A Short Proof of Gowers' Lower Bound for Reg Lemma

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Regular Bipartite Graphs

A natural property we expect to find in G(n,n,d)?

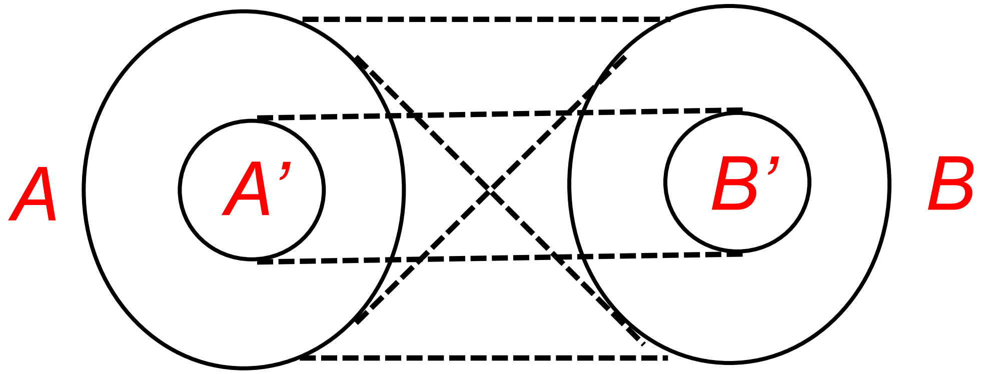
All pairs of vertex sets have "correct" density

Definition: $d(X,Y) = |E(X,Y)| / |X||Y|$

Definition: $G=(A,B,E)$ is *ε-regular* if

$$|d(A',B') - d(A,B)| \leq \varepsilon$$

for every $A' \subseteq A$ and $B' \subseteq B$ satisfying $|A'|,|B'| \geq \varepsilon n$



The Regularity Lemma

Def: A *partition* $V = \{V_1,\ldots,V_k\}$ of $V(G)$ is a ε-regular if $|V_i|=n/k$, and for every V_i all but εk of the V_j's are s.t. (V_i,V_j) is ε-regular.

Reg. Lemma [Szemerédi '78]: For any $\varepsilon > 0$, there is $M=M(\varepsilon)$ s.t. any graph has an ε-regular partition $V = \{V_1,\ldots,V_k\}$ with $k \le M$.

Main drawback: Proof gives $M(\varepsilon) \le \text{twr}(1/\varepsilon^5)$

[Gowers '96]:

1. $M(\varepsilon) \ge \text{twr}(\log(1/\varepsilon))$. Short and (relatively) simple.
2. $M(\varepsilon) \ge \text{twr}(1/\varepsilon^{1/16})$. Long and very complicated.

[Conlon-Fox '12]: $M(\varepsilon) \ge \text{twr}(1/\varepsilon)$

A Short and Simpler Proof that

$M(\varepsilon) \geq \mathrm{twr}(1/\varepsilon^c)$

Preliminary Observations

We can extend notion of density/ε-regular/ε-regular-partition to complete graphs, where each edge (i,j) has a weight $\in [0,1]$.

Claim: If we generate a random graph from a weighted complete graph, then whp they have the same regular partitions.

To prove that $M(\varepsilon) \geq \text{twr}(1/\varepsilon^{1/2})$ enough to prove that:

Theorem [Moshkovitz-S '13]: For every $\varepsilon>0$, there is a weighted graph G s.t. every ε-regular partition of G is of size $\text{twr}(1/\varepsilon^{0.5})$.

