The Green-Tao theorem and a relative Szemerédi theorem

Yufei Zhao

Massachusetts Institute of Technology

Joint work with David Conlon (Oxford) and Jacob Fox (MIT)

Simons Institute
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Green–Tao Theorem (arXiv 2004; Annals 2008)

The primes contain arbitrarily long arithmetic progressions.

Examples:

- 3, 5, 7
- 5, 11, 17, 23, 29
- 7, 37, 67, 97, 127, 157
- Longest known: 26 terms

Green–Tao Theorem (2008)

The primes contain arbitrarily long arithmetic progressions (AP).

Szemerédi's Theorem (1975)

Every subset of $\mathbb{N}$ with positive density contains arbitrarily long APs.

(upper) density of $A \subset \mathbb{N}$ is $\limsup_{N\to\infty} \frac{|A \cap [N]|}{N}$

$[N] := \{1, 2, \ldots, N\}$

$P =$ prime numbers

Prime number theorem: $\frac{|P \cap [N]|}{N} \sim \frac{1}{\log N}$

Proof strategy of Green–Tao theorem

P = prime numbers, Q = "almost primes"

$P \subseteq Q$ with relative positive density, i.e., $\dfrac{|P \cap [N]|}{|Q \cap [N]|} > \delta$

Step 1:

Relative Szemerédi theorem (informally)

If $S \subset \mathbb{N}$ satisfies certain pseudorandomness conditions, then every subset of S of positive density contains long APs.

Step 2: Construct a superset of primes that satisfies the conditions.

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What pseudorandomness conditions?

Green–Tao:

1. Linear forms condition
2. Correlation condition

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A natural question (e.g., asked by Green, Gowers, ...)

Does relative Szemerédi theorem hold with weaker and more natural hypotheses?

Relative Szemerédi theorem

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Green–Tao:

1. Linear forms condition
2. Correlation condition $\leftarrow$ no longer needed

A natural question (e.g., asked by Green, Gowers, ...)

Does relative Szemerédi theorem hold with weaker and more natural hypotheses?

Our main result

Yes! A weak linear forms condition suffices.

Szemerédi's theorem

Host set: $\mathbb{N}$

Relative Szemerédi theorem

Host set: some sparse subset of integers

Conclusion: relatively dense subsets contain long APs

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Random host set

- Kohayakawa–Łuczak–Rödl '96 — *3-AP*, $p \gtrsim N^{-1/2}$
- Conlon–Gowers '10+, Schacht '10+ — *k-AP*, $p \gtrsim N^{-1/(k-1)}$

Pseudorandom host set

- Green–Tao '08 — *linear forms + correlation*
- Conlon–Fox–Z. '13+ — *linear forms*

Conclusion: relatively dense subsets contain long APs

Roth's theorem

Roth's theorem (1952)

If $A \subseteq [N]$ is 3-AP-free, then $|A| = o(N)$.

$[N] := \{1, 2, \ldots, N\}$

3-AP = 3-term arithmetic progression

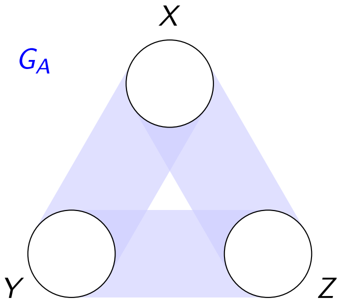
It'll be easier (and equivalent) to work in $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$.

Proof of Roth's theorem

Roth's theorem (1952)

If $A \subseteq \mathbb{Z}_N$ is 3-AP-free, then $|A| = o(N)$.

Given A, construct tripartite graph G_A with vertex sets $X = Y = Z = \mathbb{Z}_N$.



