Lower Bounds from Algorithm Design: An Overview

Ryan Williams
MIT

Course Announcement

CS294-152. Lower Bounds: Beyond the Boot Camp

Soda 405

Mondays 4:00pm to ≈ 6:30pm

(with a break in the middle)

first lecture is next week

Outline

- A High-Level View
- Algorithms versus Boolean Circuits
- Circuit Analysis => Circuit Lower Bounds
- Some Details and Some Progress:
  - NQP (Quasi-NP) is not in ACC
  - NP doesn’t have small depth-two neural nets

High-level view of algorithms and complexity

- **Algorithm designers**

- **Complexity theorists**

- **What makes some problems easy to solve?**
  When can we find an *efficient* algorithm?

- **What makes other problems difficult?**
  When can we prove that a problem is not easy?

  When can we prove a *Lower Bound on the resources (time/space/communication/etc) needed to solve a problem*?

The tasks of the algorithm designer and the complexity theorist appear to be polar opposites.

- **Algorithm designers prove upper bounds**
- **Complexity theorists prove lower bounds**

Furthermore, it's generally believed that Algorithm Design is easier than Lower Bounds

- **In Algorithm Design: find one clever algorithm**
- **In Lower Bounds: must reason about "all possible" algorithms, and argue none of them work well**

… but there are thousands of worst-case algorithms which analyze all possible finite objects of some kind…

My Opinion:
This isn't why lower bounds are hard!

Why are lower bounds hard to prove?

There are *many* known “no-go” theorems

- Relativization [70’s]
- Natural Properties [90’s]
- Algebrization [00’s]

Summary: The common proof techniques are not good enough to prove even weak lower bounds!

Great pessimism in complexity theory

How will we make progress?

There are *many* known "no-go" theorems

- Relativization [70's]
- Natural Properties [90's]
- Algebrization [00's]

Summary: The common proof techniques are not good enough to prove even weak lower bounds!

Great pessimism in complexity theory

Have to non-relativize, non-algebrize, and non-naturalize!

One Direction for Progress:
Connect Algorithm Design to Lower Bounds

Much more than *opposites*!
There are deeper connections we are slowly uncovering.

Thesis: Designing Algorithms (in some sense) *is equivalent to* Proving Lower Bounds

A typical result in Algorithm Design:
"Here is an algorithm A that solves the problem, on all possible instances of the problem"

A typical theorem from Lower Bounds:
"Here is a proof P that the problem can't be solved, by all possible algorithms of some type"

Meta-computation:
Problems whose input is the code of an algorithm

A “Plan” For Proving Lower Bounds

Want to prove results of the form:

Task A is impossible for computation model B

Find results showing (algorithm design → lower bounds):

Task A’ is possible for computation model B’
→ Task A is impossible for computation model B

Then, use results from algorithm design to show:

Task A’ is possible for computation model B’

Where do we start????

Want to prove results of the form:

Task A is impossible for computation model B

Find results showing (algorithm design ➔ lower bounds):

Define Task A' be *about* analyzing model B

????

Task A' is possible for computation model B'

➔ Task A is impossible for computation model B

????

Define Task A in terms of model B'

Then, use results from algorithm design to show:

Task A' is possible for computation model B'

(algorithm design ➔ lower bounds)?

A simple example from complexity theory:

If PSPACE = EXPTIME then PTIME $\neq$ PSPACE

PSPACE = problems solvable in polynomial space

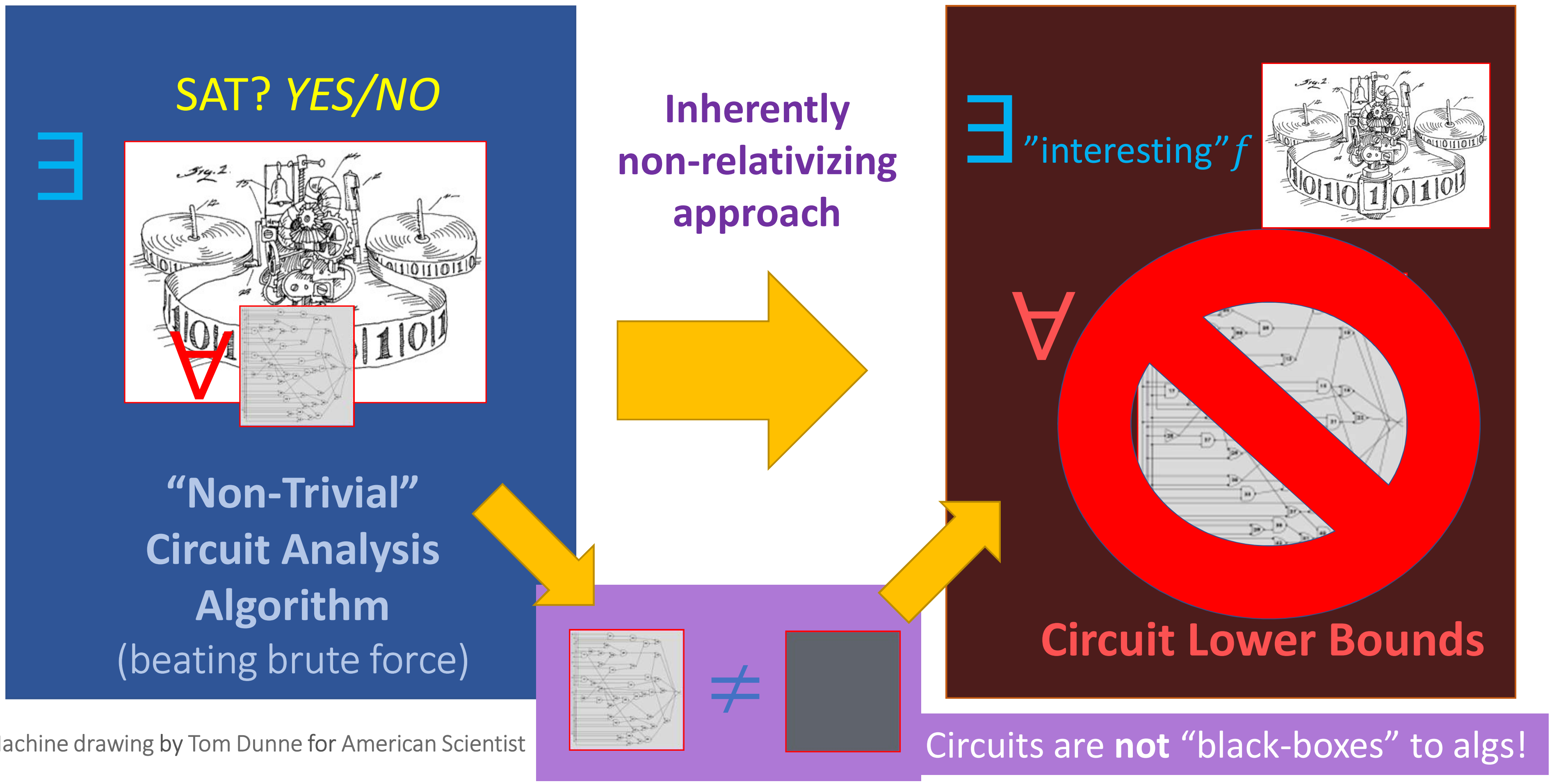
PTIME = ... in polynomial time

EXPTIME = ... in exponential time

Proof: PTIME $\neq$ EXPTIME (time hierarchy theorem)
So PTIME = PSPACE implies PSPACE $\neq$ EXPTIME. QED

Many such results can be proved....
But they do not seem useful!

