Spatial Isolation ⇒ Zero Knowledge Even in a Quantum World

TOM GUR

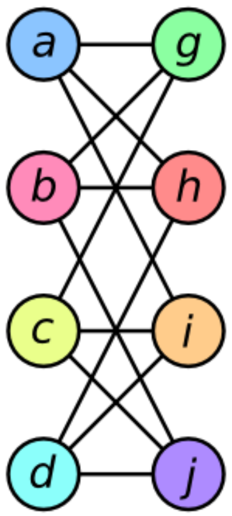
Berkeley
UNIVERSITY OF CALIFORNIA

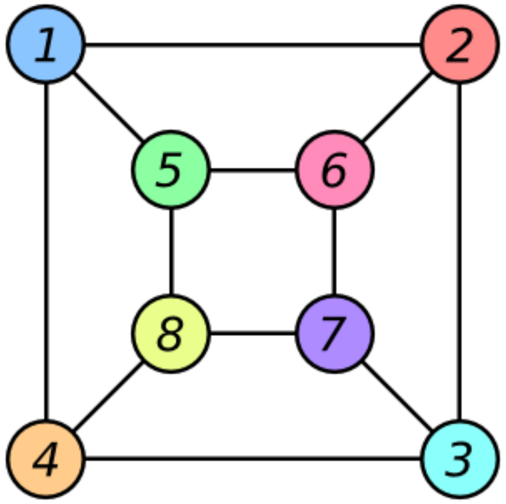
Joint work with Alessandro Chiesa, Michael Forbes, and Nicholas Spooner

The problem

Zero Knowledge

Zero Knowledge





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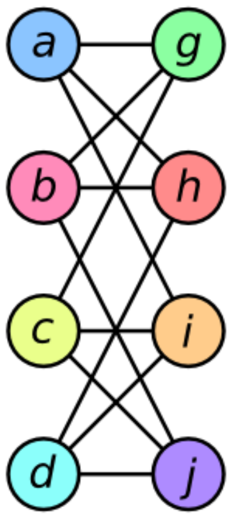

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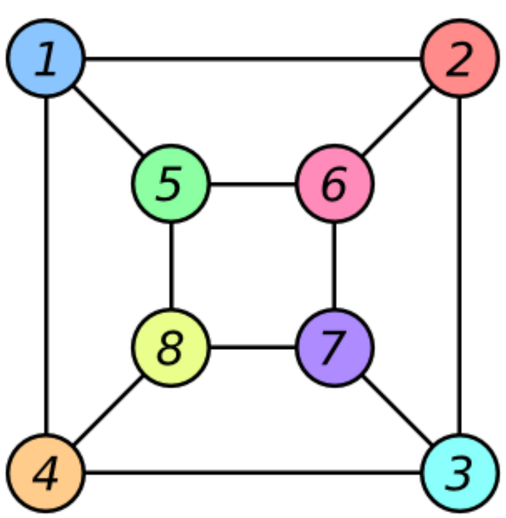

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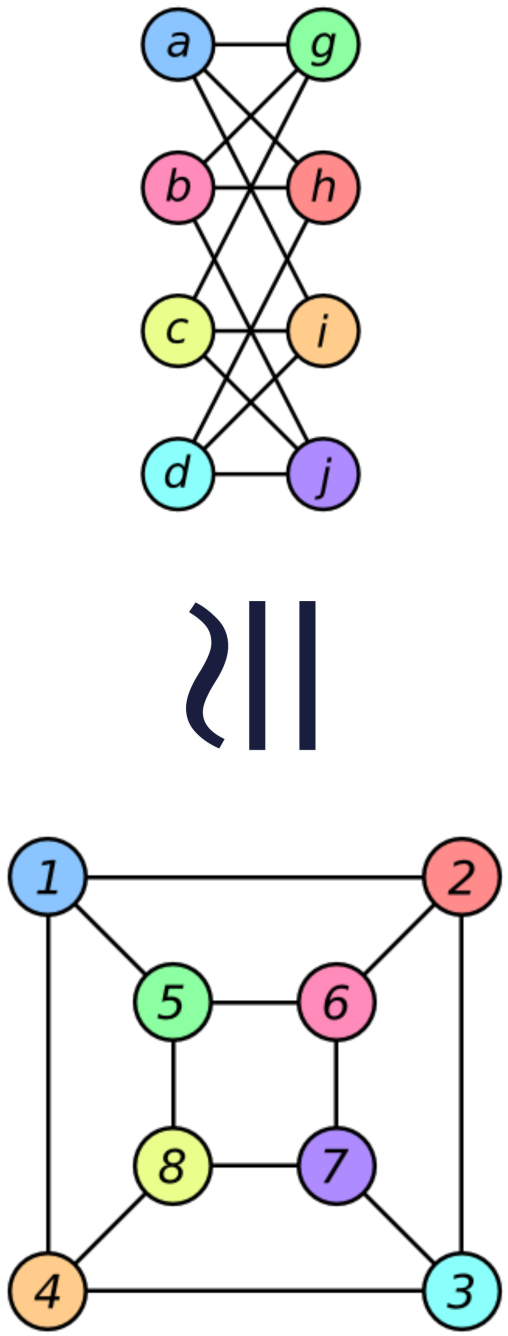
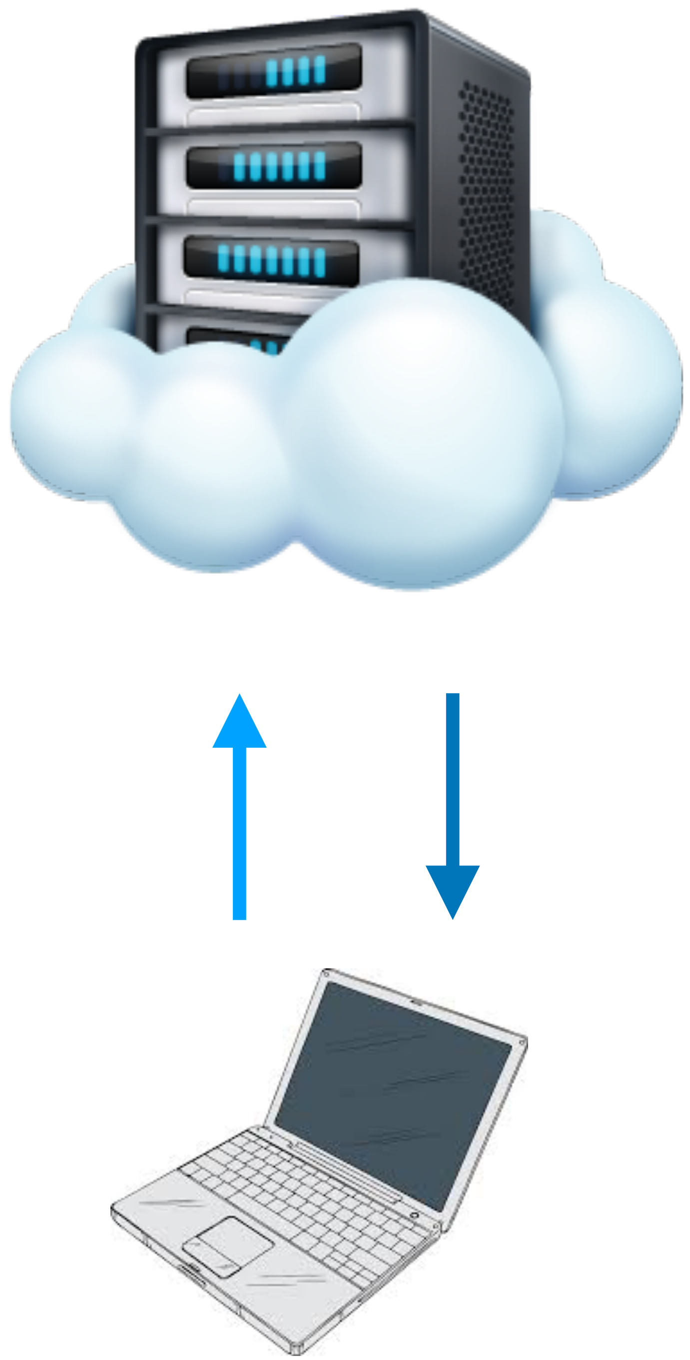
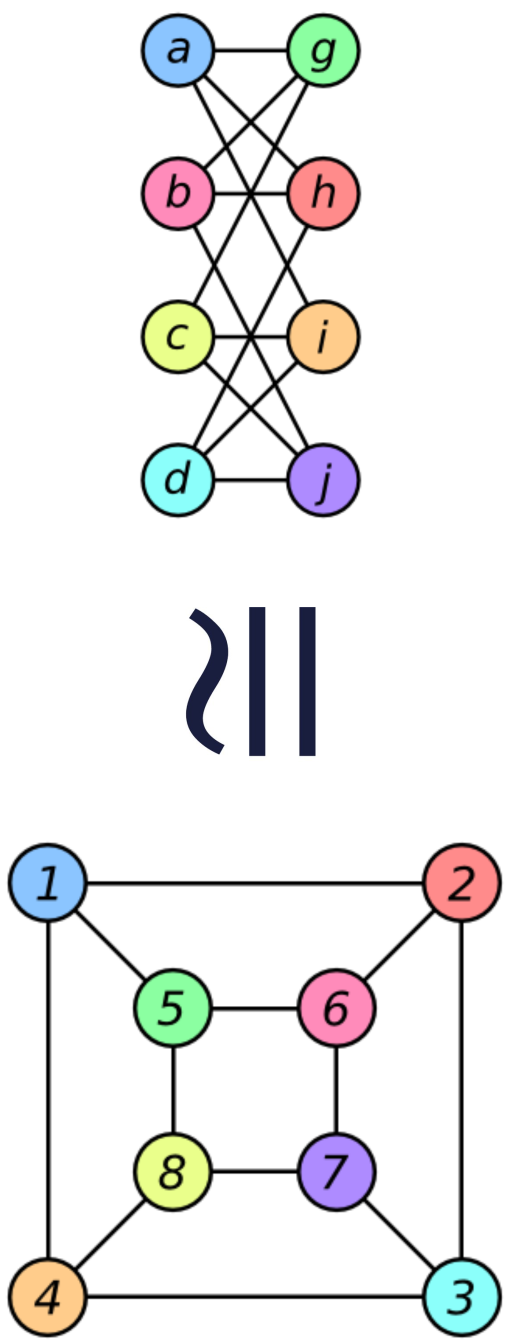


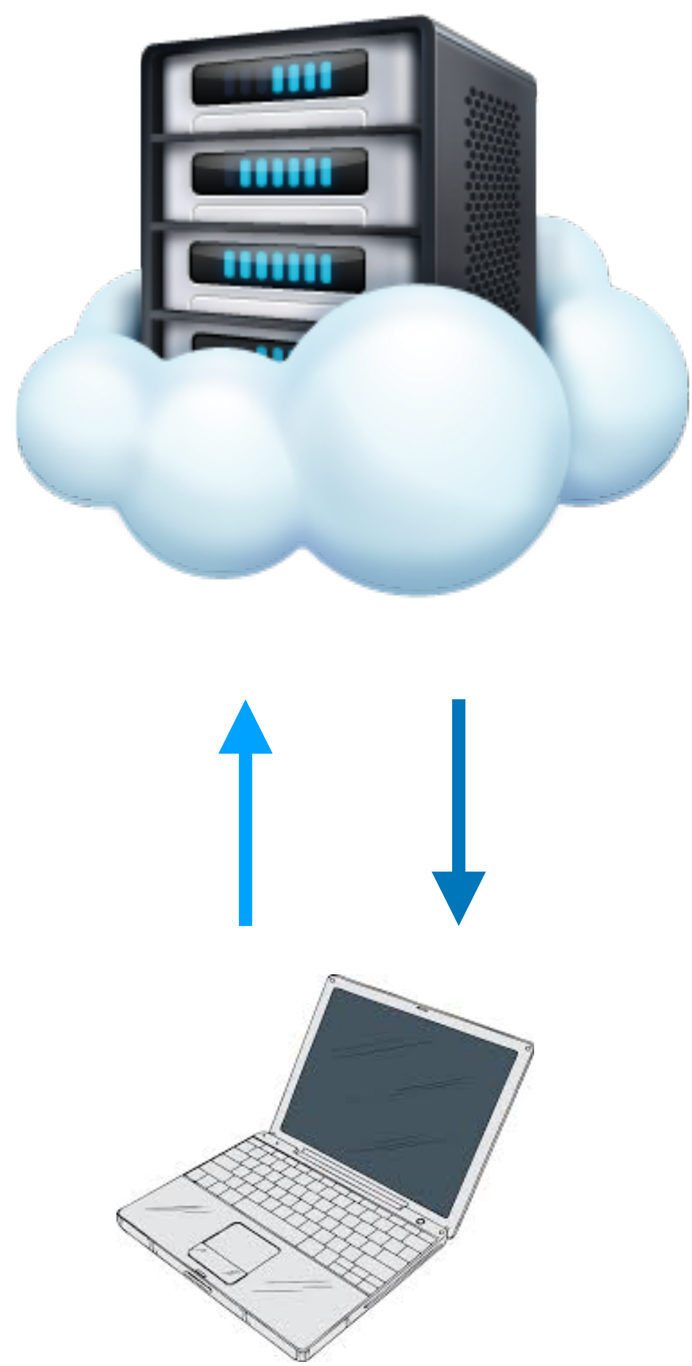


Zero Knowledge

Zero-Knowledge Proofs

[Goldwasser-Micali-Rackoff 89]