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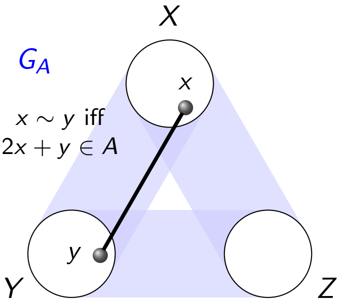

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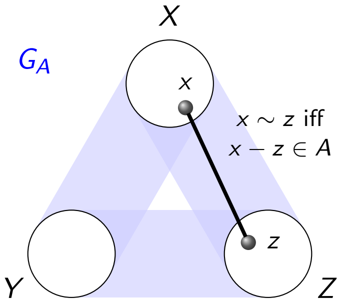

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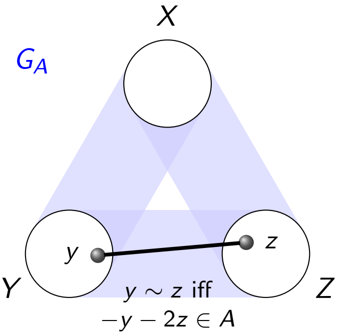

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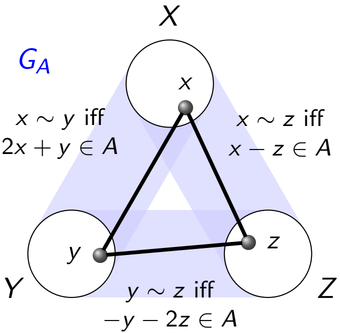

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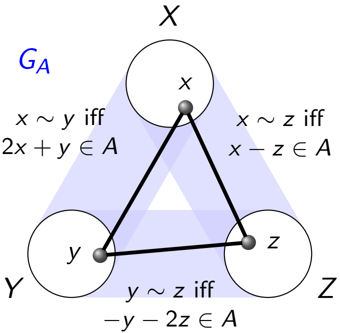
Triangle xyz in $G_A \iff$

$$2x + y,\ x - z,\ -y - 2z \ \in A$$



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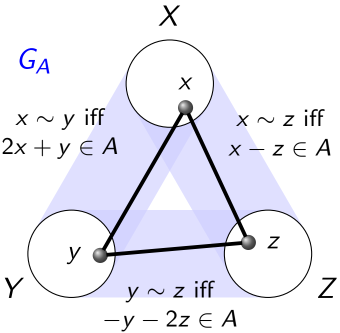
It's a 3-AP with diff $-x - y - z$



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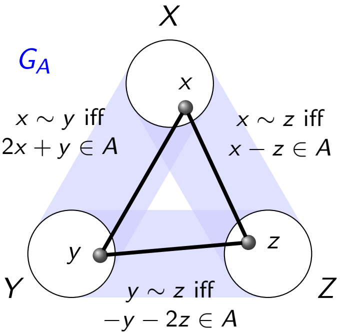

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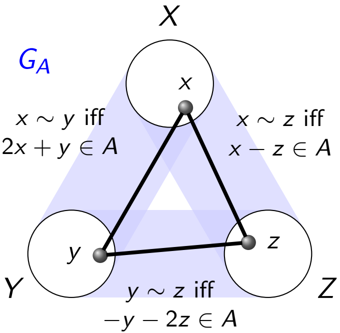

~~No triangles?~~ Only triangles $\longleftrightarrow$ trivial 3-APs with diff 0.

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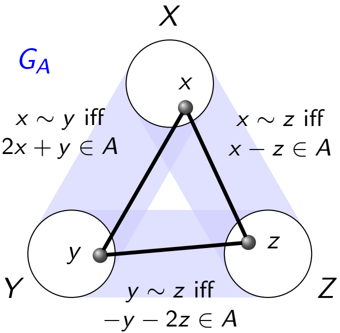

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Every edge of the graph is contained in exactly one triangle (the one with $x + y + z = 0$).

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Constructed a graph with

- $3N$ vertices
- $3N|A|$ edges
- every edge in exactly one triangle

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Theorem (Ruzsa & Szemerédi '76)

If every edge in a graph $G = (V, E)$ is contained in exactly one triangle, then $|E| = o(|V|^2)$.

(a consequence of the *triangle removal lemma*)

So $3N|A| = o(N^2)$. Thus $|A| = o(N)$.

Relative Roth theorem

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Relative Roth theorem (Conlon, Fox, Z.)

If $S \subseteq \mathbb{Z}_N$ satisfies some pseudorandomness conditions, and $A \subseteq S$ is 3-AP-free, then $|A| = o(|S|)$.