Big Idea: Interesting circuit-analysis algorithms tell us about the *limitations* of circuits in modeling algorithms



Turing Machine drawing by Tom Dunne for American Scientist

Big Idea: Interesting circuit-analysis algorithms
tell us about the *limitations* of circuits in modeling algorithms

Goal: Algorithmic task A is impossible for
"efficient" circuits (this is our model B)

Show: Non-trivial analysis of "efficient" circuits
is possible with algorithms (model B')
➔ Algorithmic Task A is impossible for
"efficient" circuits

Show: Non-trivial analysis of "efficient" circuits
is possible with algorithms

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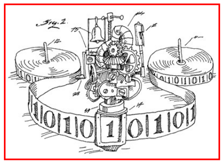
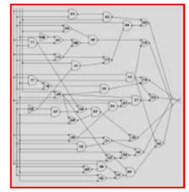
Algorithms

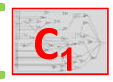


Can take in **arbitrarily long inputs** and still solve the problem
$f : \{0,1\}^* \to \{0,1\}$

(Boolean) Circuits



Only take in fixed-length inputs
$g: \{0,1\}^n \to \{0,1\}$

Circuit Family = { 

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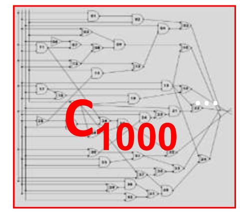
, 

, **}**

For every input length n,
a ***circuit family*** has a circuit $\mathbf{C}_n$ to be run on all inputs of length n

P/poly = **{** $f : \{0,1\}^* \to \{0,1\}$ computable by a **circuit family** $\{\mathbf{C}_n\}$
such that $(\exists k \geq 1)(\forall n)$, the ***size* of $\mathbf{C}_n$ is at most n^k }**

Each circuit is "small" relative to its number of inputs

Circuit model has "programs with ***infinite-length descriptions***"
The standard methods in computability theory are powerless…

Circuit Family = { 

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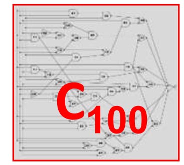
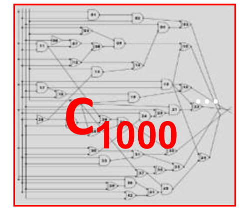
, 

, 

, }

P/poly = { $f : \{\mathbf{0},\mathbf{1}\}^* \rightarrow \{\mathbf{0},\mathbf{1}\}$ computable with a **circuit family**
{C_n} such that $(\exists k \geq 1)(\forall n)$, the ***size* of C_n is at most n^k** }

Why study this "infinite" model of computation?

1) Circuits could be easier to analyze than Turing machines!

2) Proving limitations on P/poly is a step towards *non-asymptotic complexity theory:*

Concrete limitations on computing within the known universe
"Any logic circuit solving most instances of my 1000-bit problem needs at least 10^{100} bits to be described"

Universe stores < 10^{80} bits [Bekenstein '70s] [Meyer-Stockmeyer '70s]

Algorithms versus Circuit Families

P/poly = { $\boldsymbol{f} : \{\mathbf{0},\mathbf{1}\}^* \to \{\mathbf{0},\mathbf{1}\}$ computable with a **circuit family**
$\{C_n\}$ such that $(\exists k \geq 1)(\forall n)$, the ***size* of C_n is at most $\boldsymbol{n}^k$** }

Most Boolean functions require huge circuits:

Theorem [Shannon '49] W.h.p., random $\boldsymbol{f} : \{\mathbf{0},\mathbf{1}\}^{\boldsymbol{n}} \to \{\mathbf{0},\mathbf{1}\}$ needs circuits of size at least **$2^n/n$**

Theorem [Lupanov'58] Every f has a circuit of size **$(1+o(1))2^n/n$**

Explicit (non-random) hard functions?

What "uniform" algorithms can be simulated in P/poly?
Can huge uniform classes (like PSPACE, EXP, NEXP) be simulated with small non-uniform classes (like P/poly)?

The key obstacle: Non-uniformity can be very powerful!

Algorithms versus Circuit Families

What "uniform" algorithms can be simulated in P/poly?
Can huge uniform classes (like PSPACE, EXP, NEXP)
be simulated with small non-uniform classes (like P/poly)?

RIDICULOUSLY OPEN: Is **NEXP** $\subset$ **P/poly?**

Can all problems with *exponentially-long answers checkable in exponential time* be solved with polynomial-size circuit families?

Conjecture: NP $\not\subset$ P/poly (harder than P $\neq$ NP)

OPEN: NP $\not\subset$ SIZE(O(n))? Best known: NP $\not\subset$ SIZE(5n), SIZE(3.01n)

Now, problems like NP $\not\subset$ SIZE(O(n)) may be attackable…(?)

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Generalized Circuit Satisfiability

Let **C** be a class of Boolean circuits

C = {formulas}, **C** = {arbitrary circuits}, **C** = {3CNFs}

The C-SAT Problem:
Given a circuit $\mathbf{K(x_1,\ldots,x_n)}$ from **C**, is there an assignment $(a_1, \ldots, a_n) \in \{0,1\}^n$ such that $\mathbf{K(a_1,\ldots,a_n) = 1}$?

A very "simple" circuit analysis problem!

[CL'70s] **C-SAT** is **NP-complete** for practically all interesting **C**

C-SAT is solvable in $\mathbf{O(2^n\ |K|)}$ time by brute force

Gap Circuit Satisfiability

Let **C** be a class of Boolean circuits

C = {formulas}, **C** = {arbitrary circuits}, **C** = {3CNFs}

Gap-C-SAT:
Given $K(x_1,\ldots,x_n)$ from **C**, and the **promise** that either
(a) $K \equiv 0$, or (b) $Pr_x[K(x) = 1] \geq 1/2$,
decide which is true.