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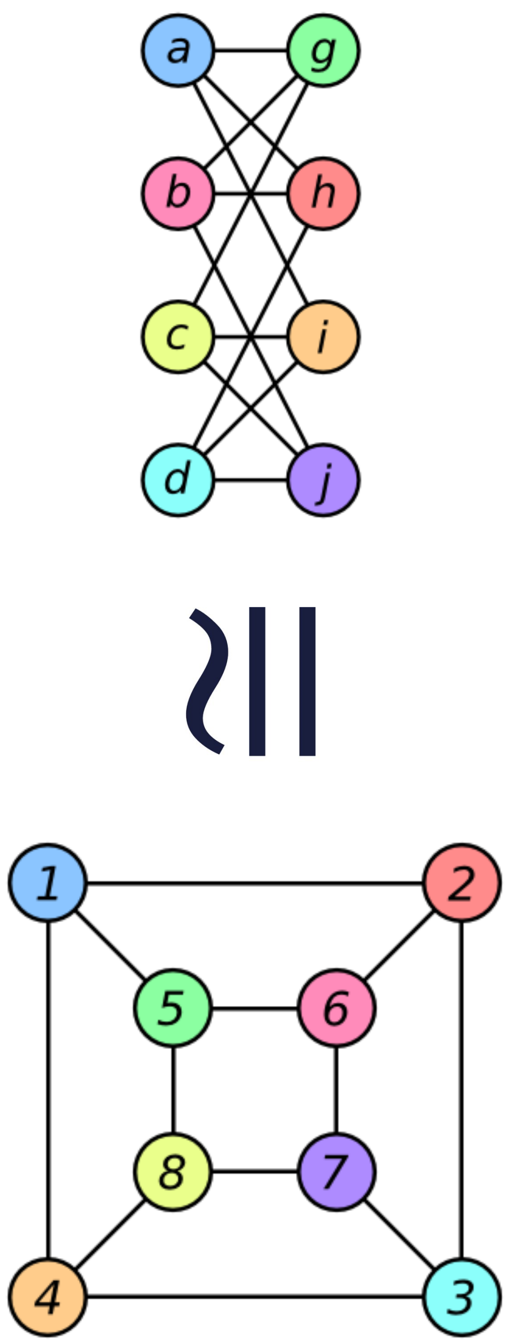

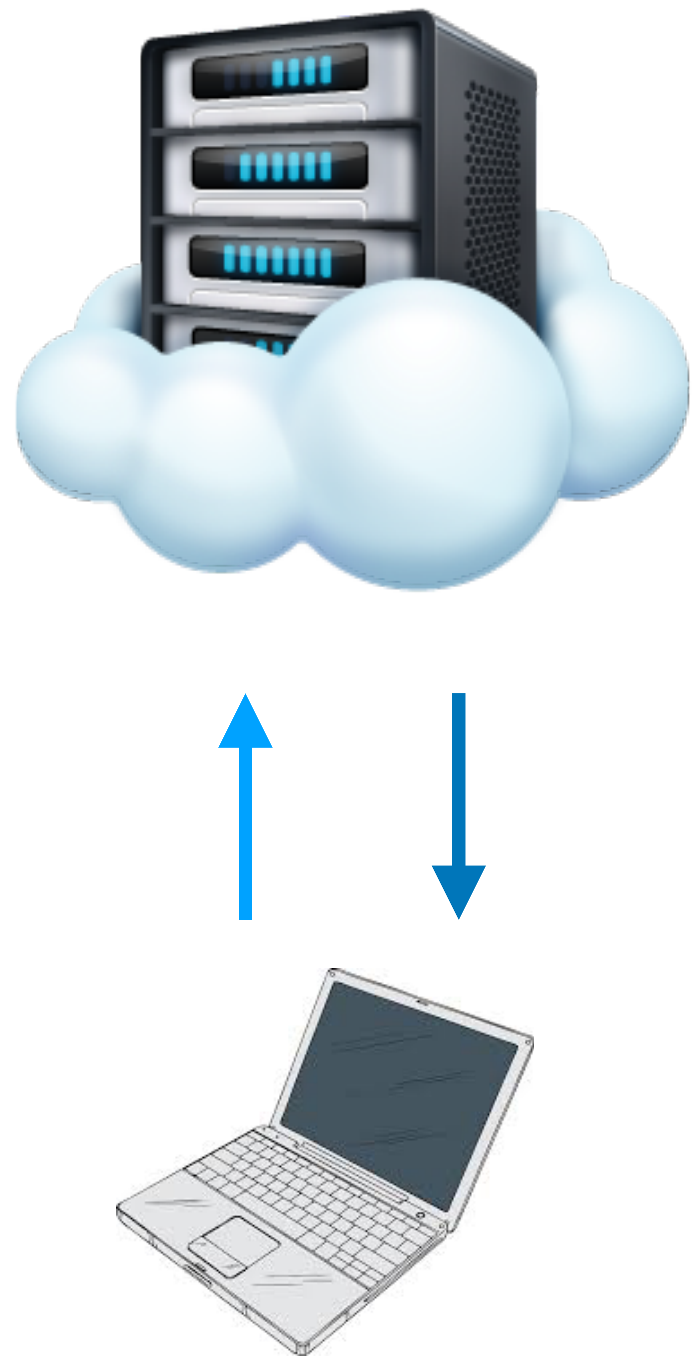

Cryptographic assumptions (OWF)

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ZK for NP

[Goldreich-Micali-Wigderson 91]

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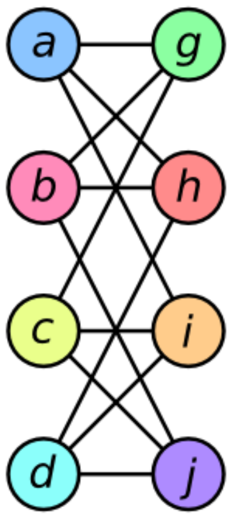

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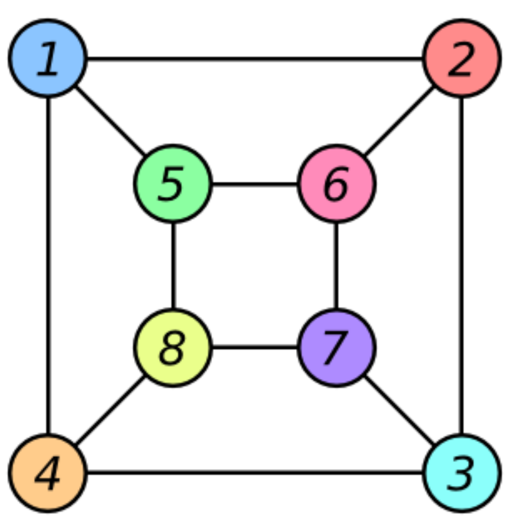

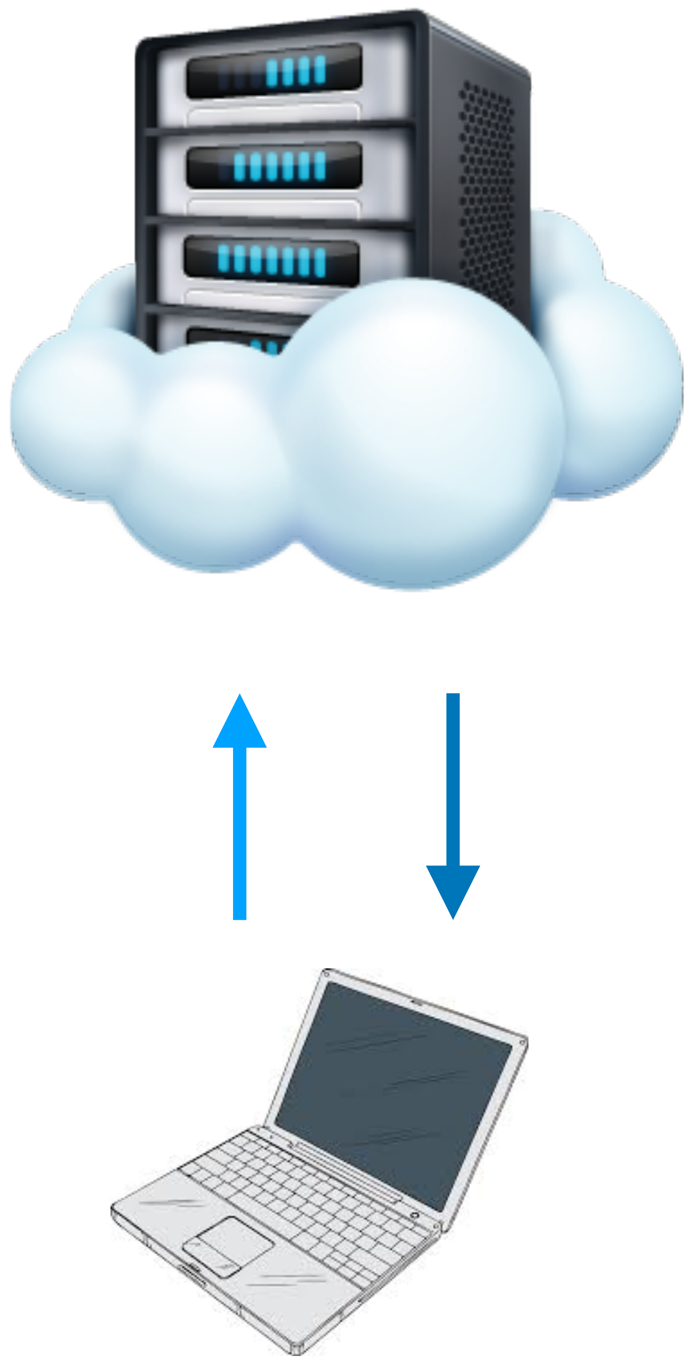

Cryptographic assumptions (OWF)

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Cryptographic assumptions are **necessary**

[Ostrovsky-Wigderson 93]

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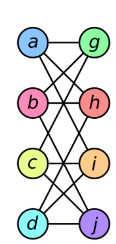
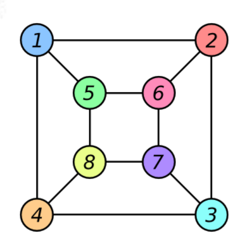
Multi-prover Interactive Proofs (MIP)

[BenOr-Goldwasser-Kilian-Wigderson 88]





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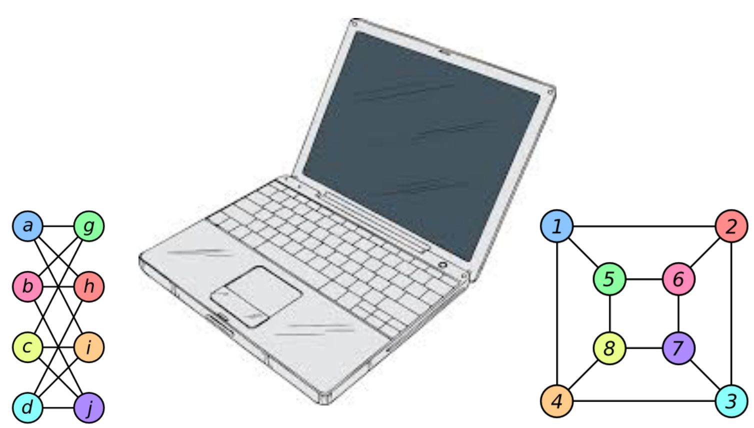