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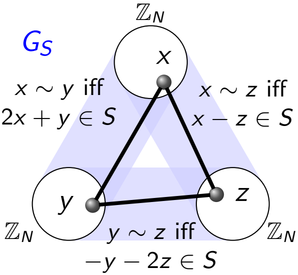

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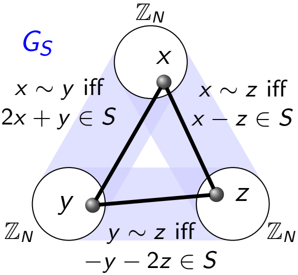
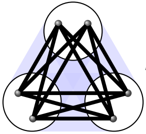
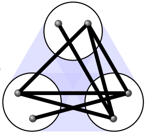


Pseudorandomness condition for S:

G_S has asymp. the expected number of embeddings of $K_{2,2,2}$ & its subgraphs (compared to random graph of same density)

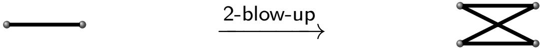


$K_{2,2,2}$ & subgraphs, e.g.,



Analogy with quasirandom graphs

Chung-Graham-Wilson '89 showed that in constant edge-density graphs, many quasirandomness conditions are equivalent, one of which is having the correct C_4 count



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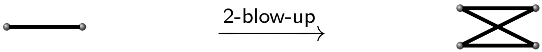

In sparse graphs, the CGW equivalences do *not* hold.

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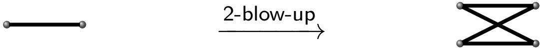

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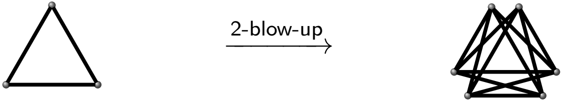
Our results can be viewed as saying that:

Many extremal and Ramsey results about H (e.g., $H = K_3$) in sparse graphs hold if there is a host graph that behaves pseudorandomly with respect to counts of the 2-blow-up of H.



Roth's theorem: from one 3-AP to many 3-APs

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$\forall \delta > 0$. Every $A \subset \mathbb{Z}_N$ with $|A| \geq \delta N$ contains a 3-AP, provided N is sufficiently large.

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$\forall \delta > 0$. Every $A \subset \mathbb{Z}_N$ with $|A| \geq \delta N$ contains a 3-AP, provided N is sufficiently large.

By an averaging argument (Varnavides), we get many 3-APs:

Roth's theorem (counting version)

$\forall \delta > 0 \; \exists c > 0$ so that every $A \subset \mathbb{Z}_N$ with $|A| \geq \delta N$ contains at least cN^2 3-APs, provided N is sufficiently large.

Transference

Start with

$$\text{(sparse)} \qquad A \subset S \subset \mathbb{Z}_N, \qquad\qquad |A| \geq \delta\,|S|$$

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Counting lemma will tell us that

$$\left(\frac{N}{|S|}\right)^3 |\{\text{3-APs in } A\}| \approx |\{\text{3-APs in } \widetilde{A}\}|$$

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$$\geq cN^2 \qquad \text{[By Roth's Theorem]}$$

$\implies$ relative Roth theorem

Roth's theorem: weighted version

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$\forall \delta > 0\ \exists c > 0$ so that every $f : \mathbb{Z}_N \to [0,1]$ with $\mathbb{E} f \geq \delta$ satisfies

$$\mathbb{E}_{x,d \in \mathbb{Z}_N}[f(x)f(x+d)f(x+2d)] \geq c$$

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Sparse setting: some sparse host set $S \subset \mathbb{Z}_N$. More generally, use a normalized measure:

$$\nu\colon \mathbb{Z}_N \to [0,\infty) \qquad \text{with} \qquad \mathbb{E}\nu = 1.$$

E.g., $\nu = \frac{N}{|S|} 1_S$ normalized indicator function.

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The subset $A \subset S$ with $|A| \geq \delta |S|$ corresponds to

$$f \colon \mathbb{Z}_N \to [0,\infty), \qquad \mathbb{E}f \geq \delta$$

and f majorized by ν, meaning that $f(x) \leq \nu(x)\ \forall x \in \mathbb{Z}_N$.