Even simpler! In randomized polynomial time

[Folklore?] If **Gap-Circuit-SAT** ∈ **P** then **P = RP**
[Hirsch, Trevisan, …] **Gap-kSAT is P for all k**

Faster **C**-SAT ⟹ Circuit Lower Bounds for **C**

Slightly Faster Circuit-SAT

[R.W. '10,'11]

Deterministic algorithms for:

- Circuit SAT in $O(2^n/n^{10})$ time with n inputs and n^k gates
- Formula SAT in $O(2^n/n^{10})$ time
- $\mathcal{C}$-SAT in $O(2^n/n^{10})$ time
- Gap-$\mathcal{C}$-SAT is in $O(2^n/n^{10})$ **time** on n^k size

(Easily solved w/ randomness!)

No "Circuits for NEXP"

Would imply:

- NEXP $\not\subset$ P/poly
- NEXP $\not\subset$ Poly-size formulas
- NEXP $\not\subset$ *poly-size* $\mathcal{C}$

NEXP $\not\subset$ *poly-size* $\mathcal{C}$

Concrete LBs

$\mathcal{C}$ = ACC [W'11]

$\mathcal{C}$ = ACC of THR [W'14]

Even Faster SAT $\Longrightarrow$ Stronger Lower Bounds

Somewhat Faster Circuit SAT
[Murray-W. '18]

Det. algorithm for some $\epsilon > 0$:

- Circuit SAT in $O(2^{n-n^{\epsilon}})$ time with n inputs and $2^{n^{\epsilon}}$ gates
- Formula SAT in $O(2^{n-n^{\epsilon}})$ time
- C-SAT in $O(2^{n-n^{\epsilon}})$ time
- Gap-C-SAT is in $\mathbf{O(2^{n-n^{\epsilon}})}$ **time** on $2^{n^{\epsilon}}$ gates

No "Circuits for Quasi-NP"

Would imply:

- NTIME$[n^{polylog\, n}] \not\subset$ P/poly
- NTIME$[n^{polylog\, n}] \not\subset$ NC1
- NTIME$[n^{polylog\, n}] \not\subset C$

NTIME$[n^{polylog\, n}] \not\subset C$

C = ACC of THR [MW'18]

Even Faster SAT $\Longrightarrow$ Stronger Lower Bounds

Note: Would refute Strong ETH!

"Fine-Grained" SAT Algorithms
[Murray-W. '18]

Det. algorithm for some $\epsilon > 0$:

- Circuit SAT in $O(2^{(1-\epsilon)n})$ time on n inputs and $2^{\epsilon n}$ gates
- FormSAT in $O(2^{(1-\epsilon)n})$ time
- C-SAT in $O(2^{(1-\epsilon)n})$ time

Strongly believed to be true...

- Gap-C-SAT is in $O(2^{(1-\epsilon)n})$ **time** on $2^{\epsilon n}$ gates

(Implied by **PromiseRP** in **P**)

No "Circuits for NP"

Would imply:

- NP $\not\subset$ SIZE(n^k) for all k
- NP $\not\subset$ Formulas of size n^k
- NP $\not\subset$ C-SIZE(n^k) for all k

NP $\not\subset$ C-SIZE(n^k) for all k

C = SUM of THR
C = SUM of ReLU
C = SUM of POL
[W'18]

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Some Lower Bounds by Algorithm Design

ACC^0: circuits of **polynomial** size and **constant** depth, with AND, OR, and MODm gates for some constant m.
$ACC^0 \subset$ P/poly, probably a proper subset!

Annoying Circuit Class to prove lower bounds for, proposed in 1986 (and it is the 0^{th} such class)

Thm [R.W.'11]: NEXP $\not\subset$ ACC^0

Thm [Murray-W'18]: NTIME[$n^{poly(\log n)}$] $\not\subset$ ACC^0 of THR

ACC ∘ THR: ***Annoying Circuits with Linear Threshold Gates*** at the bottom

Progress Report

[W'14, Murray-W'18] **Quasi-NP does not have ACC ∘ THR circuits of polynomial size**

SAT algorithm uses a new depth-two representation of **ACC ∘ THR**

and *fast rectangular matrix multiplication* to evaluate the representation quickly

Improving the lower bounds to multiple layers of THR gates is an open frontier:

[Tamaki'16, Alman-Chan-W'16] **E^{NP} does not have ACC ∘ THR ∘ THR circuits of subquadratic size**

Uses recent probabilistic polynomials for THR [Srinivasan'13, Alman-W'15]

Open: Quasi-NP does not have THR ∘ THR circuits of subquadratic size

[S.Chen-Papakonstantinou'16] **Better size-depth tradeoff lower bound for NEXP vs ACC**

[R.Chen-Oliveira-Santhanam'18] **Average Case: NEXP doesn't have poly-size ACC circuits**

computing a $\frac{1}{2} + \frac{1}{poly(\log n)}$ fraction of n-bit inputs correctly

Carefully applies coding-theoretic techniques on top of the framework

[W'18] **NP does not have $O(n^{100})$-size depth-two neural networks**

with sign activation function, nor with ReLU activation functions

At the heart: [Horowitz-Sahni 70s] **Counting subset sum solutions on n items is in $\sim 2^{n/2}$ time!**

New lower bounds from an old algorithm!

Progress Report

[W'14, Murray-W'18] Quasi-NP does not have ACC $\circ$ THR circuits of polynomial size

SAT algorithm uses a new depth-two representation of **ACC $\circ$ THR**
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At the heart: [Horowitz-Sahni 70s] Counting subset sum solutions on n items is in $\sim 2^{n/2}$ time!

New lower bounds from an old algorithm!

Lower Bounds for
NEXP, Quasi-NP, and NP
From Nontrivial Gap-SAT Algorithms

How **NEXP $\not\subset$ ACC^0** Was Proved

Let $\mathbb{C}$ be a "typical" circuit class (like ACC^0)

Thm A [W'11] (algorithm design → lower bounds)

If for all k, Gap-$\mathbb{C}$-SAT on n^k-size is in $O(2^n/n^k)$ time, then NEXP does not have poly-size $\mathbb{C}$-circuits.

Thm B [W'11] (algorithm)

$\exists\ \varepsilon$, ACC^0-SAT on 2^{n^ε} size is in $O(2^{n-n^\varepsilon})$ time.

(Used a well-known representation of ACC^0 from 1990, that people long suspected should imply lower bounds)

Note the inefficiency!

Theorem B gives a much stronger algorithm than is necessary in Theorem A.

This is exactly the starting point of [Murray-W'18]…

Idea of Theorem A

Let $\mathbb{C}$ be some circuit class (like ACC^0)

Thm A [W'11] (algorithm design ➔ lower bounds)
If for all k, Gap $\mathbb{C}$-SAT on n^k-size is in $O(2^n/n^k)$ time, then NEXP does not have poly-size $\mathbb{C}$-circuits.