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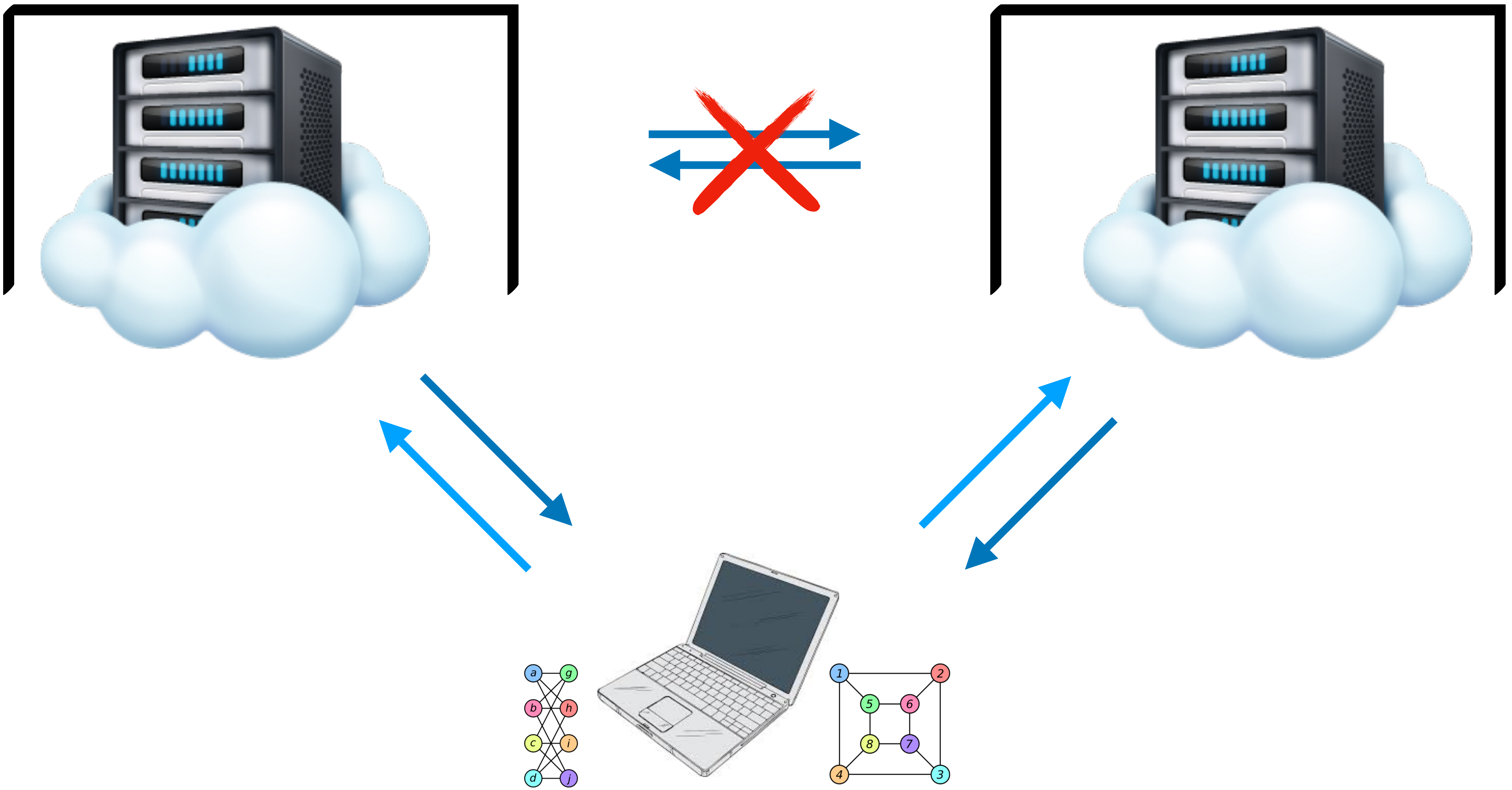

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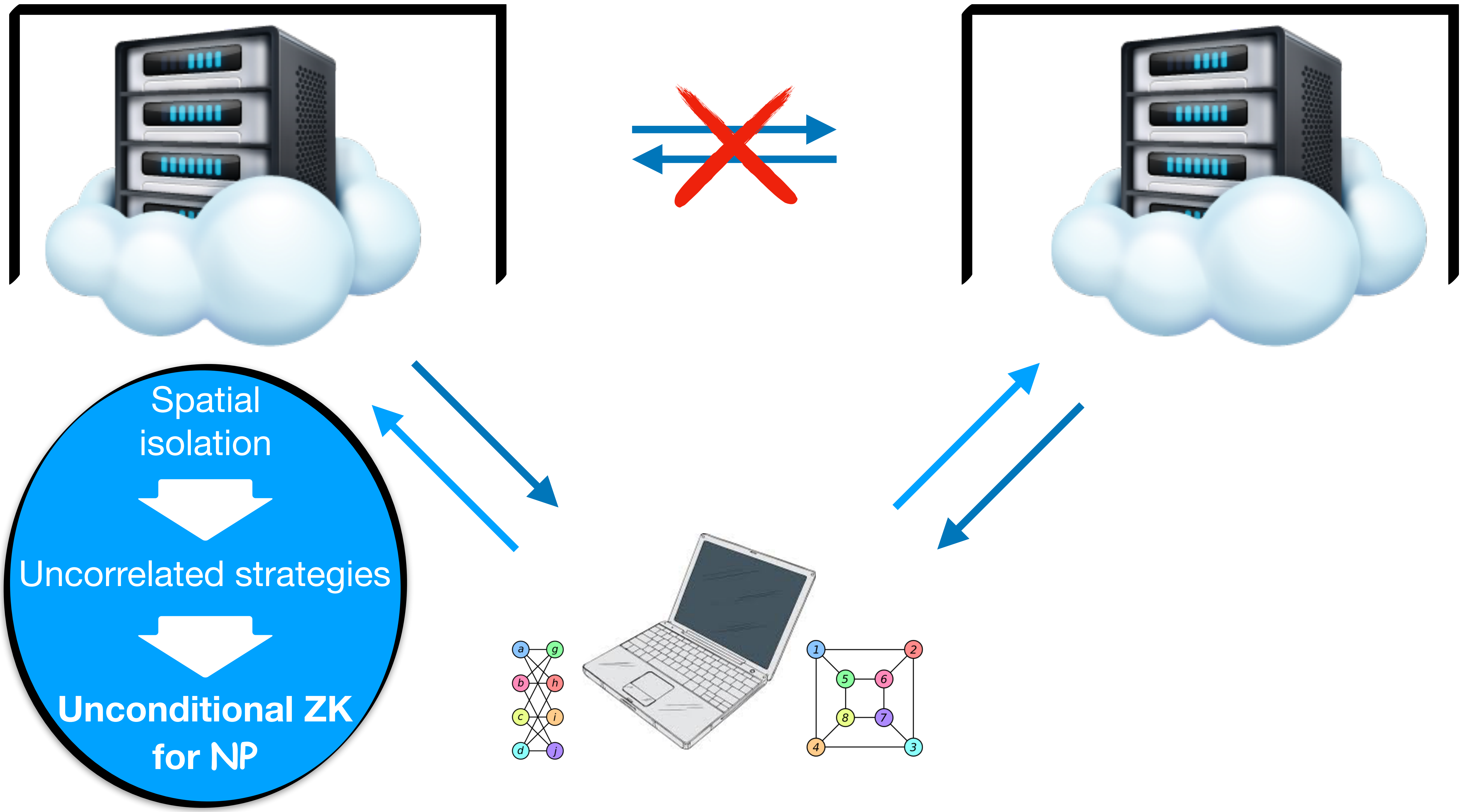

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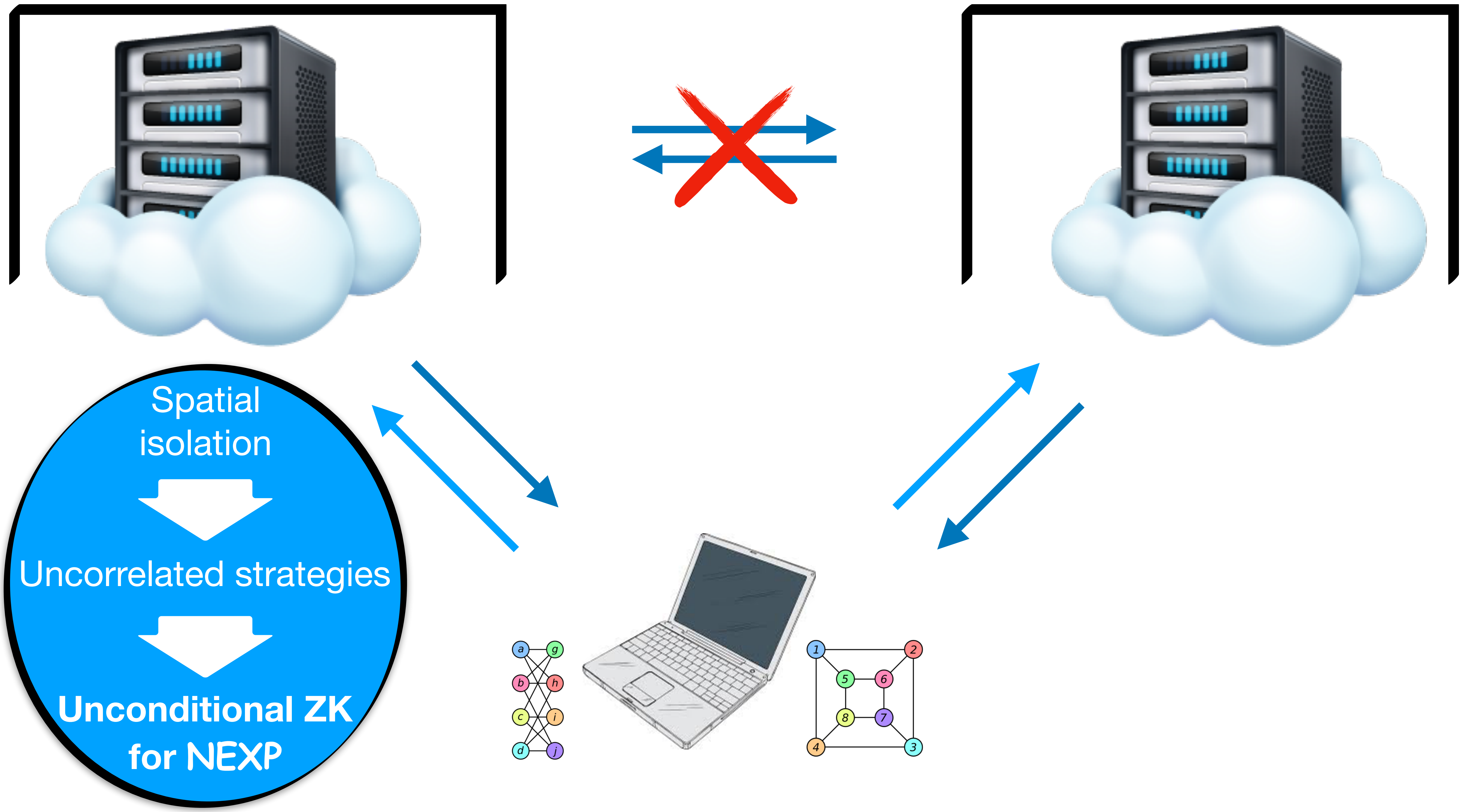

