expansion

- Combinatorial ("small" sets have "large" boundaries)
- Linear Algebraic (large spectral gap)
- Probabilistic (random walks converge rapidly)

(One of) The combinatorial definitions

Definition

A graph G = (V,E) is said to be ϵ - **edge expanding** if for all subsets S of V of size $\leq |V|/2$, the number of cross edges $(e(S, V\backslash S))$ is large. That is,

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In this sense the edge expansion $h(G)$ of a graph is defined as

$$h(G) = min_{S \in V, |S| \leq |V|/2} \frac{e(S, V\backslash S)}{|S|}$$

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- The highest absolute eigenvalue of a matrix is called its **spectral radius**

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Theorem (Cheeger's Inequality)

Let G be a d-regular graph with spectrum as defined above. Then

$$\frac{d-\lambda_2}{2} \leq h(G) \leq \sqrt{2d(d-\lambda_2)}$$

How much expansion can we expect?

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Theorem (Alon-Bopanna)

For a d-regular graph G

$$\lambda_2 \geq 2(\sqrt{d-1}) - o_n(1)$$

The term $o_n(1)$ goes to zero as $n \to \infty$

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Do such graphs exist with arbitrarily large size?

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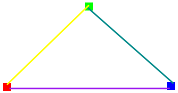
- But no construction other than LPS is known.
- Friedman suggested building expanders by "lifting" the original graph

What are lifts?

Figure : Base Graph



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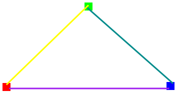

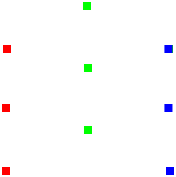

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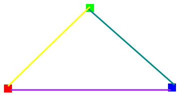


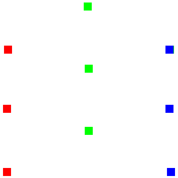
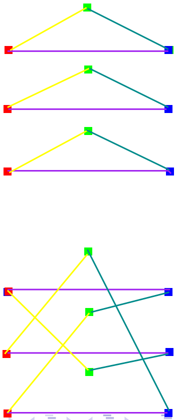


Figure : Lifted Graphs



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- The lift of a d-regular graph is d-regular

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- The number of vertices in S_H increases k times and so does the number of edges going out of S_H.
- It is also known that the expansion does not go down arbitrarily as we increase the degree of the lifts.

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- Take any eigenvector f for G and a construct f' by repreating over the whole fiber the value $f(v)$. The resulting vector f' is an eigenvector of H with the same eigenvalue.

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    - Take a k-lift
    - Repeat

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- Addario-Berry and Griffits improved it to $\lambda(H) = O(d^{1/2})$ which is the best possible upto a constant
- Recently, Puder gave an almost-optimal result of $\lambda(H) = 2\sqrt{d-1} + 1$.

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- In a recent breakthrough, Marcus-Spiemlan-Srivastava showed that for every bipartite base graph exists a 2-lift with $\lambda(H) = 2\sqrt{d-1}$.
- This is optimal, but we still don’t know what happens on average (w.h.p over random lifts), nor do we know how to construct them.

Our Results

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- We also give a new characterization of the spectrum of shift k-lifts.

Our Results

Theorem 1

Let G be a d-regular graph with non-trivial eigenvalues at most λ in absolute value, and H be a (uniformly random) 2-lift of G. Let λ_{new} be the largest in absolute value new eigenvalue of H. Then

$$\lambda_{new} \leq \mathcal{O}(\lambda)$$

with probability at least $1 - e^{-\Omega(n/d^2)}$. Moreover, if G is moderately expanding such that $\lambda \leq \frac{d}{\log d}$, then

$$\lambda_{new} \leq \lambda + \mathcal{O}(\sqrt{d})$$

with probability at least $1 - e^{-\Omega(n/d^2)}$

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Theorem 2

Let G be a d -regular graph with non-trivial eigenvalues at most λ *in absolute value, and H be a random shift k-lift of G. Let* λ_{new} *be the be the largest in absolute value new eigenvalue of H. Then*