Idea. Show that if we assume both:

(1) NEXP has poly-size $\mathbb{C}$-circuits,
AND
(2) a faster Gap $\mathbb{C}$-SAT algorithm

Then we can show NTIME[2^n] $\subseteq$ NTIME[o(2^n)]
(contradicts the nondeterministic time hierarchy!)

Proof Ideas in Theorem A

Idea. Assume

(1) NEXP has poly-size $\mathbb{C}$-circuits, AND

(2) there's a faster Gap $\mathbb{C}$-SAT algorithm

Show that NTIME[2^n] $\subseteq$ NTIME[o(2^n)]

Take any problem L in **nondeterministic 2^n time**
Given an input x, we "compute" L on x by:

1. Guessing a witness y of O(2^n) length.
2. Checking y is a witness for x in O(2^n) time.

Want to "speed-up" both parts 1 and 2, using the above assumptions

Proof Ideas in Theorem A

Idea. Assume

(1) NEXP has poly-size $\mathbb{C}$-circuits, AND

(2) there's a faster Gap $\mathbb{C}$-SAT algorithm

Show that NTIME$[2^n]$ $\subseteq$ NTIME$[o(2^n)]$

Take any problem L in **nondeterministic 2^n time**
Given an input x, we **will** "compute" L on x by:

1. **Use (1) to guess a witness y of o(2^n) length**
   (**Easy Witness Lemma [IKW02]**:
   if NEXP is in P/poly, then L has "small witnesses")
2. **Use (2) to check y is a witness for x in o(2^n) time**
   Technical: Use a highly-structured PCPs for NEXP
   [W'10, BV'14] to reduce the check to **Gap $\mathbb{C}$-SAT**

Proof Ideas in Theorem A

Idea. Assume

(1) NEXP has poly-size $\mathbb{C}$-circuits, AND

(2) there's a faster Gap $\mathbb{C}$-SAT algorithm

Show that NTIME[2^n] $\subseteq$ NTIME[o(2^n)]

Take any problem L in **nondeterministic 2^n time**
Given an input x, we **will** "compute" L on x by:

1. **Use (1) to guess a witness y of o(2^n) length**
   (Easy Witness Lemma [IKW02]:
   if NEXP is in P/poly, then L has "small witnesses")
2. **Use (2) to check y is a witness for x in o(2^n) time**
   Technical: Use a highly-structured PCPs for NEXP [W'10, BV'14] to reduce the check to **Gap $\mathbb{C}$-SAT**

Guessing Short Witnesses

1. Guess a witness y of $O(2^n)$ length.

Definition. An NTIME[2^n] problem L has ***easy witnesses*** if

$\exists c \geq 1$, $\forall$ Verifiers **V** for L, **if** $\exists\, y \in \{0,1\}^{2^{|x|+d}}$ **s.t. V(x,y) accepts**, **then**
$\exists$ circuit D_x of $|x|^c$ size and $|x| + d$ inputs s.t. V(x,$tt(D_x)$) accepts,
where $tt(D_x)$ **= Truth Table of circuit D_x.**

Easy Witness Lemma [IKW'02]:
If **NEXP is in P/poly** then all NEXP problems have ***easy witnesses***

Small circuits for solving NEXP problems
→ Small circuits for *solutions* to NEXP problems

Replace 1 with: 1'. Guess poly($|x|$)-size circuit D_x

Proof Sketch of Theorem A

Idea. Assume

(1) NEXP has poly-size $\mathbb{C}$-circuits, and

(2) there's a faster Gap $\mathbb{C}$-SAT algorithm

Show that NTIME[2^n] ⊆ NTIME[o(2^n)]

Take any problem L in **nondeterministic 2^n time.**
Given an input x, we compute L on x by:

1. **Guessing a circuit D_x of poly(|x|) size (Easy Witness Lemma, using (1))**
2. **Using (2) to check D_x encodes a witness for x in o(2^n) time (Nice PCPs for L)**

Improving Theorem A [MW'18]

Let $\mathbb{C}$ be a "typical" circuit class (like ACC^0)

Thm A+ [MW18] If there is an ε>0 such that
Gap-$\mathbb{C}$-SAT on $2^{n^{\varepsilon}}$-size circuits is in O($2^{n-n^{\varepsilon}}$) time
then NTIME[$2^{(\log n)^{O(1)}}$] doesn't have poly-size $\mathbb{C}$-circuits

Thm A++ [MW18] If there is an ε>0 such that
Gap-$\mathbb{C}$-SAT on $2^{\varepsilon n}$-size circuits is in O($2^{n(1-\varepsilon)}$) time
then for all k, NP doesn't have n^k-size $\mathbb{C}$-circuits

and NTIME[$n^{\log^* n}$] doesn't have poly-size $\mathbb{C}$-circs [Tell'18]

Proof of Theorem A++?

Approach: Want to show that given

(1) NP has n^k-size $\mathbb{C}$-circuits, and

(2) Gap-$\mathbb{C}$-SAT algorithm running in $2^{(1-\varepsilon)n}$ time

Then NTIME[n^d] ⊆ NTIME[o(n^d)] for some d

Let $L \in$ NTIME[n^d]. To solve L faster on input x,

1. ~~**Guess a witness circuit C_x of o(n^d) size**~~
2. **Check C_x encodes witness for x in o(n^d) time**
   (Use nice PCP; this still works, if part 1 works)

Easy Witness Lemma only works for NEXP!

New Easy Witness Lemma [MW'18]

NTIME[t(n)] has s(n)-size witness circuits if
$\forall L \in$ NTIME[t(n)], $\forall$ Verifiers V, $\forall x \in L$,
$\exists$ s(n)-size circuit D_x such that V(x, tt(D_x)) accepts.

Old Easy Witness Lemma [IKW02]:

If every problem in NEXP has poly(n)-size circuits,
then NEXP has poly(n)-size witness circuits.

New Easy Witness Lemma (Special Case of [MW'18]):
If every problem in NP has n^k-size circuits,
then NP has $n^{O(k^3)}$-size witness circuits.
Similar statement for NTIME[$n^{polylog\, n}$].

Proof of Theorem A++?

Approach: Want to show that given

(1) NP has n^k-size $\mathbb{C}$-circuits, and

(2) Gap-$\mathbb{C}$-SAT algorithm for $2^{\epsilon n}$ size, in $2^{n(1-\epsilon)}$ time

Then NTIME[n^{k^4}] ⊆ NTIME[o(n^{k^4})]

Let L ∈ NTIME[n^{k^4}]. To solve L faster on input x,

1. **Guess circuit C_x of O(n^{k^3}) size with $k^4 \log n$ inputs, encoding witness y of length n^{k^4}**
   (Use (1) and New Easy Witness Lemma)
2. **Check C_x encodes witness for x in o(n^{k^4}) time**
   (Use (2) and nice PCP)

Contradiction!