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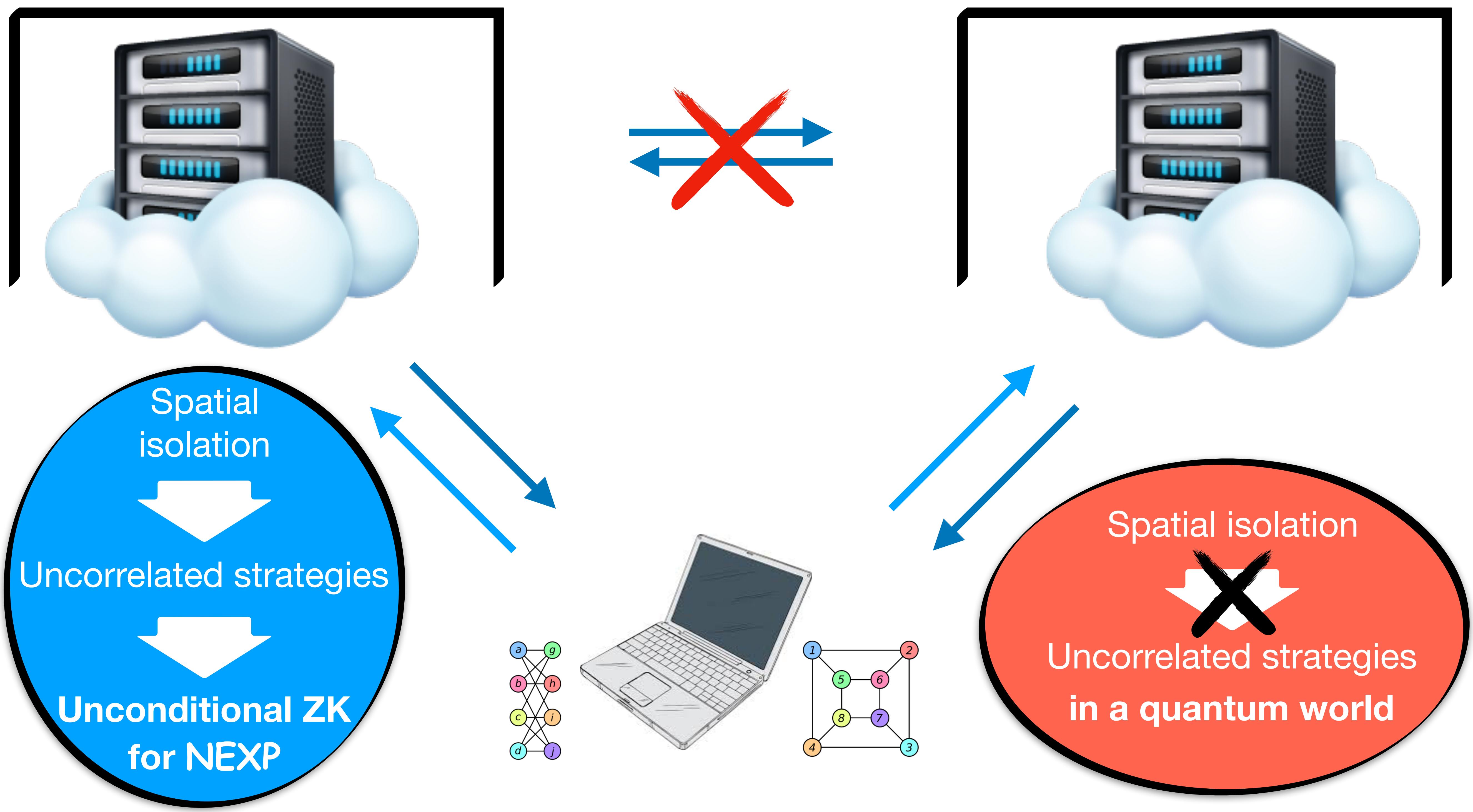
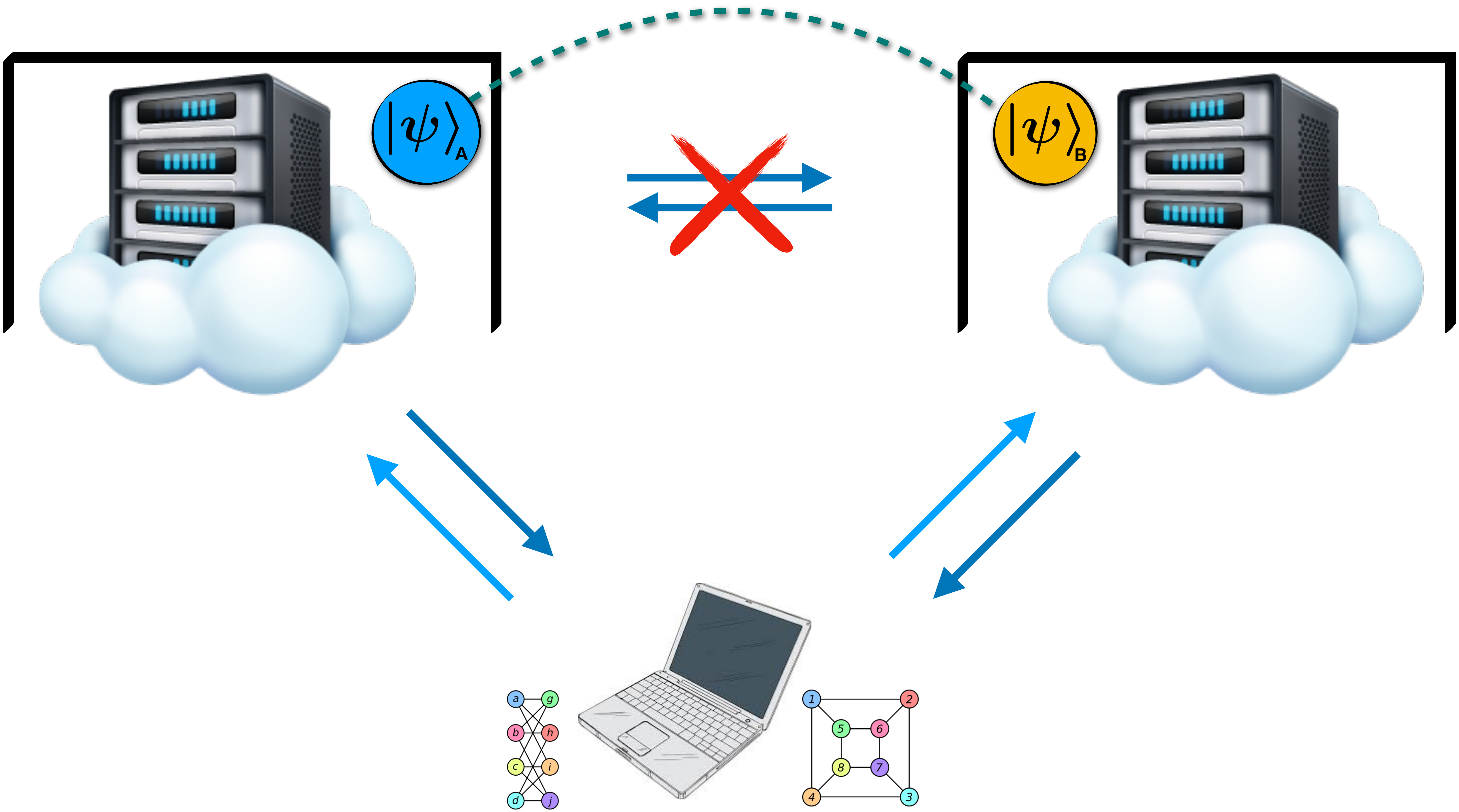


Quantum Entanglement

MIP*

[Cleve-Hoyer-Toner-Watrous 04]



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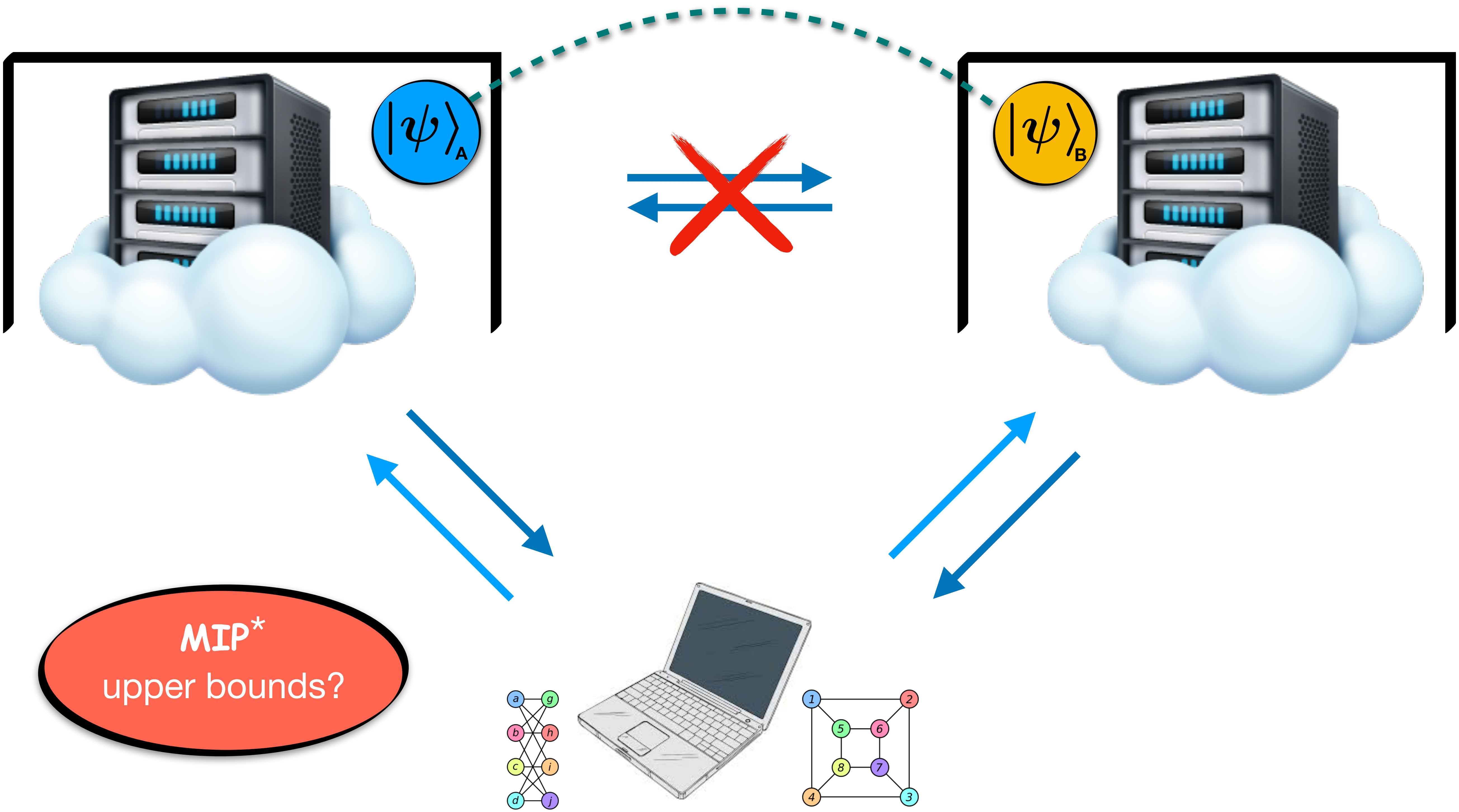

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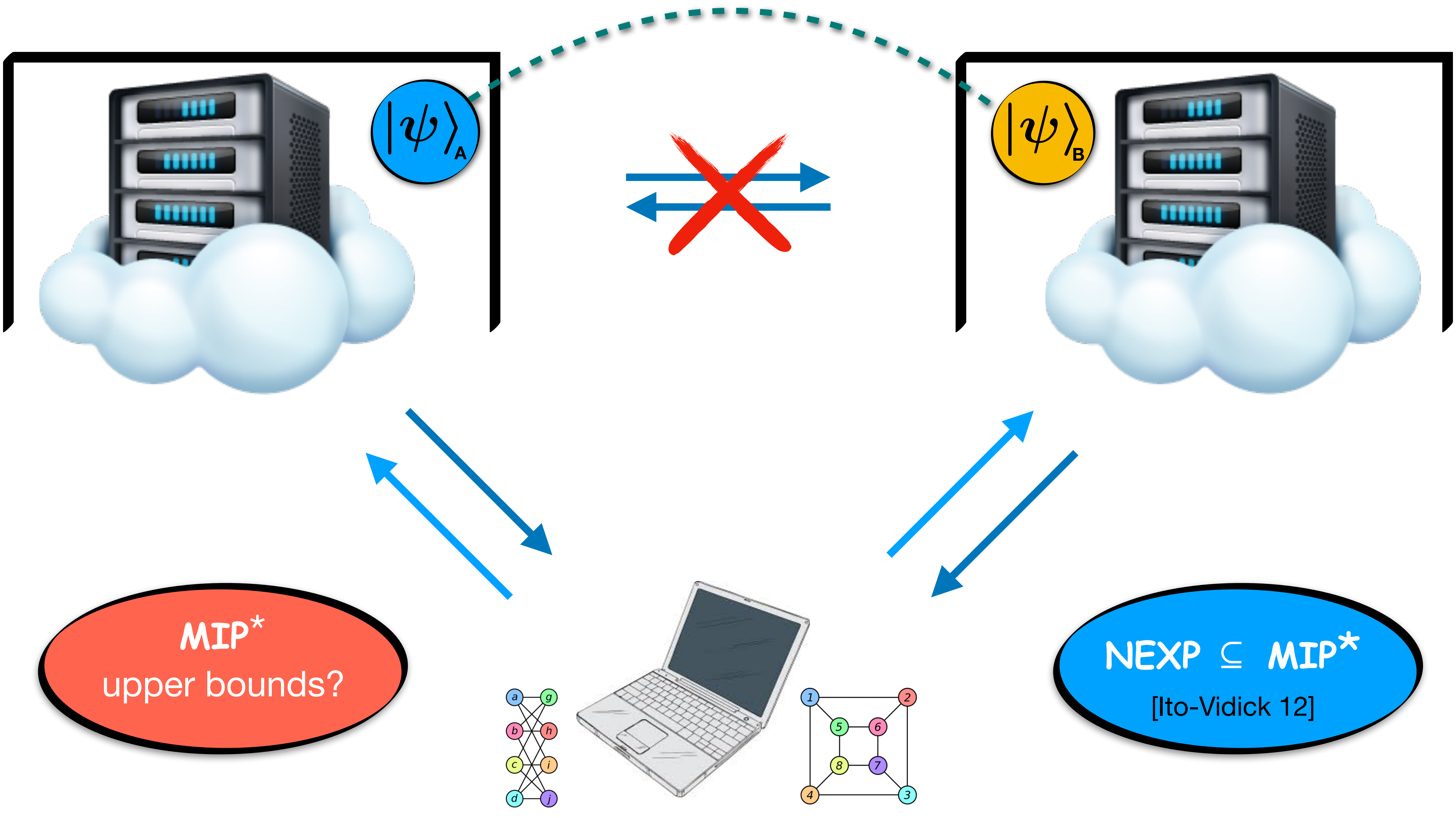