	Sets	Functions
Dense setting	$A \subset \mathbb{Z}_N$ $\lvert A \rvert \geq \delta$	$f : \mathbb{Z}_N \to [0, 1]$ $\mathbb{E} f \geq \delta$
Sparse setting	$A \subset S \subset \mathbb{Z}_N$ $\lvert A \rvert \geq \delta \lvert S \rvert$	$f \leq \nu : \mathbb{Z}_N \to [0, \infty)$ $\mathbb{E} f \geq \delta$, $\mathbb{E}\nu = 1$

(sparse with $\nu \equiv 1 \longrightarrow$ dense setting)

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Relative Roth theorem (Conlon, Fox, Z.)

$\forall \delta > 0\ \exists c > 0$ so that if

- $\nu : \mathbb{Z}_N \to [0,\infty)$ satisfies the 3-linear forms condition, and
- $f : \mathbb{Z}_N \to [0,\infty)$ majorized by ν and $\mathbb{E} f \geq \delta$, then

$$\mathbb{E}_{x,d \in \mathbb{Z}_N}[f(x)f(x+d)f(x+2d)] \geq c$$

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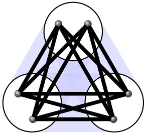
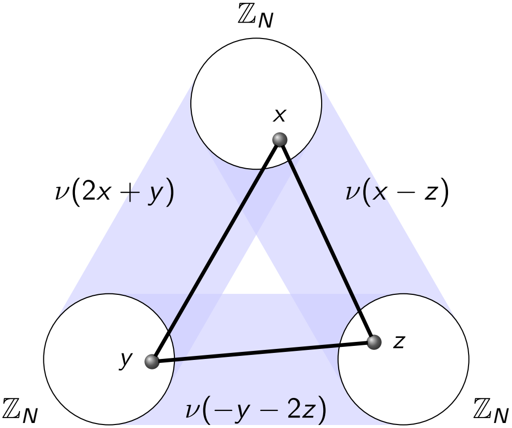
3-linear forms condition

The density of $K_{2,2,2}$  in



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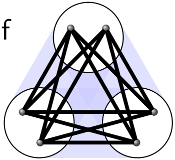
$\nu\colon \mathbb{Z}_N \to [0,\infty)$ satisfies the 3-linear forms condition if



$$\mathbb{E}[\nu(2x+y)\nu(2x'+y)\nu(2x+y')\nu(2x'+y')\cdot$$
$$\nu(x-z)\nu(x'-z)\nu(x-z')\nu(x'-z')\cdot$$
$$\nu(-y-2z)\nu(-y'-2z)\nu(-y-2z')\nu(-y'-2z')] = 1 + o(1)$$

as well as if any subset of the 12 factors were deleted.

Relative Szemerédi theorem (Conlon, Fox, Z.)

$\forall \delta > 0, k \in \mathbb{N}\ \exists c(k,\delta) > 0$ so that if

- $\nu \colon \mathbb{Z}_N \to [0,\infty)$ satisfies the *k-linear forms condition*, and
- $f \colon \mathbb{Z}_N \to [0,\infty)$ majorized by ν and $\mathbb{E} f \geq \delta$, then

$$\mathbb{E}_{x,d \in \mathbb{Z}_N}[f(x)f(x+d)f(x+2d)\cdots f(x+(k-1)d)] \geq c(k,\delta)$$

provided N is sufficiently large.

$k = 4$: build a weighted 4-partite 3-uniform hypergraph

on $W \times X \times Y$: $\nu(3w + 2x + y \qquad)$
on $W \times X \times Z$: $\nu(2w + x \qquad - z)$
on $W \times Y \times Z$: $\nu(w \qquad - y - 2z)$
on $X \times Y \times Z$: $\nu(\qquad -x - 2y - 3z)$

common diff: $-w-x-y-z$

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$k = 4$: build a weighted 4-partite 3-uniform hypergraph

on $W \times X \times Y$: $\nu(3w + 2x + y)$
on $W \times X \times Z$: $\nu(2w + x - z)$
on $W \times Y \times Z$: $\nu(w - y - 2z)$
on $X \times Y \times Z$: $\nu(-x - 2y - 3z)$

common diff: $-w-x-y-z$

4-linear forms condition: correct count of the 2-blow-up of the simplex $K_4^{(3)}$ (as well as its subgraphs)

Two approaches

Conlon, Fox, Z.