IKW's Easy Witness Lemma

Easy Witness Lemma [IKW02]:
NTIME[2^n] ⊂ SIZE[n^k] for some k

$\Longrightarrow$ NTIME[2^n] has n^c-size witness circuits for some c.

Strategy: Assume the negation, prove a contradiction!

(1) $\exists k$ NTIME[2^n] ⊂ SIZE[n^k] **and**

(2) ∀c, NTIME[2^n] **DOESN'T** have n^c-size witness circuits

IKW start with $L_{hard} \in$ SPACE[n^{k+1}] / i.o.-SIZE[n^k]

and show how assumptions (1) and (2) imply:

SPACE[n^{k+1}] ⊆ MA ⊆ i.o.-NTIME[2^n]$_{/n}$ ⊆ i.o.-SIZE[n^k]

Merlin-Arthur protocols

infinitely often, with n bits of advice

Proof of IKW's Easy Witness Lemma

(1) $\exists k$ NTIME$[2^n] \subset$ SIZE$[n^k]$ **and**

(2) $\forall$c, NTIME$[2^n]$ **DOESN'T** have n^c-size witness circuits

Show how assumptions (1) and (2) imply:

SPACE$[n^{k+1}]$ ⊆ MA ⊆ i.o.-NTIME$[2^n]_{/n}$ ⊆ i.o.-SIZE$[n^k]$

MA: Merlin-Arthur = NP with probabilistic verification

L is in MA means there's a polytime V such that

x ∈ L ➔ there is a y such that V(x,y) always accepts

x ∉ L ➔ for every y, V(x,y) rejects with prob > ¾

Merlin **Arthur**

Proof of IKW's Easy Witness Lemma

(1) $\exists k$ NTIME[2^n] ⊂ SIZE[n^k] **and**

(2) ∀c, NTIME[2^n] **DOESN'T** have n^c-size witness circuits

Show how assumptions (1) and (2) imply:

SPACE[n^{k+1}] ⊆ MA ⊆ i.o.-NTIME[2^n]$_{/n}$ ⊆ i.o.-SIZE[n^k]

(1) NTIME[2^n] ⊂ SIZE[n^k]

⇒ SPACE[O(n)] ⊂ P/poly

⇒ PSPACE ⊂ P/poly

⇒ **PSPACE = MA [BFNW'93]**
Use the fact that PSPACE = IP **[Shamir]**:
Guess a small circuit encoding the prover's strategy,
then run the interactive protocol with that circuit

Proof of IKW's Easy Witness Lemma

(1) $\exists k$ NTIME$[2^n] \subset$ SIZE$[n^k]$ **and**

(2) $\forall$c, NTIME$[2^n]$ **DOESN'T** have n^c-size witness circuits

Show how assumptions (1) and (2) imply:

SPACE$[n^{k+1}]$ ⊆ MA ⊆ i.o.-NTIME$[2^n]_{/n}$ ⊆ i.o.-SIZE$[n^k]$

(1) NTIME$[2^n] \subset$ SIZE$[n^k]$

$\Rightarrow$ i.o.-NTIME$[2^n]/n \subset$ i.o.-SIZE$[n^k]$
(Hard-code the advice in the circuit)

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Show how assumptions (1) and (2) imply:

SPACE[n^{k+1}] $\subseteq$ MA $\subseteq$ i.o.-NTIME[2^n]$_{/n}$ $\subseteq$ i.o.-SIZE[n^k]

(2) NTIME[2^n] DOESN'T have n^c-size witness circuits:

$\neg$($\forall L \in$ NTIME[2^n], $\forall$ Verifiers V, for all but finitely many $x \in L$,

$\exists y$ s.t. V(x, y) accepts and (Circuit complexity of y) $\leq n^c$)

Proof of IKW's Easy Witness Lemma

(1) $\exists k$ NTIME$[2^n] \subset$ SIZE$[n^k]$ **and**

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Show how assumptions (1) and (2) imply:

SPACE$[n^{k+1}]$ ⊆ MA ⊆ i.o.-NTIME$[2^n]_{/n}$ ⊆ i.o.-SIZE$[n^k]$

(2) NTIME$[2^n]$ DOESN'T have n^c-size witness circuits:

$\exists L \in$ NTIME$[2^n]$, ∃ Verifier V, ∃ **infinitely many** $x \in L$,
such that $\forall y$ [V(x, y) accepts ⇒ (Circuit complexity of y) > n^c]

Given a 'bad' input x as advice, can use verifier V to guess-and-check a function with circuit complexity > n^c in $O(2^n)$ time

Can nondeterministically generate hard functions!

Proof of IKW's Easy Witness Lemma

(1) $\exists k$ NTIME$[2^n] \subset$ SIZE$[n^k]$ **and**

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Show how assumptions (1) and (2) imply:

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(2) NTIME$[2^n]$ DOESN'T have n^c-size witness circuits:

$\exists L \in$ NTIME$[2^n]$, $\exists$ Verifier V, $\exists$ **infinitely many** $x \in L$,

such that $\forall y$ [V(x, y) accepts $\Rightarrow$ (Circuit complexity of y) > n^c]

Thm [Hardness-to-PRGs] *There's an $\alpha > 0$ and **$O(2^n)$-time** computable $\boldsymbol{F}$ such that, **given a string y with circuit complexity > n^c**, $\boldsymbol{F}$ outputs a set of $O(2^n)$ strings which "fool" all circuits of size $\boldsymbol{n^{\alpha c}}$*

Use F to derandomize $n^{O(c)}$-time Merlin-Arthur protocols in $O(2^n)$ time, on *infinitely many* input lengths, with *n bits of advice*

Scaling Down to NP?

New Easy Witness Lemma (Special Case)
If **NP** has n^k-size circuits,
then **NP** has $n^{O(k^3)}$-size witness circuits.

Idea: Derive a contradiction from assuming that

$$\mathbf{NP} \subset \mathbf{SIZE}[n^k]$$

and

$\forall c$, NP does NOT have n^c-size witness circuits.

Scaling Down to NP?

What happens when we try to follow the IKW proof?

We want to derive something like:

$$\textbf{PSPACE} \subseteq \textbf{MA} \subseteq \textbf{i.o.NP}_{/n} \subseteq \textbf{i.o.SIZE}[n^k]$$

These two inclusions are OK!

They follow from NP $\subset$ SIZE$[n^k]$

and

NP does NOT have n^c-size witness circuits

Scaling Down to NP?

What happens when we try to follow the IKW proof?