$$\lambda_{new} \leq \mathcal{O}(\lambda)$$

with probability at least $1 - k \cdot e^{-\Omega(n/d^2)}$*. Moreover, if G is moderately expanding such that* $\lambda \leq \frac{d}{\log d}$*, then*

$$\lambda_{new} \leq \lambda + \mathcal{O}(\sqrt{d})$$

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- First, can we get rid of the dependency on λ of the base graph and obtain bounds that depend on d like is the case for large lifts?
- NO! The dependency on λ is necessary.
- Let G be a disconnected graph on n vertices that consists of $n/(d+1)$ copies of K_{d+1}, and let H be a random 2-lift of G. Then the largest non-trivial eigenvalue of G is $\lambda = d$ and it can be shown that with high probability, $\lambda_{new} = \lambda = d$ (noted by BL).

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- Thus, our results are nearly optimal, maybe we can improve the constant!

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Some Definitions

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- The signed adjacency matrix of G, denoted by $A_s(G)$ is its adjacency matrix with edge e replaced by $s(e)$
- A 2-lift corresponding to a signing s can be defined by letting the edges in the fiber of edge (x, y) be $(x_0, y_0), (x_1, y_1)$ if $s(x, y) = 1$ and $(x_1, y_0), (x_0, y_1)$ otherwise.

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Lemma

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To see this note that $2 * A(H) =$

$$\begin{pmatrix} A + A_s & A - A_s \\ A - A_s & A + A_s \end{pmatrix}$$

Now for an eigenvector u of $A_s(G)$ the eigenvector $(u, -u)$ is an eigenvector of $A(H)$ with the same eigenvalue.

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- In light of the previous observation we want to be able to claim that for base graph G, the spectral radius of a **typical** signing is proportional to $\lambda(G)$.
- Thus, we need a high probability bound on $||A_s|| = \max_{x \in R^n} \frac{|x^T A_s x|}{||x||^2}$.

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Lemma (Bilu-Linial)

- *A be an $n \times n$ real symmetric matrix with zeros on the diagonal*
- *The l_1 norm of each row in A is at most d*
- *For all vectors $u, v \in \{0,1\}^n$ the following holds*

$$\frac{|u^T A v|}{\|u\|\|v\|} \leq \alpha$$

- *Then the spectral radius of A is $O(\alpha(\log(d/\alpha)+1))$*

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- Conclude by Converse EML that there exists an A_s with spectral radius $O(\sqrt{d \log^3 d})$.

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- We need a more delicate analysis of the spectral norm, which we show next.

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- Now it is easy to see that $|y^T A_s y| = |\sum_{i,j} (2^{-i} u_i)^T A_s (2^{-j} u_j)|$.

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$$\Pr[|u_i^T A_s u_j| \geq \sqrt{d \log d |S(u_i)||S(u_j)|}] \leq d^{-(|S(u_i)|+|S(u_j)|)}$$

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- So far, this is what BL have also used
- But now we are faced with two significant challenges mentioned above, the high probability and the $\log d$ loss.

Small Support Sets: The High Probability Remedy

- When The support of vectors u_i, u_j are small, then the probability $d^{-(|S(u_i)|+|S(u_j)|)}$ is not enough for our goal.

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- Note that we already argued that the dependence on λ in Theorem 1 cannot possibly be improved since for two small sets the number of edges is essentially governed by λ in the base graph. This explains intuitively the choice of dealing with small sets separately first.

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- Note that we already argued that the dependence on λ in Theorem 1 cannot possibly be improved since for two small sets the number of edges is essentially governed by λ in the base graph. This explains intuitively the choice of dealing with small sets separately first.
- Once we are left with sets of large support, then we can get good probability bounds.

Large Support Sets: The $\log d$ factors Remedy

- Number of terms in the sum $|y^T A_s y| = |\sum_{i,j} (2^{-i} u_i)^T A_s (2^{-j} u_j)|$ is at most $E(S(u_i), S(u_j))$.