A relative Szemerédi theorem. 20pp

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An arithmetic transference proof of a relative Sz. theorem. 6pp

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Two approaches

Conlon, Fox, Z.

A relative Szemerédi theorem. 20pp

- Transfer hypergraph removal lemma

More general

Z.

An arithmetic transference proof of a relative Sz. theorem. 6pp

- Transfer Szemerédi's theorem

More direct

Transference

Start with $f \leq \nu$

$$\text{(sparse)} \qquad f : \mathbb{Z}_N \to [0, \infty) \qquad \mathbb{E} f \geq \delta$$

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$\Longrightarrow$ relative Roth theorem

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- Our approach: cut norm (aka discrepancy)

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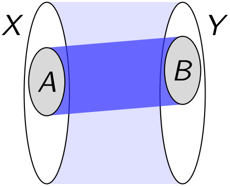
- Our approach: cut norm (aka discrepancy)

Using cut norm:

- Cheaper dense model theorem
- Trickier counting lemma

Cut norm

Weighted bipartite graphs $g, \widetilde{g} \colon X \times Y \to \mathbb{R}$



Cut norm (Frieze-Kannan): $\|g - \widetilde{g}\|_\square \le \epsilon$

means that for all $A \subset X$ and $B \subset Y$:

$$\left| \mathbb{E}_{\substack{x \in X \\ y \in Y}} [(g(x,y) - \widetilde{g}(x,y)) 1_A(x) 1_B(y)] \right| \le \epsilon$$

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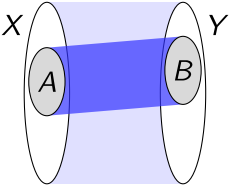

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For $\mathbb{Z}_N$: $f, \widetilde{f} \colon \mathbb{Z}_N \to \mathbb{R}$ being ϵ-close in cut norm means:
for all $A, B \subset \mathbb{Z}_N$

$$\left| \mathbb{E}_{x,y \in \mathbb{Z}_N} [(f(2x+y) - \widetilde{f}(2x+y)) 1_A(x) 1_B(y)] \right| \le \epsilon.$$

(weaker than being close in Gowers uniformity norm)

Dense model theorem

Theorem (Dense model)

If $\nu : \mathbb{Z}_N \to [0, \infty)$ *is close to 1 in cut norm then*

$\forall\ f : \mathbb{Z}_N \to [0, \infty)$ *majorized by* ν

$\exists\ \widetilde{f} : \mathbb{Z}_N \to [0, 1]$ *s.t.* f *is close to* $\widetilde{f}$ *in cut norm and* $\mathbb{E}f = \mathbb{E}\widetilde{f}$

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Proof approaches

1. Regularity-type energy-increment argument
   (Green–Tao, Tao–Ziegler)
2. Separating hyperplane theorem / LP duality
   + Weierstrass polynomial approximation theorem
   (Gowers & Reingold–Trevisan–Tulsiani–Vadhan)

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Counting lemma

Weighted graphs $g, \widetilde{g} \colon (X \times Y) \cup (X \times Z) \cup (Y \times Z) \to \mathbb{R}$

Triangle counting lemma (dense setting)

Assume $0 \le g, \widetilde{g} \le 1$. If $\|g - \widetilde{g}\|_\square \le \epsilon$, then

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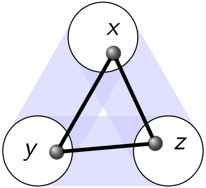

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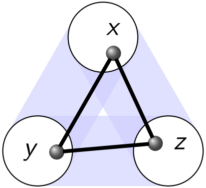

$$|\mathbb{E}[(g(x,y) - \widetilde{g}(x,y))1_A(x)1_B(y)]| \le \epsilon \quad \forall A \subset X, B \subset Y$$

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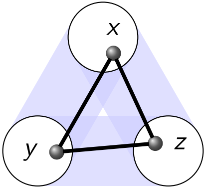