We want to derive something like:

PSPACE $\subseteq$ MA $\subseteq$ i.o.NP$_{/n}$ $\subseteq$ i.o.SIZE$[n^k]$

Problem: Can't conclude PSPACE is in MA from assuming NP $\subset$ SIZE$[n^k]$ and NP does NOT have n^c-size witness circuits!

Possible fix: Use another circuit lower bound?

Thm [San07] MA$_{/1}$ $\not\subset$ SIZE$[n^k]$

Scaling Down to NP?

What happens when we try to follow the IKW proof?

We want to derive something like:

$$\mathbf{MA}_{/1} \subseteq \mathbf{i.o.NP}_{/n+1} \subseteq \mathbf{i.o.SIZE}[n^k]$$

New problem: We only know $\mathbf{MA}_{/1} \not\subset \mathbf{SIZE}[n^k]$

Don't know if $\mathbf{MA}_{/1} \not\subset \mathbf{i.o.SIZE}[n^k]$

Possible fix: Prove a stronger MA lower bound?
Turns out we don't need an
"almost-everywhere" lower bound…

New Lower Bound for Merlin-Arthur Protocols

Thm [MW'18] For all k, there is an $L \in$ **MA-TIME**$[n^{k^2}]_{/O(\log n)}$ such that **for all but finitely many** input lengths n,

either L_n has circuit complexity at least n^k

or L_{n^k} has circuit complexity at least n^{k^2}

Our proof of the new EWL shows:

If every problem in NP has n^k-size circuits
and some NP problem doesn't have $n^{O(k^3)}$-size witnesses,
then the above Merlin-Arthur lower bound is contradicted!

Sketch of the New Easy Witness Lemma

Start with $\boldsymbol{L} \in$ **MA-TIME[$\boldsymbol{n^{k^2}}$]$_{/\mathbf{O(log\ n)}}$** from our new circuit lower bound.

Assuming **some NP problem doesn't have $\boldsymbol{n^{O(k^3)}}$-size witnesses**,
we derive a partial derandomization of the MA protocol for L:

For infinitely many $\boldsymbol{n}$, there is an NP$_{/O(n)}$ algorithm computing
$\boldsymbol{L}$ correctly on all inputs of length $\boldsymbol{n}$ AND of length $\boldsymbol{n^k}$.

Assuming **NP has $\boldsymbol{n^k}$-size circuits**, we can derive:

For infinitely many $\boldsymbol{n}$,
$\boldsymbol{L_n}$ has an $\boldsymbol{n^k}$-size circuit AND $\boldsymbol{L_{n^k}}$ has an $\boldsymbol{n^{k^2}}$-size circuit.

This directly contradicts our lower bound for $\boldsymbol{L}$!

More Details on Derandomizing MA

Assume: NP does NOT have n^{k^3}-size witness circuits.
Let V be a "bad" verifier (for inf. many x, every witness for x is not easy)

How to derive MA $_{/O(\log n)} \subseteq$ i.o.NP$_{/n+O(\log n)}$

Given a 'bad' x_w as advice,

Guess a 'bad' y such that V(x_w,y) accepts
// y encodes a function with circuit complexity > n^{k^3}

Stick y into a PRG that fools $n^{\Omega(k^3)}$-size circuits

Use PRG to derandomize an m-time MA protocol
(Guess Merlin's message, construct a circuit of size m^2 that takes Arthur's message as input)

This works as long as $m^2 << n^{O(k^3)}$

More Details on Derandomizing MA

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If NP does not have n^{k^3}-size witness circuits,
the *same* advice x_w can be used to derandomize MA
for *all* running times up to $m = n^{O(k^3)}$

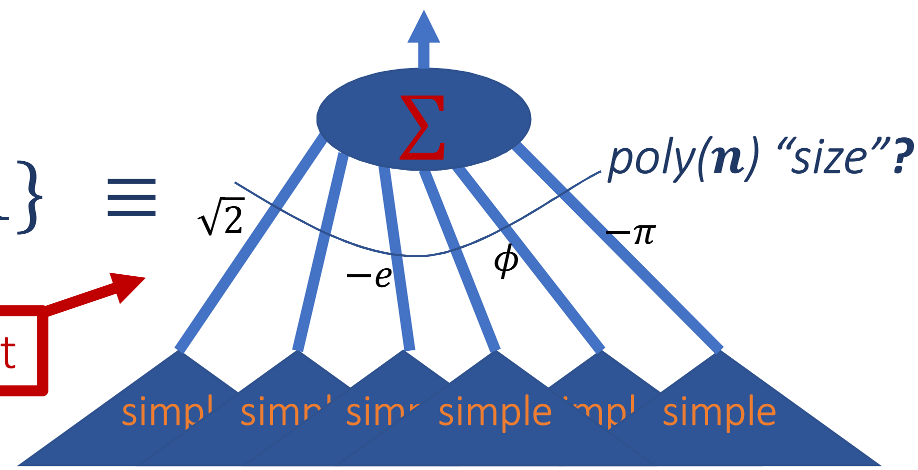
Lower Bounds for NP
Against Some Depth-Two Classes

The $\mathbb{R}$-linear Representation Problem

Let $\boldsymbol{\mathcal{C}}$ be a class of “simple” functions
(take Boolean inputs, but need not be Boolean-valued)

> Which “interesting” functions $\boldsymbol{f}$ can(not) be represented by “short” $\mathbb{R}$-linear combinations of functions from $\boldsymbol{\mathcal{C}}$?

$$f : \{0,1\}^n \to \{0,1\} \quad \equiv$$



Call this a $\sum \circ \boldsymbol{\mathcal{C}}$ circuit

If $\boldsymbol{\mathcal{C}}$ spans the vector space of all functions $\boldsymbol{f} : \{\mathbf{0}, \mathbf{1}\}^{\boldsymbol{n}} \to \mathbb{R}$
then there is always some $\sum \circ \boldsymbol{\mathcal{C}}$ circuit of $\leq \mathbf{2}^{\boldsymbol{n}}$ size…

The $\mathbb{R}$-linear Representation Problem

Which "interesting" functions $\boldsymbol{f}$ can(not) be represented by "short" $\mathbb{R}$-linear combinations of functions from $\boldsymbol{\mathcal{C}}$?