Quasi-random set partitions

Lemma 1: If $M=2^{m/100}$ then there are m bipartitions of $\{1,\ldots,M\}$, denoted $(A_1,B_1),\ldots,(A_m,B_m)$, so that:

1. For every $1 \le i \le m$, we have $|A_i|=|B_i|=M/2$
2. For every $\lambda=(\lambda_1,\ldots, \lambda_M)$ satisfying

   $$\lambda_i \ge 0 \ , \ ||\lambda||_1 = 1 \ , \ ||\lambda||_\infty \le 1\text{-}8\delta$$

   there are $m/6$ bipartitions (A_i,B_i) satisfying

   $$\min\left(\sum_{t\in A_i} \lambda_t \, , \sum_{t\in B_i} \lambda_t \right) \ge \delta$$

A Hard Graph for the Reg Lemma

Recall: We need to define a weighted complete n-vertex graph G s.t. every ε-regular partition of G has order $\mathrm{twr}(1/\varepsilon^{0.5})$.

We assume henceforth that $\varepsilon \le \varepsilon_0$ and that $n \ge n_0(\varepsilon)$.

We define a sequence of partitions X_r of a set of n vertices.

1. X_0 is the entire vertex set.
2. X_{r+1} is obtained from X_r by partitioning each of its clusters into $2^{|X_r|/100}$ sub-clusters.

Then, $|X_0| = 1$, $|X_{r+1}| = |X_r| \cdot 2^{|X_r|/100}$, so $|X_r| = \mathrm{twr}(r)$.

A Hard Graph for Reg Lemma

X_{r+1} obtained from X_r by partitioning each cluster into $2^{|Xr|/100}$ sub-clusters.

Definition of G: For $r = 0,1,2,\ldots 1/\varepsilon^{0.5}$ do the following

1. Let $X_r = \{X^1,\ldots,X^m\}$ and $X_{r+1} = \{X^1{}_1,\ldots,X^1{}_M ,\ldots, X^m{}_1,\ldots,X^m{}_M\}$
2. Let $(A_1,B_1),\ldots,(A_m,B_m)$ be a sequence of partitions of $[M]$ satisfying the properties of Lemma 1.
3. For every $X^i, X^j \in X_r$ do the following
   - Let $(A^{i,j},B^{i,j})$ be the "natural" partition of X^i defined by (A_j,B_j) using the subsets $X^i{}_1,\ldots,X^i{}_M$
   - Let $(A^{j,i},B^{j,i})$ be the "natural" partition of X^j defined by (A_i,B_i) using the subsets $X^j{}_1,\ldots,X^j{}_M$
   - Add weight $\varepsilon^{0.5}$ to edges in $(A^{i,j} ,A^{j,i})\cup(B^{i,j} ,B^{j,i})$.

A Couple of Observations

1. Iteration r prevents X_r from being an ε*-regular* partition of G.
2. After iterations $1,\ldots r\text{-}1$, for any $X^i, X^j \in X_r$ all edges in (X^i, X^j) have the same weight.

   In particular, finest partition (partition $X_{1/\sqrt{\varepsilon}}$) is $o(1)$-regular.
3. For every $X \in X_r$ and any vertex u the total density added at iteration r to $d(u,X)$ is exactly $0.5\varepsilon^{0.5}$.
4. Same holds for every $X \in X_r$ and set of vertices U.

Corollary: For every $X \in X_r$ and vertex set U, the total density added to $d(U,X)$ in iterations $r,\ldots,\varepsilon^{-0.5}$ is exactly $0.5\varepsilon^{0.5}(\varepsilon^{-0.5}-r+1)$.

The Key Lemma

Definition: $A \subset_\alpha B$ if $|A \cap B| \geq (1-\alpha)|A|$. (so $A \subset_0 B$ iff $A \subseteq B$)
If $\mathcal{Z}$ and $\mathcal{X}$ are partitions of $V(G)$ then $\mathcal{Z} \subset_\alpha \mathcal{X}$, if for every $Z \in \mathcal{Z}$ there is $X \in \mathcal{X}$ such that $Z \subset_\alpha X$. (so $\mathcal{Z} \subset_0 \mathcal{X}$ iff $\mathcal{Z}$ refines $\mathcal{X}$)

Key Lemma: If $\mathcal{Z}$ is an ε*-regular* partition of G and $\mathcal{Z} \subset_\alpha \mathcal{X}_r$ then $\mathcal{Z} \subset_{\alpha+8\varepsilon} \mathcal{X}_{r+1}$ (assuming $\alpha \leq \varepsilon^{0.5}$).