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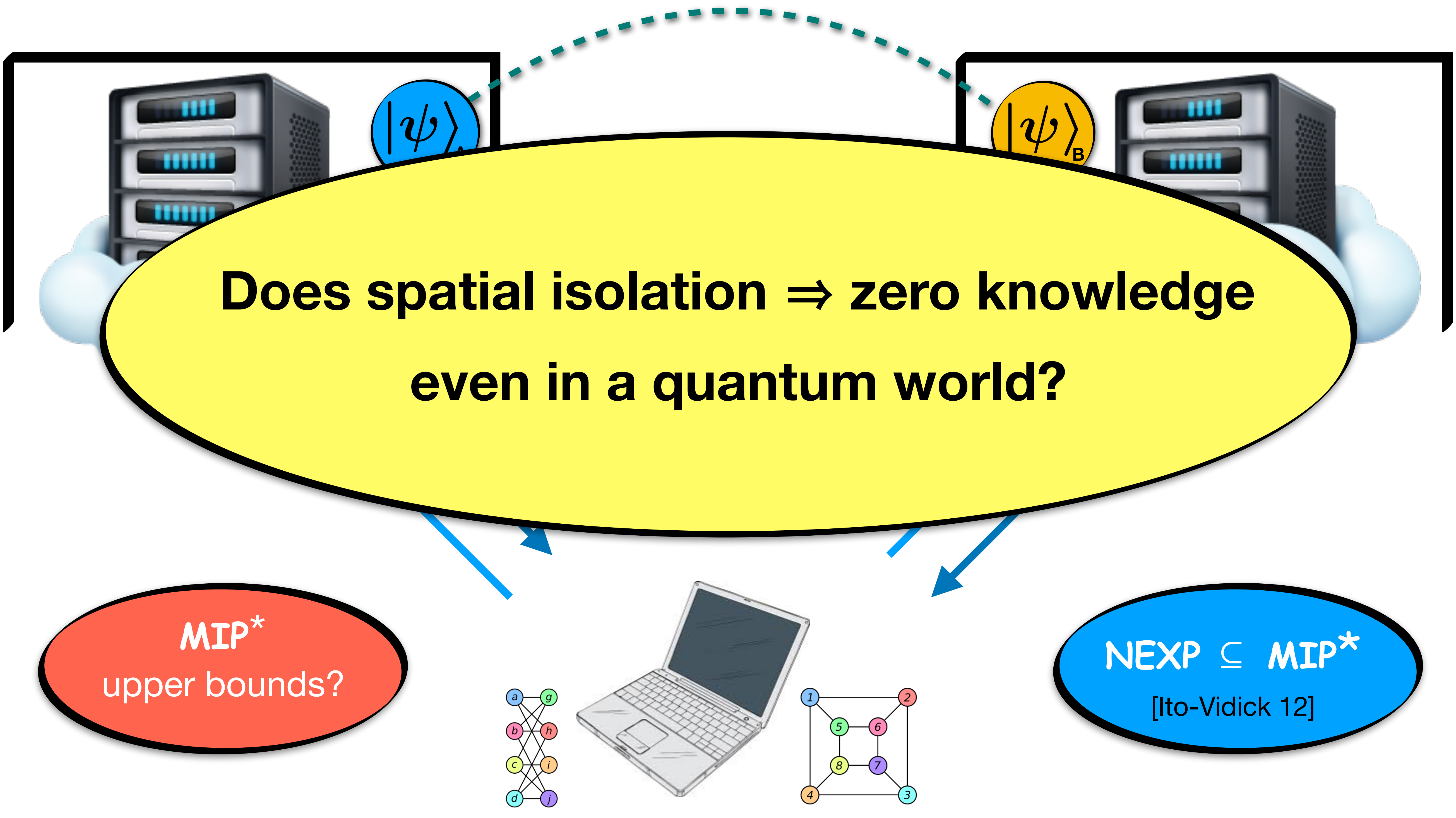

Spatial isolation => zero knowledge
even in a quantum world

Yes!

Spatial isolation => zero knowledge **even in a quantum world**

Theorem: $NEXP \subseteq ZK\text{-}MIP^*$

Spatial isolation => zero knowledge
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Theorem: NEXP ⊆ ZK-MIP*

The challenge

Spatial isolation => zero knowledge
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Theorem: **NEXP ⊆ ZK-MIP***

The challenge

We know that: **NEXP ⊆ MIP*** [IV12]

Spatial isolation => zero knowledge
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Technique Incompatibility

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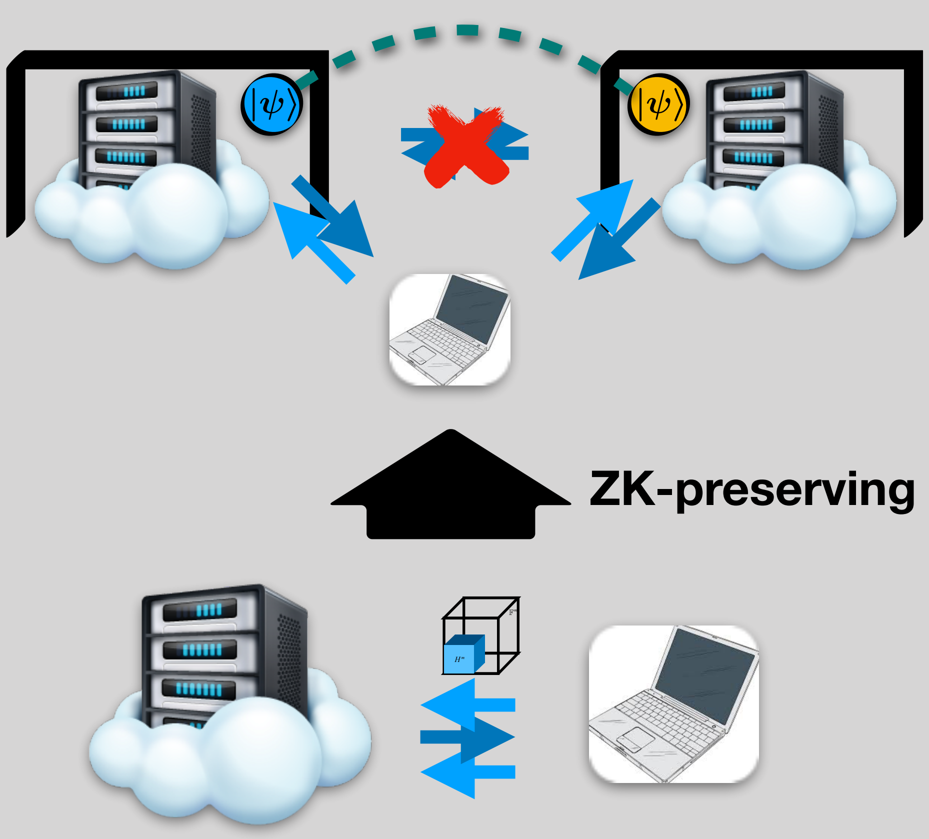
Proof in 2 steps:

Lifting lemma



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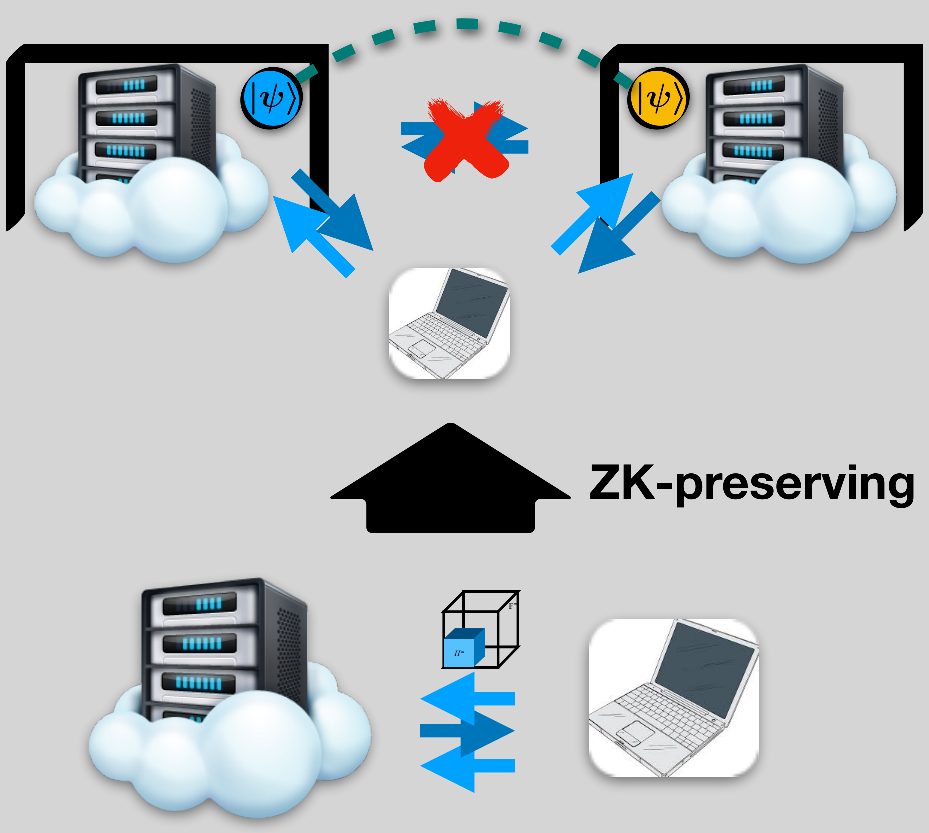
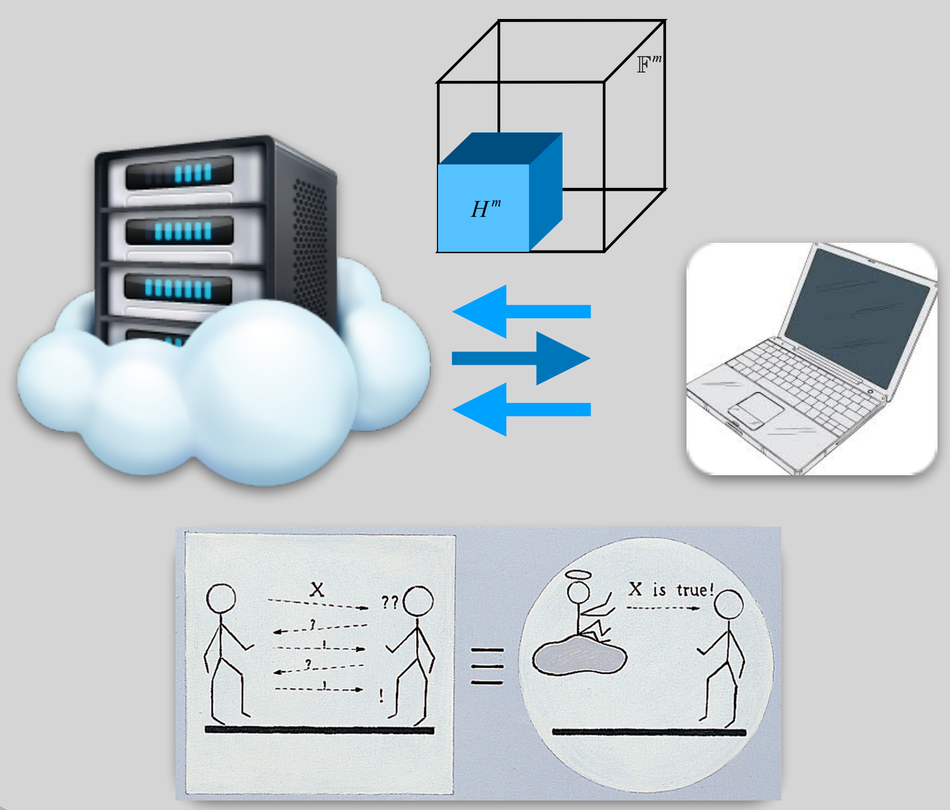