If $\boldsymbol{\mathcal{C}}$ is the class of $\mathbf{2^n}$ $\boldsymbol{AND}$ functions on $\boldsymbol{n}$ variables:

$\sum \circ \boldsymbol{AND} \equiv \mathbf{0/1}$ polynomials over $\mathbb{R}$

If $\boldsymbol{\mathcal{C}}$ is the class of $\mathbf{2^n}$ $\boldsymbol{PARITY}$ functions on $\boldsymbol{n}$ variables:

$\sum \circ \boldsymbol{PARITY} \equiv \mathbf{-1/1}$ polynomials over $\mathbb{R}$

(Fourier analysis of Boolean functions)

These are well-understood:

$\boldsymbol{\mathcal{C}}$ is a basis for the vector space of functions $f : \{0,1\}^n \to \mathbb{R}$

$\Rightarrow$ the $\mathbb{R}$-linear representation of $\boldsymbol{f}$ is *unique,*

so the "shortest" is also the "longest"…

More interesting cases: representations are *not* unique

[W'18] Three Simple Classes

1. Linear Threshold Functions $[\boldsymbol{LTF}]$
2. Rectified Linear Units $[\boldsymbol{ReLU}]$
3. $\boldsymbol{GF(p)}$-Polynomials of Degree-$\boldsymbol{d}$ $[\boldsymbol{POLYd[p]}]$
   ($\boldsymbol{p}$ prime and $\boldsymbol{d \geq 2}$)

For all three classes:

- There are $\gg \boldsymbol{2^n}$ functions on $\boldsymbol{n}$ variables, so $\mathbb{R}$**-linear** representations are not unique
  $\boldsymbol{2^{\Theta(n^2)}}$ LTFs, $\boldsymbol{p^{\Theta(n^d)}}$ degree-$\boldsymbol{d}$ polys, ∞ ReLU functions
- **$\mathbb{R}$-linear Representations have been studied!**
  $\sum \circ \boldsymbol{LTF}$ = **Special Case of Depth-2 Threshold Circuits**
  $\sum \circ \boldsymbol{ReLU}$ = **"Depth-2 Neural Net with ReLU activation"**
  $\sum \circ \boldsymbol{POLYd[p]}$ = **"Higher-Order" Fourier Analysis for $\boldsymbol{d \geq 2}$**

Sums of Linear Threshold Functions

Def. $f_n: \{0,1\}^n \to \{0,1\}$ is an LTF if $\exists\, w_1, \ldots w_n, t \in \mathbb{R}$ such that $\forall\, (x_1, \ldots, x_n) \in \{0,1\}^n$, $\boldsymbol{f(x_1, \ldots, x_n) = 1 \iff \sum_i w_i x_i \geq t}$

Depth-Two LTF Circuits ($\boldsymbol{LTF \circ LTF}$): Major problem to find "nice" functions without n^k-gate $LTF \circ LTF$ circuits, for all k

[Hajnal et al.'91] **exp(n)** depth-two lower bounds for ***small*** w_i's

[Roychowdhury-Orlitsky-Siu'94] **What about $\sum \circ \boldsymbol{LTF}$?**

Special case of $\boldsymbol{LTF \circ LTF}$:

the linear form for output LTF must always evaluate to 0 or 1

Still, no $\boldsymbol{n^{1.5}}$-gate lower bounds were known for $\sum \circ \boldsymbol{LTF}$!

We prove:

Thm $\forall \boldsymbol{k}$, $\exists \boldsymbol{f_k} \in \boldsymbol{NP}$ **without $\boldsymbol{n^k}$-size $\sum \circ \boldsymbol{LTF}$**

Thm $\exists \boldsymbol{f} \in \boldsymbol{NTIME[n^{\log^* n}]}$ **without $\boldsymbol{poly(n)}$-size $\sum \circ \boldsymbol{LTF}$**

Note: It is a *major* open problem to prove
$\exists \boldsymbol{f} \in \boldsymbol{NP}$ without $\boldsymbol{n^k}$-size (unrestricted) circuits

Sums of ReLUs

Def. $f_n: \mathbb{R}^n \to \mathbb{R}^+$ is a ReLU if $\exists\, w_1, \ldots w_n, t \in \mathbb{R}$ such that
$\forall\, (x_1, \ldots, x_n) \in \mathbb{R}^n,\ \ \boldsymbol{f(x_1, \ldots, x_n) = \max(0, \sum_i w_i x_i + t)}$

$\sum \circ\, \boldsymbol{ReLU}$ generalizes $\sum \circ\, \boldsymbol{LTF}$

$\sum \circ\, \boldsymbol{ReLU}$ = **"Depth-Two Neural Nets with ReLU Activations"**

Very widely studied, thousands of references

Several recent references [see paper] give lower bounds
for some "weird" $\boldsymbol{f}: \mathbb{R}^n \to \mathbb{R}$ which vary sharply / sensitive

No lower bounds known for discrete-domain / Boolean functions
(note: "most sensitive" Boolean fn PARITY has O(n)-size $\sum \circ\, \boldsymbol{LTF}$)

We can generalize the $\sum \circ\, \boldsymbol{LTF}$ limits to $\sum \circ\, \boldsymbol{ReLU}$:

Thm $\forall \boldsymbol{k}, \exists \boldsymbol{f_k} \in \boldsymbol{NP}$ without $\boldsymbol{n^k}$-size $\sum \circ\, \boldsymbol{ReLU}$

Thm $\exists \boldsymbol{f} \in \boldsymbol{NTIME}[\boldsymbol{n^{\log^* n}}]$ **without** $\boldsymbol{poly(n)}$**-size** $\sum \circ\, \boldsymbol{ReLU}$

Sums of Low-Degree GF(p)-Polys

$\sum\circ\, \boldsymbol{POLYd[p]}$: Linear combination of $f: \{0,1\}^n \to \{0,1,\ldots,p-1\}$ where for every f there is a degree-d polynomial $q(x)$ such that

$\forall x \in \{0,1\}^n$, $\boldsymbol{f(x) = q(x)}$ mod $\boldsymbol{p}$

Case of $\boldsymbol{d = 2, p = 2}$ is already very interesting!

Compelling Conjecture ["Degree-Two Uncertainty Principle"]:

$\boldsymbol{AND}$ (on $\boldsymbol{n}$ inputs) requires $\boldsymbol{n^{\omega(1)}}$-size $\sum\circ\, \boldsymbol{POLY2[2]}$

Known: $\boldsymbol{AND}$ requires $\Omega(2^n)$-size $\sum\circ\, \boldsymbol{POLY1[2]}$

$\boldsymbol{AND}$ has $O(2^{n/2})$-size $\sum\circ\, \boldsymbol{POLY2[2]}$

No non-trivial lower bounds were known for $\sum \circ\, \boldsymbol{POLY2[p]}$

We prove:

Thm $\forall \boldsymbol{d, k}, \forall \boldsymbol{p}$ prime, $\exists \boldsymbol{f_k} \in \boldsymbol{NP}$ without $\boldsymbol{n^k}$-size $\sum\circ\, \boldsymbol{POLYd[p]}$

Thm $\exists \boldsymbol{f} \in \boldsymbol{NTIME[n^{\log^* n}]}$ **without** $\boldsymbol{poly(n)}$**-size** $\sum\circ\, \boldsymbol{POLYd[p]}$ **for all fixed** $\boldsymbol{d}$ **and fixed prime** $\boldsymbol{p}$

Key Theorem

A new instance of "Circuit Analysis Algorithms ⇒ Circuit Lower Bounds"

Key Theorem: Let $\mathcal{C}$ be a class of functions $f : \{0,1\}^n \to \mathbb{R}$.
Assume: there is an $\varepsilon > 0$ and an algorithm A so that
for any given $f_1, \ldots, f_4 \in \mathcal{C}$, A can compute the "sum-product"

$$\sum_{a \in \{0,1\}^n} \prod_{i=1}^{4} f_i(a)$$

in $2^{n(1-\varepsilon)}$ time.