Corollary: If $\mathcal{Z}$ is an ε*-regular* partition of G then $\mathcal{Z} \subset_{8\sqrt{\varepsilon}} \mathcal{X}_{1/\sqrt{\varepsilon}}$

Proof: $\mathcal{Z} \subset_0 \mathcal{X}_0$ so we can repeatedly apply the Key Lemma.

Corollary: $|\mathcal{Z}| \geq |\mathcal{X}_{1/\sqrt{\varepsilon}}|/2 = \mathrm{twr}(1/\varepsilon^{0.5})$.

Proof of Key Lemma

Definition: $A \subset_\alpha B$ if $|A \cap B| \geq (1-\alpha)|A|$. If Z and X are partitions of $V(G)$ then $Z \subset_\alpha X$, if for every $Z \in Z$ there is $X \in X$ such that $Z \subset_\alpha X$.

Key Lemma: If Z is an ε*-regular* partition of G and $Z \subset_\alpha X_r$ then $Z \subset_{\alpha+8\varepsilon} X_{r+1}$ (assuming $\alpha \leq \varepsilon^{0.5}$).

Proof: Suppose $Z_0 \in Z$ satisfies $Z_0 \subset_\alpha X^i \in X_r$ but does not satisfy $Z_0 \subset_{\alpha+8\varepsilon} X^i_t$ for all sets $X^i_1, \ldots, X^i_M \in X_{r+1}$. We need to find εk sets Z so that (Z_0, Z) is not ε*-regular*.

Claim: At least $m/6$ of the sets $X^j \in X_r$ satisfy

$$\min(|Z_0 \cap A^{i,j}|, |Z_0 \cap B^{i,j}|) \geq \varepsilon\, |Z_0| \qquad (*)$$

Proof of Key Lemma

Claim: At least $m/6$ of the sets $X^j \in X_r$ satisfy

$$\min(|Z_0 \cap A^{i,j}|, |Z_0 \cap B^{i,j}|) \geq \varepsilon |Z_0| \qquad (*)$$

Def: A vertex $u \in Z \cap X$ is *useful* if $Z \subset_\alpha X$

Def: A set $X^j \in X_r$ is *useful* if it satisfies (*) and at least $|X^j|(1-12\alpha)$ of its vertices are useful

Claim 1: At least $m/12$ of the sets $X^j \in X_r$ are useful.

Proof: At least $m/6$ of the sets $X^j \in X_r$ satisfy (*) and at most $m/12$ of them have more than $12\alpha|X^j|$ unfriendly vertices.

Proof of Key Lemma

Recall: Assuming $Z_0 \in Z$ satisfies $Z_0 \subset_\alpha X^i \in X_r$ but does not satisfy $Z_0 \subset_{\alpha+8\varepsilon} X^i_t$ for all sets $X^i_1, \ldots, X^i_M \in X_{r+1}$, we need to find εk sets Z so that (Z_0, Z) is not ε-*regular*.

Claim 1: At least $m/12$ of the sets $X^j \in X_r$ are useful.

Claim 2: If $X^j \in X_r$ is useful then there are $12\varepsilon k/m$ sets Z so that $Z \subset_\alpha X^j$ and (Z_0, Z) is not ε-*regular*.

Claim 1 and *Claim 2* give the Key Lemma.

Proof of Key Lemma

Claim 2: If $X^j \in X_r$ is useful then there are $12\varepsilon k/m$ sets Z so that $Z \subset_\alpha X^j$ and (Z_0,Z) is not ε-*regular*.