Algebraic ZK



Interactive PCP

$\Downarrow$

MIP*

From Classical to Quantum

Lifting Lemma: **Any PCP** ➡ **MIP* with similar parameters**

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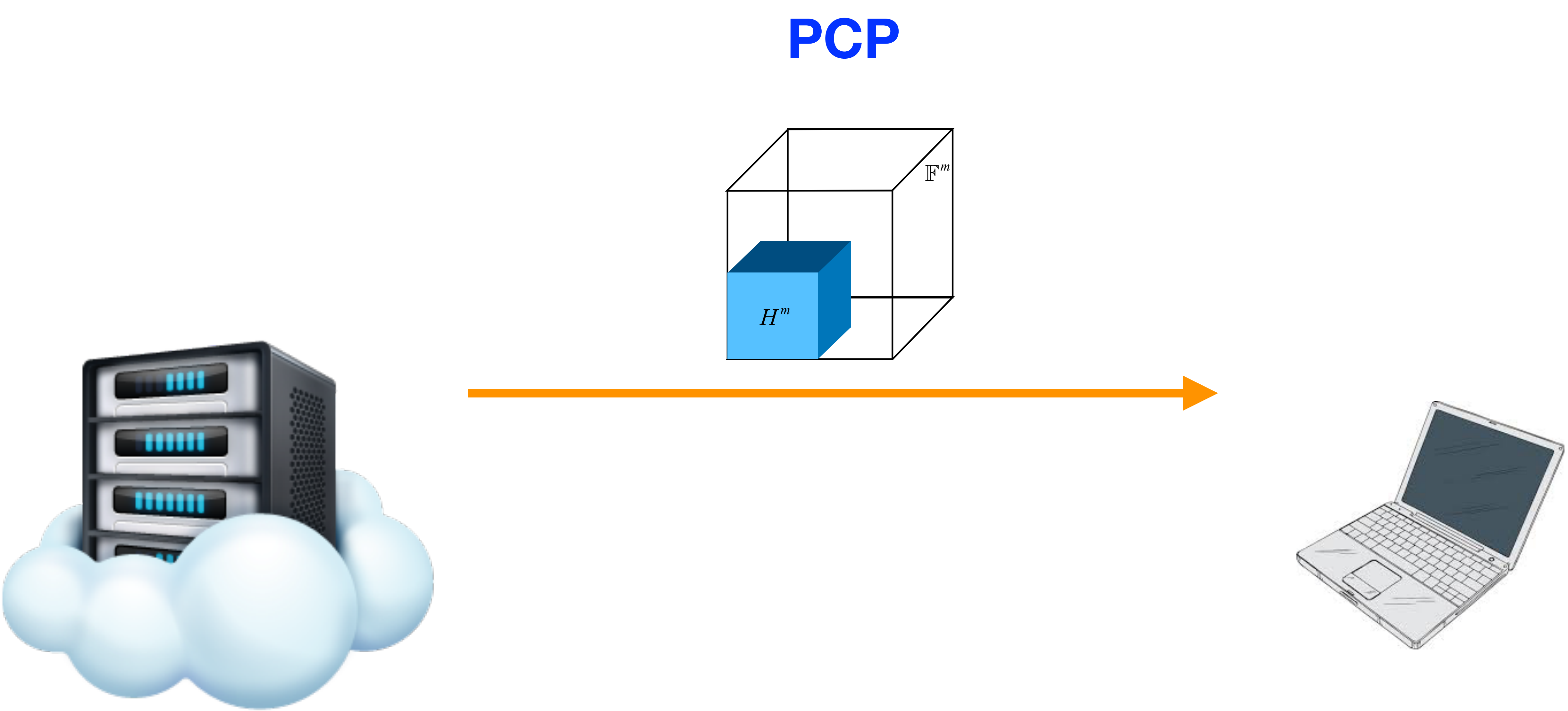

From Classical to Quantum

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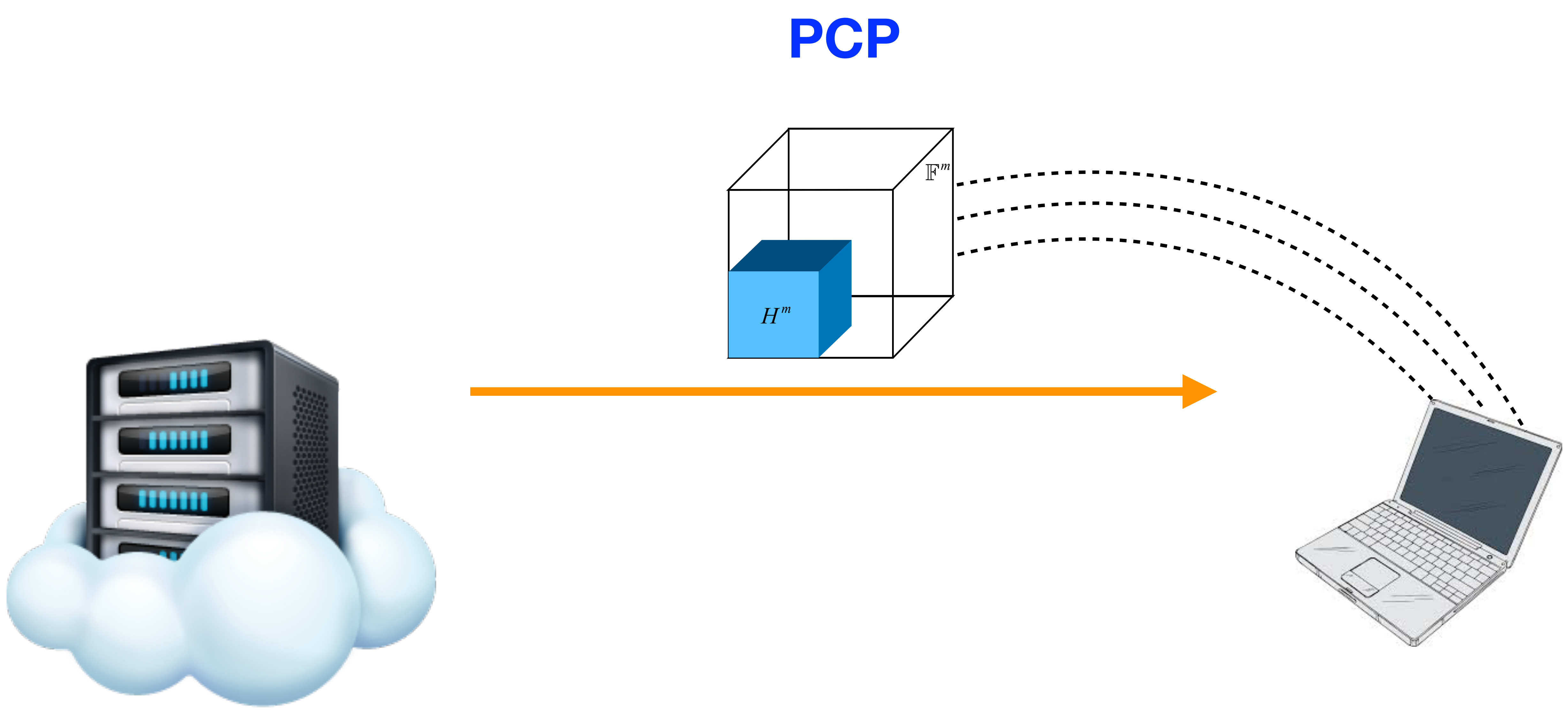

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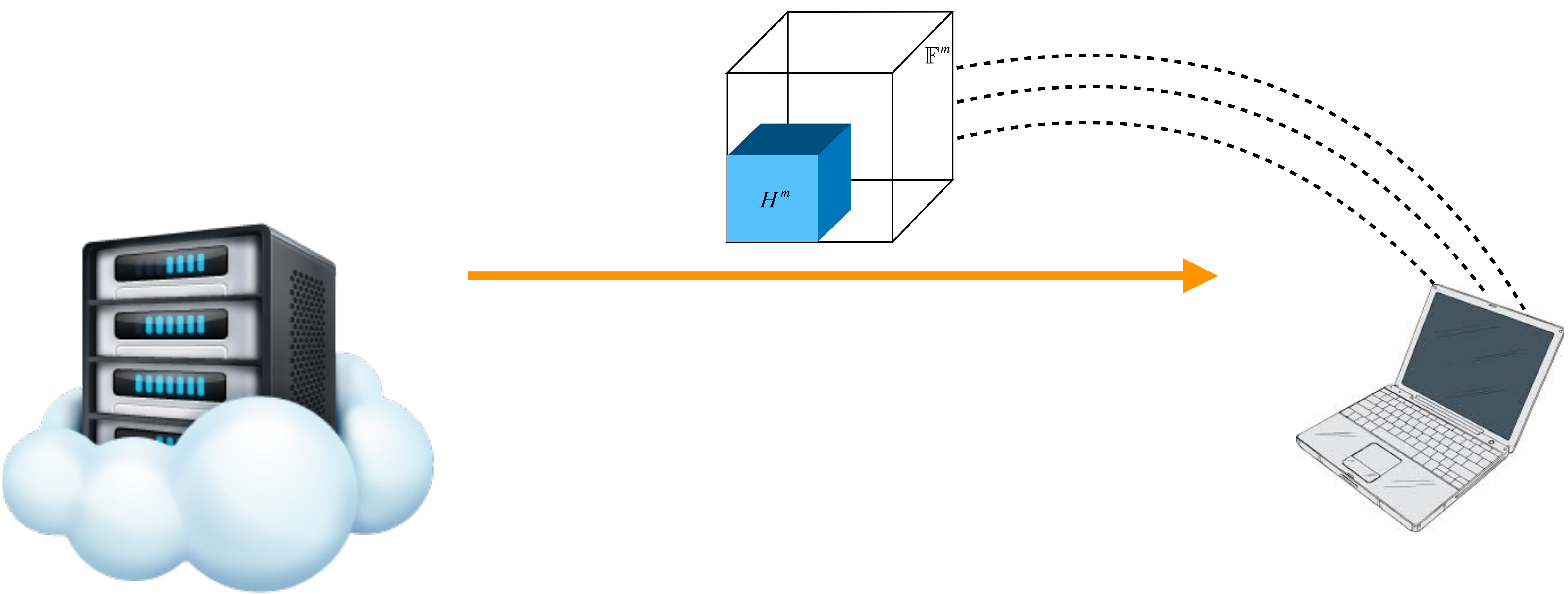
PCP