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- We approximate them by using Expander Mixing Lemma by $d|S(u_i)||S(u_j)|/n + \lambda\sqrt{|S(u_i)||S(u_j)|}$.
- To make the analysis easier we consider two cases according to which of the two terms in EML dominates the other.

The Expander Mixing Lemma

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Lemma (Expander Mixing Lemma)

For any two vertex subsets $S, T \in V$ of a graph G we have that

$$|E(S,T) - \frac{d \cdot |S||T|}{n}| \leq \lambda(\sqrt{|S| \cdot |T|})$$

Case 1: $\lambda\sqrt{|S(u_i)||S(u_j)|} \leq d|S(u_i)||S(u_j)|/n$

- Remember the original Chernoff bound: $\Pr[|u_i^T A_s u_j| \geq \sqrt{d \log d |S(u_i)||S(u_j)|}] \leq d^{-(|S(u_i)|+|S(u_j)|)}$

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- Specifically, we show that with probability at least $1 - e^{-\Omega(\frac{n}{d^2})}$ we have for each relevant term of the sum:

$$|u_i^T A_s u_j| \leq 8\sqrt{\lambda\sqrt{|S(u_i)||S(u_j)|}|S(u_j)|\log(\frac{2d|S(u_i)|}{|S(u_j)|})}$$

- This turns out to be exactly what is needed for a union bound to go through.

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- This turns out to be exactly what is needed for a union bound to go through.
- Also counters the discrepancy between the sizes of the sets $S(u_i)$ and $S(u_j)$. Bilu-Linial ended up losing a lot when one set was much smaller than the other.

Case 2: $\lambda\sqrt{|S(u_i)||S(u_j)|} \geq d|S(u_i)||S(u_j)|/n$.

- "Easy" case when $|i-j| > \frac{1}{2}\log d$. Focus on the part where $|i-j| \leq \frac{1}{2}\log d$.

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$$|\sum_i u_i^T A_s u_j| \leq 8\sqrt{1/n * d|S(u_j)|^2(\sum_i |S(u_i)|2^{2i})\log(\frac{2n}{|S(u_j)|})}$$

- This can be done because all these $S(u_i)$ have no intersection giving us independence to apply a Chernoff bound on a sum of them.

Putting the Large Support Terms Together

Lemma

Let $u_1, u_2, \ldots \in \{0, \pm 1\}^n$, $v_1, v_2 \ldots \in \{0, \pm 1\}^n$ be two families of vector sets such that for all (i,j), $S(u_i) \cap S(u_j) = S(v_i) \cap S(v_j) = \emptyset$ and either for all i, $|S(v_i)| > \frac{n}{d^2}$ or for all i, $|S(u_i)| > \frac{n}{d^2}$. Let A_s be a random signing matrix. The following holds with high probability over random choices of signing.

$$|\sum_{i \leq j} (2^{-i} * u_i^T) A_s (2^{-j} * v_j)| \leq$$

$$\mathcal{O}(\max(\sqrt{\lambda \log d}, \sqrt{d})) \sum_i |S(u_i)| 2^{-2i} + (\frac{\lambda}{5} + \mathcal{O}(\sqrt{d})) \sum_j |S(v_j)| 2^{-2j}$$

Theorem 2 and Shift Lifts

Definition

Shift lift of a graph G is obtained by replacing each vertex of G by k vertices (fibre) and replacing each edge by a shift permutation between the corresponding fibres.

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- e.g. for an edge (x, y) we would have a permutation of the form $x - y = c \mod k$.
- This can be seen as a generalization of 2-lift.

Shift Lifts: Spectral Characterization

- Given a graph G, sign each edge by $+1$ or -1 depending on whether the permutation in the 2-lift is identity permutation or a cross permutation. Then, new eigenvalues of the lift are eigenvalues of the signed adjacency matrix A_s.

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- Now, Theorem 2 can (almost) reduce to Theorem 1.

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- How can we use our results to build good expanders?

Thank you