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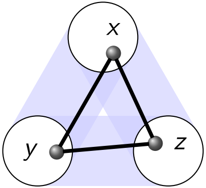

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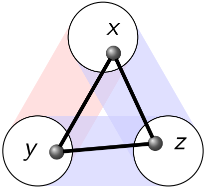

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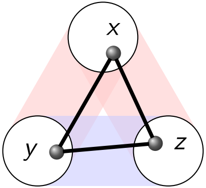

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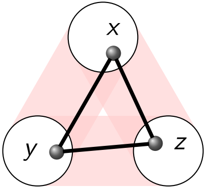

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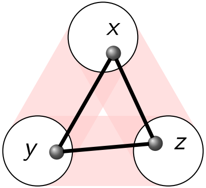

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This argument doesn't work in the sparse setting (g unbounded)

Sparse counting lemma

Sparse triangle counting lemma (Conlon, Fox, Z.)

Assume that ν satisfies the 3-linear forms condition.
If $0 \le g \le \nu$, $0 \le \widetilde{g} \le 1$ and $\|g - \widetilde{g}\|_\square = o(1)$, then

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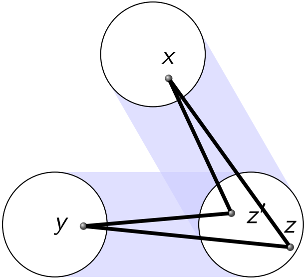
Proof ingredients

1. Cauchy-Schwarz
2. **Densification**
3. Apply cut norm/discrepancy (as in dense case)

Densification



$$\mathbb{E}[g(x,z)g(y,z)g(x,z')g(y,z')]$$

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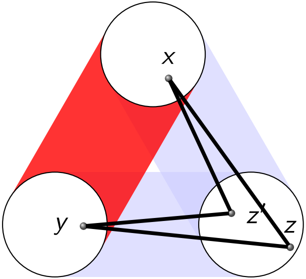

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$g'(x,y) \lesssim 1$ for almost all (x,y)

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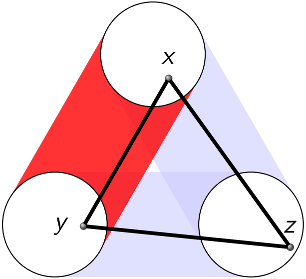

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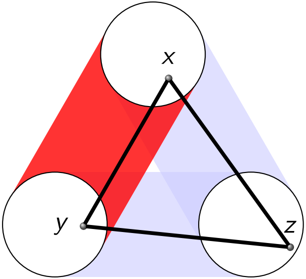

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Made $X \times Y$ dense. Now repeat for $X \times Z$ & $Y \times Z$.
Reduce to dense setting.

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$\implies$ relative Roth theorem

COMING SOON

The Green-Tao theorem: an exposition

Coming Soon

The Green-Tao theorem: an exposition

- A gentle exposition giving a complete & self-contained proof of the Green-Tao theorem (other than a black-box application of Szemerédi's theorem)
- $\sim$ 25 pages

Relative Szemerédi theorem (Conlon, Fox, Z.)

If $\nu\colon \mathbb{Z}_N \to [0,\infty)$ satisfies the k-linear forms condition, then any f with $0 \le f \le \nu$ and

$$\mathbb{E}_{x,d\in\mathbb{Z}_N}[f(x)f(x+d)f(x+2d)\cdots f(x+(k-1)d)] = o(1)$$

must satisfy $\mathbb{E}f = o(1)$.

3-linear forms condition: $(x, x', y, y', z, z' \sim \mathrm{Unif}(\mathbb{Z}_N))$

$$\begin{aligned}\mathbb{E}[&\nu(2x+y)\nu(2x'+y)\nu(2x+y')\nu(2x'+y')\cdot\\ &\nu(x-z)\nu(x'-z)\nu(x-z')\nu(x'-z')\cdot\\ &\nu(-y-2z)\nu(-y'-2z)\nu(-y-2z')\nu(-y'-2z')] = 1+o(1)\end{aligned}$$

as well as if any subset of the 12 factors were deleted.

4-linear forms condition: $\mathbb{E}[\nu(3w+2x+y)\cdots] = 1+o(1)$

THANK YOU!