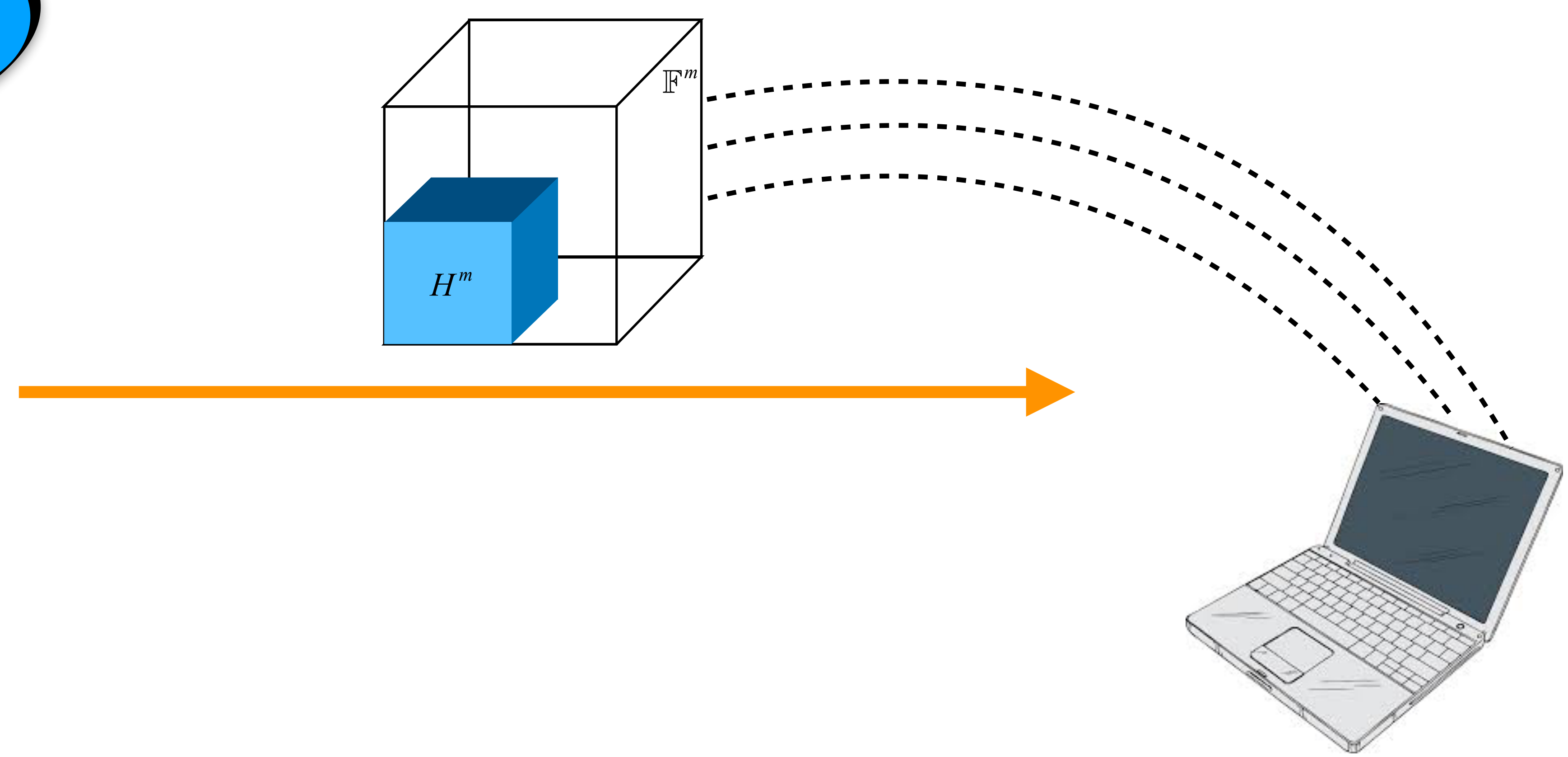
All machines are CLASSICAL

From Classical to Quantum

Lifting Lemma: **Any PCP** ➡ **MIP* with similar parameters**

Abstraction of IV12's NEXP ⊆ MIP*

PCP



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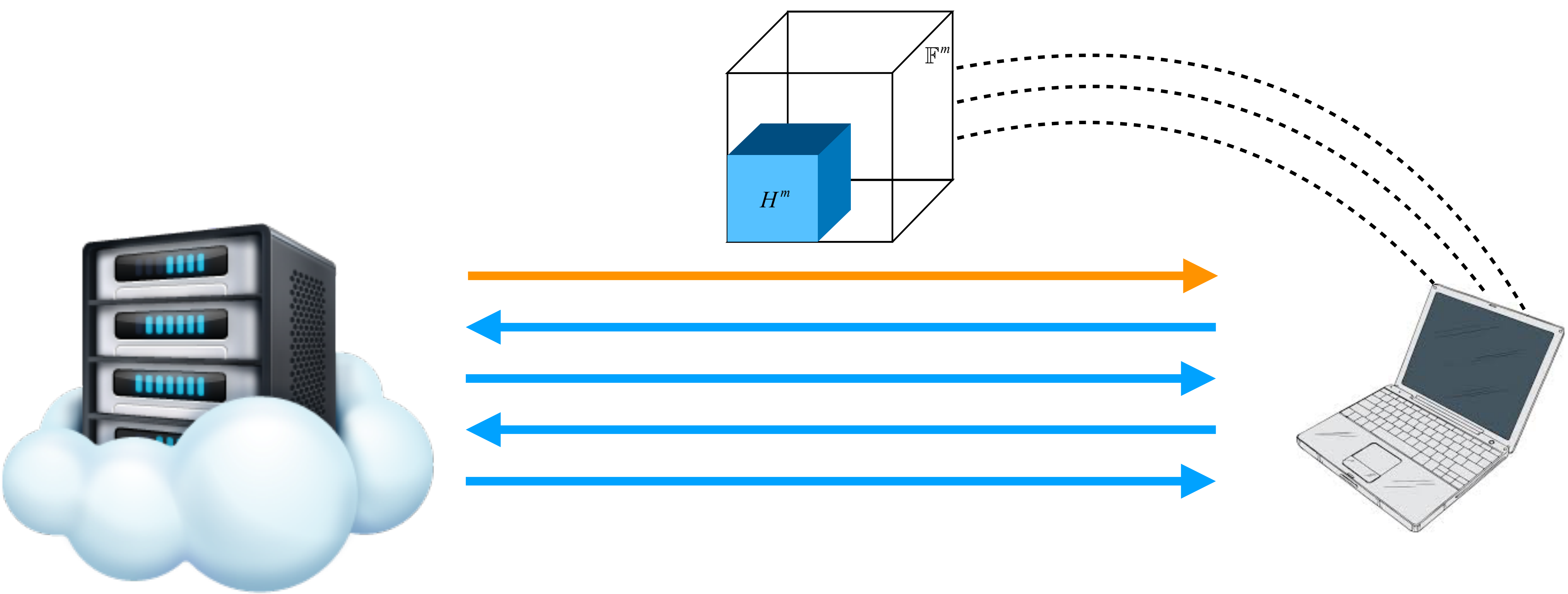
Interactive PCP

[Kalai-Raz 08]



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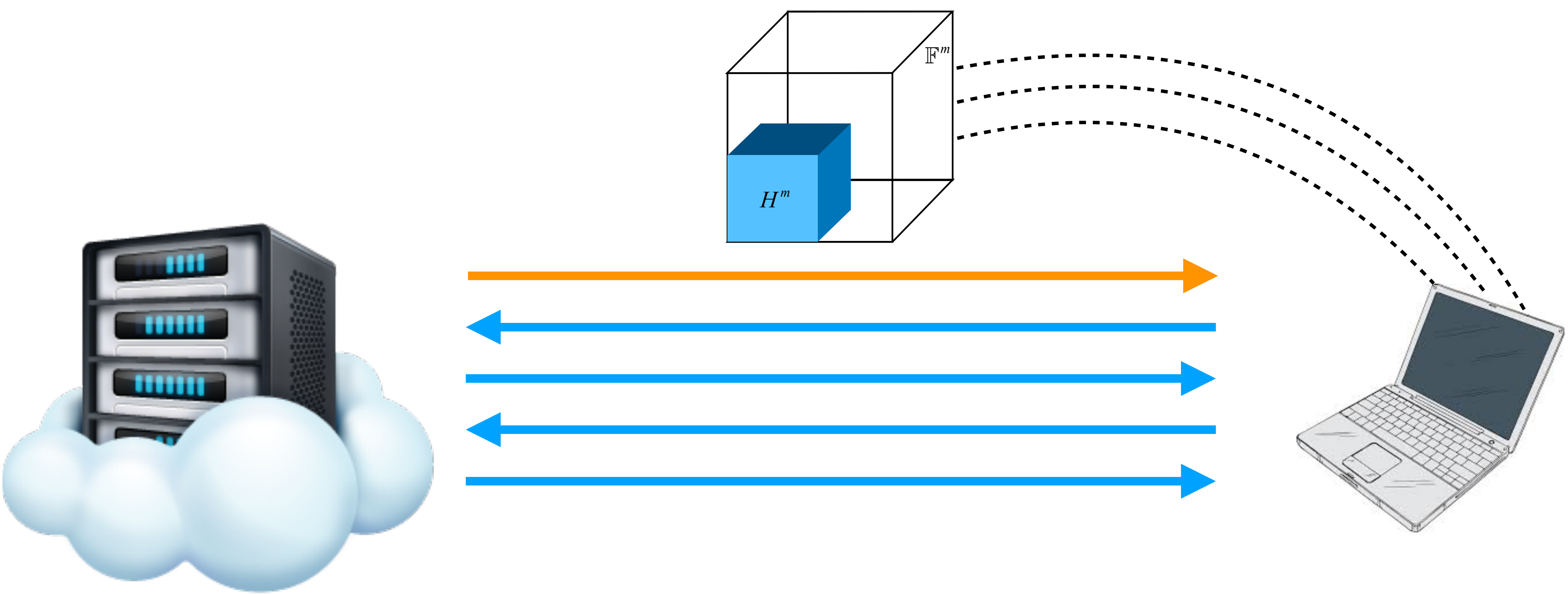
Lifting Lemma: Any "low-degree" interactive PCP ➡ MIP* with similar parameters

PRESERVING ZK

Low-degree Interactive PCP

[Kalai-Raz 08]



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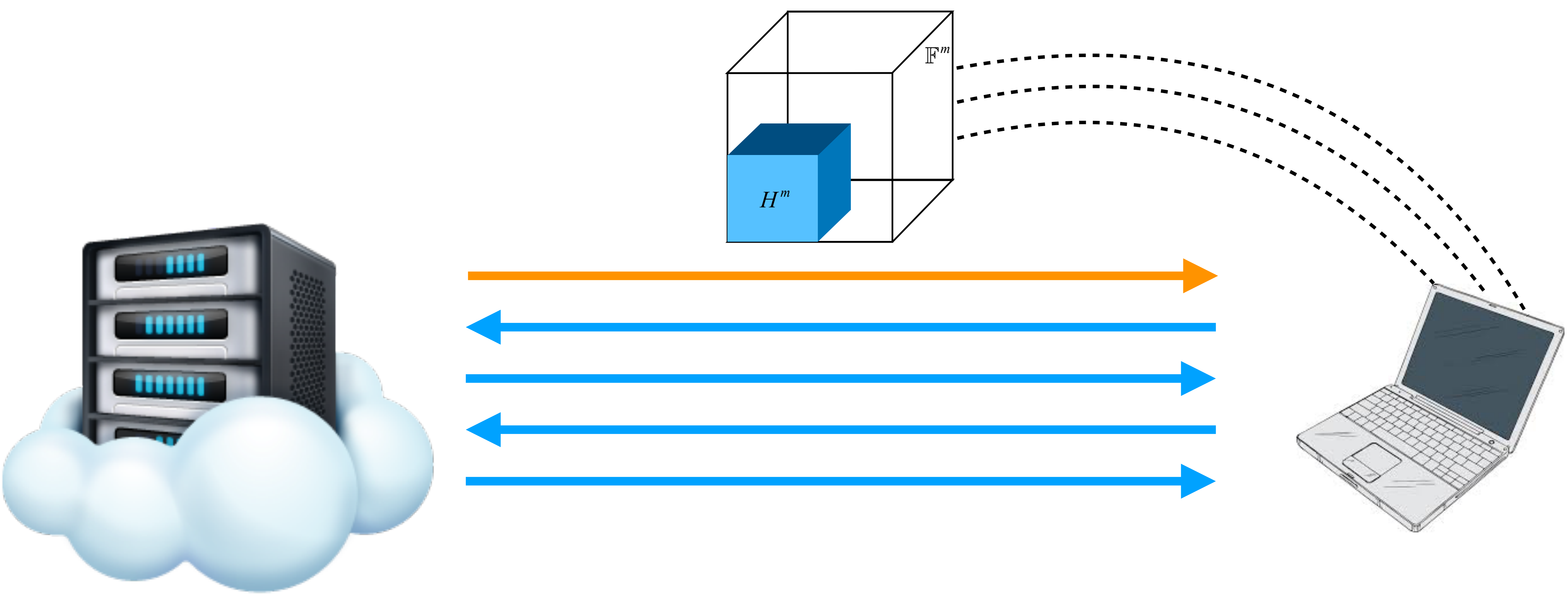

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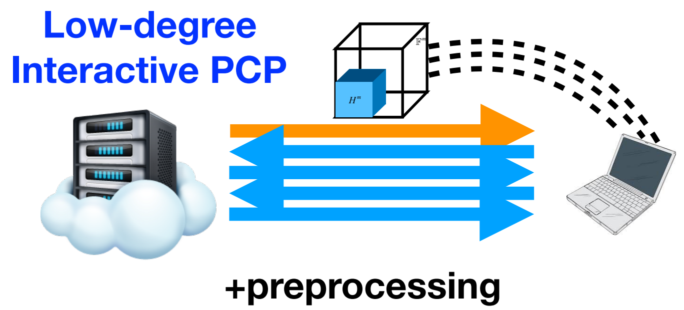
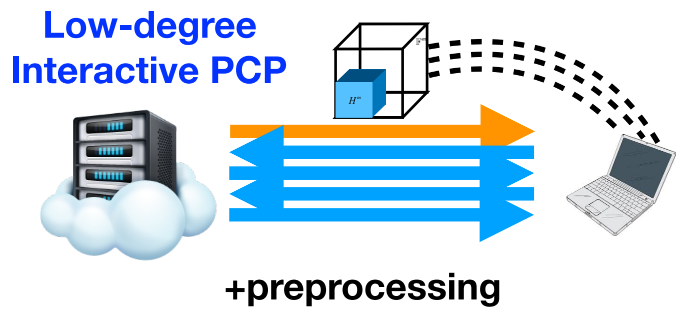
Overview of the Lifting Lemma



Overview of the Lifting Lemma

w.p. 1/2: **MIP* point-vs-plane Low-degree test**
[Natarajan-Vidick 18]