Then: $\forall k$, $\exists f \in NP$ without n^k-size $\sum\circ\, \mathcal{C}$, and
$\exists f \in NTIME\left[n^{\log^* n}\right]$ without $poly(n)$-size $\sum\circ\, \mathcal{C}$

Solving a generalization of #SAT for $\mathcal{C}$
→ Strong lower bounds for $\sum\circ\, \mathcal{C}$

Applies new Easy Witness Lemma [Murray-W'18]

We show how to compute sum-products in $2^{n(1-\varepsilon)}$ time
for LTFs, ReLUs, and low-degree polynomials

Major Ideas in the Key Theorem

Assume: (1) There is a $2^{n(1-\varepsilon)}$-time sum-product algorithm A for $\mathcal{C}$

(2) For some fixed k, all $f \in NP$ have n^k-size $\sum\circ\,\mathcal{C}$ **Goal: Derive a contradiction.**

(1) and (2) $\Rightarrow$ Given (unrestricted) Boolean circuit T with n inputs and m size, we can guess-and-check an m^k-size $\sum\circ\,\mathcal{C}$ computing T, in $2^{n(1-\varepsilon)}m^{O(1)}$ time

Notes: (a) Checking that a given $\sum\circ\,\mathcal{C}$ is Boolean-valued is the hardest part.
(b) In order to guess the $\sum\circ\,\mathcal{C}$ circuit, we need that the coefficients in our linear combinations have "small" bit complexity, WLOG

(1) $\Rightarrow$ **Can solve #Circuit-SAT in *nondeterministic* $2^{n(1-\varepsilon)}m^{O(1)}$ time**

Idea: given (unrestricted) circuit T, guess-and-check an equivalent m^k-size $\sum\circ\,\mathcal{C}$ computing T. Then, #SAT(T) is equiv. to $\sum_{a\in\{0,1\}^n}(\sum\circ\,\mathcal{C}\,(a)) = \sum\sum_a \mathcal{C}(a)$.

[Murray-W'18] + **#Circuit-SAT algorithm** $\Rightarrow \forall k, \exists f \in NP$ without n^k-size unrestricted circuits

Contradicts (2) when $\sum\circ\,\mathcal{C}$ can be simulated by Boolean circuits!

The proof crucially relies on the $\sum\circ\,\mathcal{C}$ circuit computing an arbitrary circuit *exactly*

Sum-Product Algorithm for LTF

Uses (old) fact that #Subset-Sum is solvable in $poly(n) \cdot 2^{n/2}$ time!

Thm [HS'76] #Subset-Sum on n numbers is in $poly(n) \cdot 2^{n/2}$ **time**

Proof Given $w_1, \ldots, w_n, t$, we want to know the number of $S \subseteq [n]$ such that $\sum_{i \in S} w_i = t$

1. Enumerate all possible $2^{n/2}$ subsets S of $\{w_1, \ldots, w_{n/2}\}$.

Make a list L_1 of the $2^{n/2}$ subset sums, and SORT all sums in L_1

2. Enumerate all possible $2^{n/2}$ subsets T of $\{w_{n/2+1}, \ldots, w_n\}$.

For each T summing to a value v,

BINARY SEARCH for a value v' in L_1 such that $v + v' = t$

3. To compute the total number of subsets summing to t:

For each sum value v' appearing in L_1,

store the *number* $n_{v'}$ *of subsets in* L_1 which have value v'.

Later, if value v' is found in the binary search,

add $n_{v'}$ to a running sum.

Takes $poly(n) \cdot 2^{n/2}$ time in total

Sum-Product Algorithm for LTF

Uses (old) fact that #Subset-Sum is solvable in $poly(n) \cdot 2^{n/2}$ time!

Thm For any $f_1, \ldots, f_4 \in LTF$, we can compute

$$\sum_{a \in \{0,1\}^n} \prod_{i=1}^{4} f_i(a) \qquad \text{in } poly(n) \cdot 2^{n/2} \text{ time.}$$

Proof **An *Exact* LTF ($ELTF$) g has the form $g(x) = 1 \Leftrightarrow \sum_i w_i x_i = t$**

#Subset-Sum in $poly(n) \cdot 2^{n/2}$ time $\Rightarrow \sum_a g(a)$ in $poly(n) \cdot 2^{n/2}$ time

[HP'10]: Every LTF on n inputs can be written as $\sum_{poly(n)} ELTF$

So we can write $\sum_{a \in \{0,1\}^n} \prod_{i=1}^{4} f_i(a) = \sum_{a \in \{0,1\}^n} \prod_{i=1}^{4} \left(\sum_{poly(n)} g_{i,j}(a) \right)$ **for $ELTF$s $g_{i,j}$**

Simple algebra: $= \sum_{a \in \{0,1\}^n} \sum_{poly(n)} \prod_{i=1}^{4} g_{i,j'}(a) = \sum_{poly(n)} \sum_{a \in \{0,1\}^n} \prod_{i=1}^{4} g_{i,j'}(a)$

Each $\prod_{i=1}^{4} g_{i,j'}(x) = h(x)$ for some $ELTF\ h$

Can compute in $poly(n) \cdot 2^{n/2}$ time!

Open Problems

Know: For each k, there is an $f \in NTIME\left[n^{O(k^4)}\right]$ without n^k-size $\sum \circ LTF$

Show SAT requires n^k-size $\sum \circ LTF$, for *all* k

Show Quasi-NP does not have THR ∘ THR circuits of subquadratic size

---

Show there's a function in E^{NP} without $6n$ size circuits

I know how to solve #SAT for $\sum \circ POLY2[2]$ in poly-time.
Thus this class should not even represent CNF. Prove that!

If $SAT \in P$, then $TIME\left(n^{\log n}\right)$ is not in $P/poly$.
If SAT is in $n^{polylog\, n}$ time, then Quasi-P is not in $P/poly$.
Is such a connection true for Gap-Circuit-SAT?
[IW97] ($TIME[2^{O(n)}]$ not in $2^{n/100}$ size) ⇒ Gap-Circuit-SAT is in P

Thank you!