Proof: Suppose claim is false. Set $Z^1 = Z_0 \cap A^{i,j}$, $Z^2 = Z_0 \cap B^{i,j}$.
$F(u,v)$ is weight added to (u,v) at iterations $r+1, r+2, \ldots$

Define $A \subseteq A^{j,i}$ as follows. Suppose $u \in Z \subset_\alpha X^j$. Put u in A if

1. (Z_0,Z) is not ε-*regular*
2. (Z_0,Z) is ε-*regular*, but

$$d_F(u,Z^2) < d_F(u,Z^1) + 0.75\varepsilon^{0.5}$$

Claim: $|A| \leq 0.5\ \varepsilon^{0.5}|A^{j,i}|$

Conclusion: $d_F(A^{j,i},Z^2) - d_F(A^{j,i},Z^1) > (1-\varepsilon^{0.5}/2)\ \tfrac{3}{4}\varepsilon^{0.5} - \varepsilon^{0.5}/2 > 0$

Concluding Remarks

Gowers '97, Conlon-Fox '12: $\mathrm{twr}(1/\varepsilon^c)$ lower bounds for weaker versions of regularity lemma.

Find simple/short proofs of these results.

Thank You

Quasi-random set partitions

Definition: A sequence of bipartitions $(A_1,B_1),\ldots,(A_m,B_m)$ of $[M]$ *is c-balanced* if

1. For every $1 \le i \le m$, we have $|A_i|=|B_i|=M/2$
2. For every $t,t' \in [M]$, at most $(½ + c)m$ of the bipartitions (A_i,B_i) are such that t,t' belong to the same set (A_i or B_i).

Lemma: If $M=2^{m/100}$ then there is a sequence of m bipartitions of M that is ¼-balanced.

Proof: Random bipartitions.

Quasi-random set partitions

Lemma 1: If $(A_1,B_1),\ldots,(A_m,B_m)$ is a ¼-*balanced* sequence of bipartitions of $[M]$, then for every $\lambda=(\lambda_1,\ldots,\lambda_M)$ with

$$\lambda_i \geq 0 \ , \ ||\lambda||_1 = 1 \ , \ ||\lambda||_\infty \leq 1-8\delta$$

then there is a bipartition (A_i,B_i) so that

$$\min\left(\sum_{t\in A_i} \lambda_t \, , \sum_{t\in B_i} \lambda_t \right) \geq \delta$$

Proof: Pick a random (A_i,B_i)

Set $Y_t = 1/-1$ if $t \in A_i / B_i$, and $Y = \sum_t \lambda_t Y_t$

Then $E[Y^2_t] = 1$ and $E[Y_t Y_{t'}] \leq ½$, implying that $E[Y^2] \leq 1-4\delta$

Hence $E[|Y|] \leq 1-2\delta$, so there is a partition (A_i,B_i) satisfying

$$\left| \sum_{t\in A_i} \lambda_t - \sum_{t\in B_i} \lambda_t \right| \leq 1-2\delta.$$

Amplifying Lemma 1

Lemma 1: If $(A_1,B_1),\ldots,(A_m,B_m)$ is a 1/4-*balanced* sequence of partitions of $[M]$, then for every $\lambda=(\lambda_1,\ldots,\lambda_M)$ s.t. … there is a bipartition (A_i,B_i) s.t …

Lemma 2: If $(A_1,B_1),\ldots,(A_m,B_m)$ is a 0.1-*balanced* sequence of bipartitions of $[M]$, then for every $\lambda=(\lambda_1,\ldots,\lambda_M)$ with … there are *m/6* bipartitions (A_i,B_i) so that …

Proof: Repeatedly apply Lemma 1. Since the sequence was initially *0.1-balanced*, then as long as we remove less than *m/6* partitions, the remaining sequence is still *1/4-balanced*.

Coro: If $M=2^{m/100}$ then there are *m* bipartitions of $[M]$ s.t. for every $\lambda\in R^M$ satisfying… there are *m/6* bipartitions satisfying…