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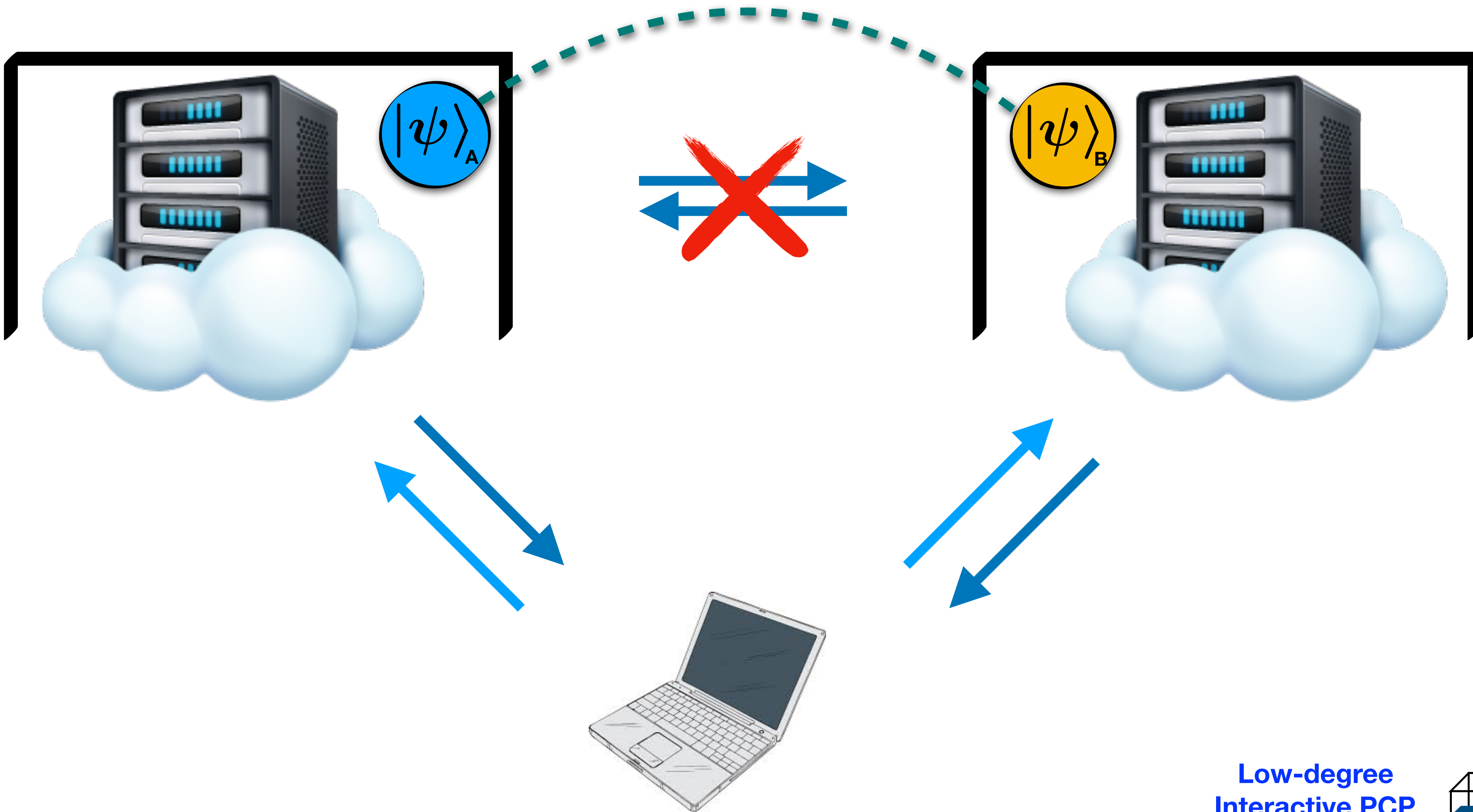

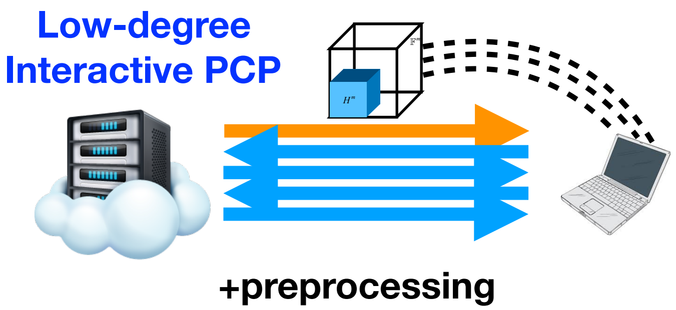

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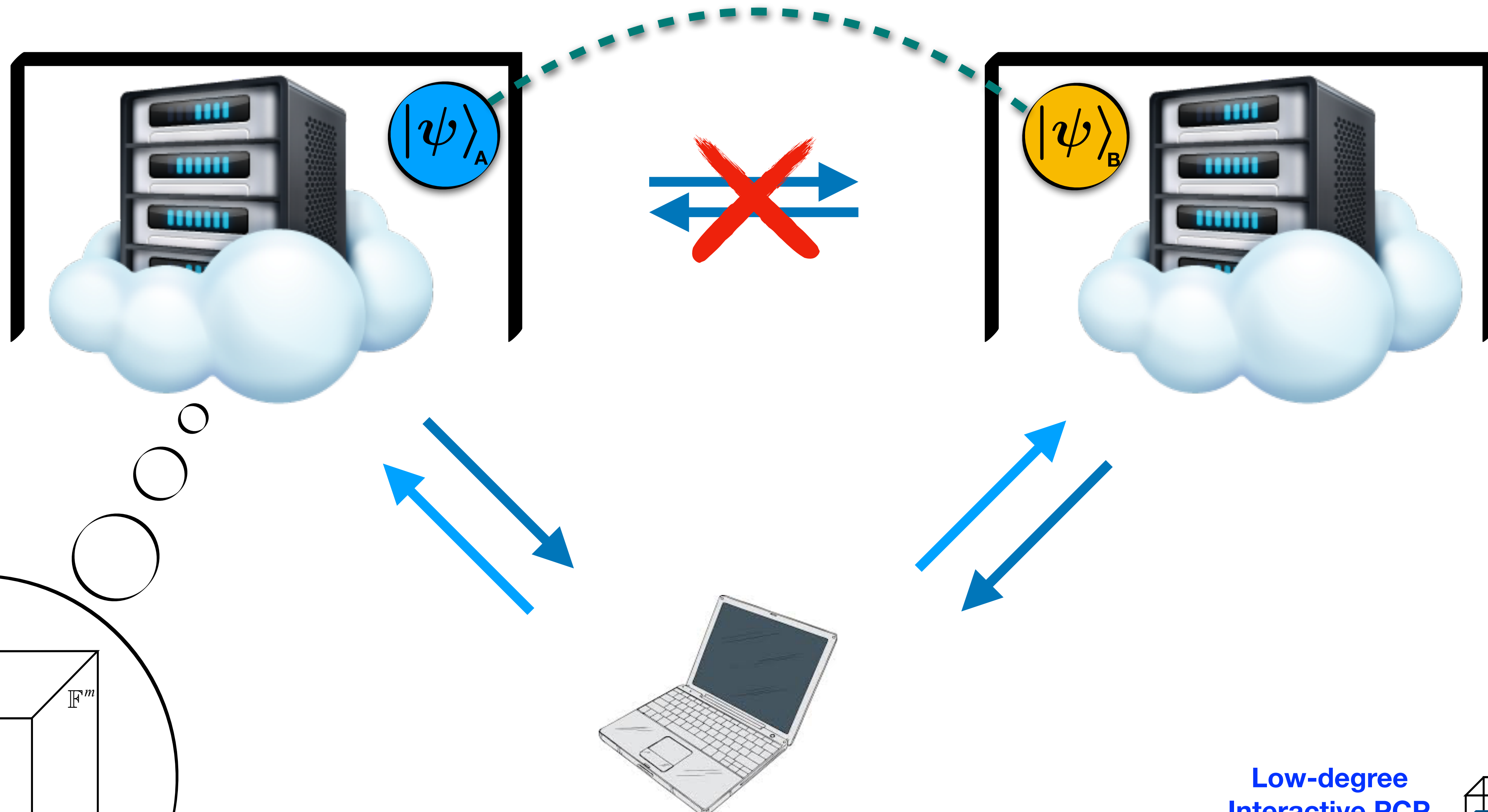

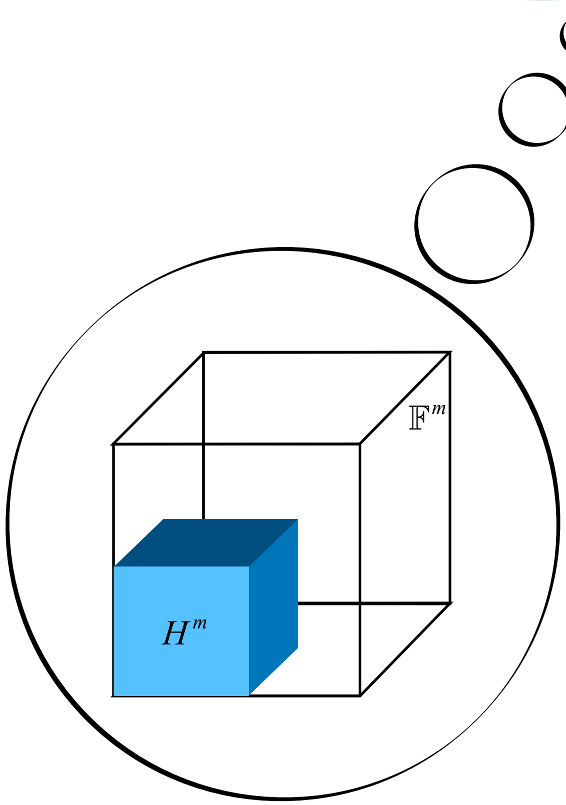

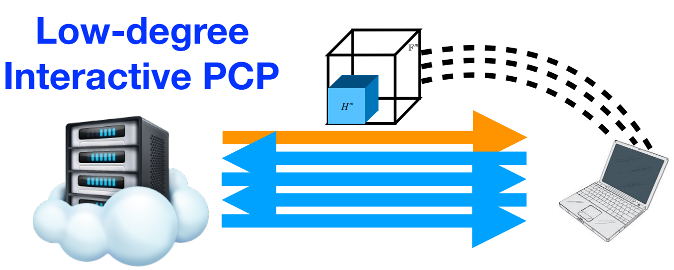

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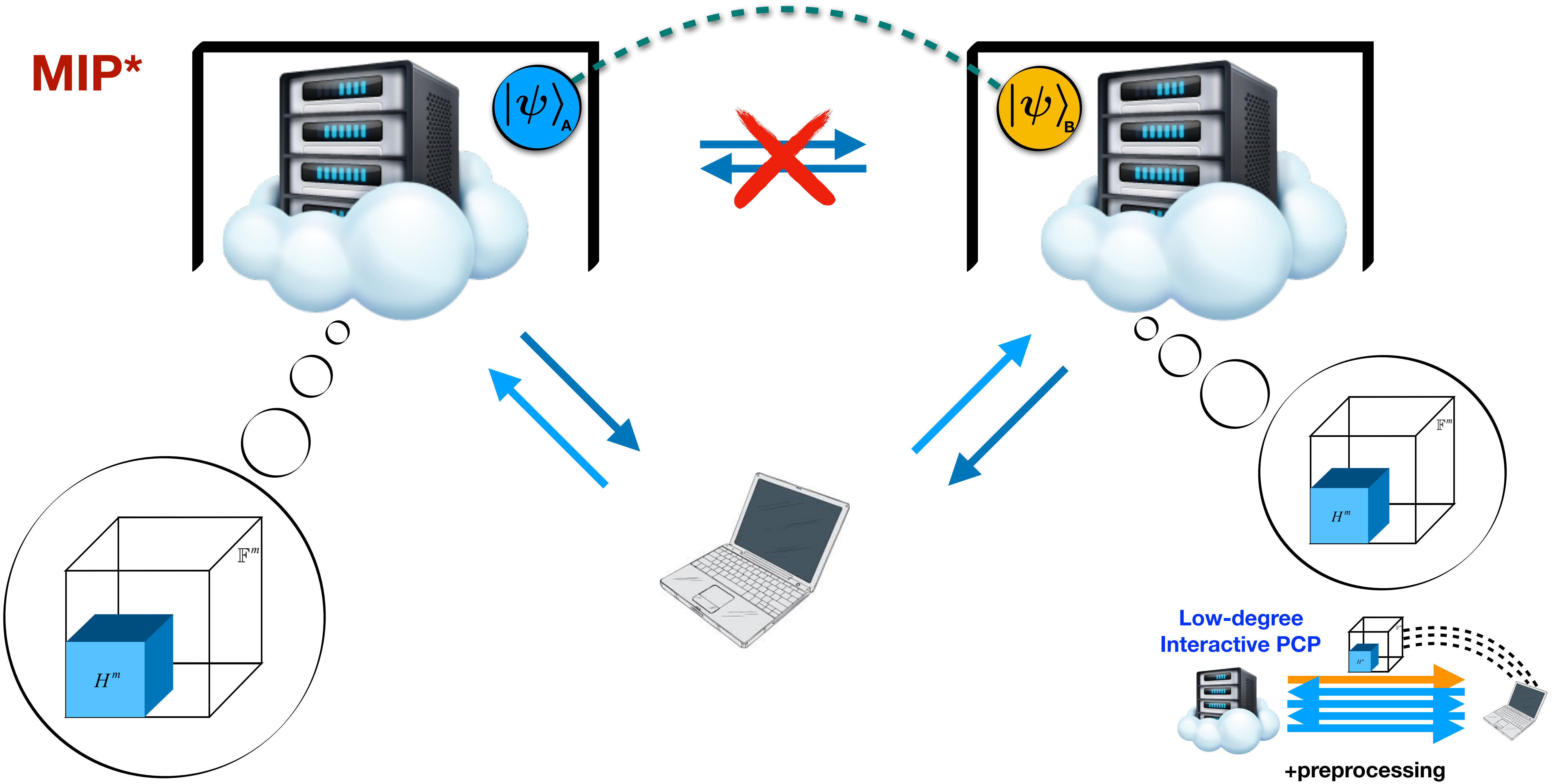

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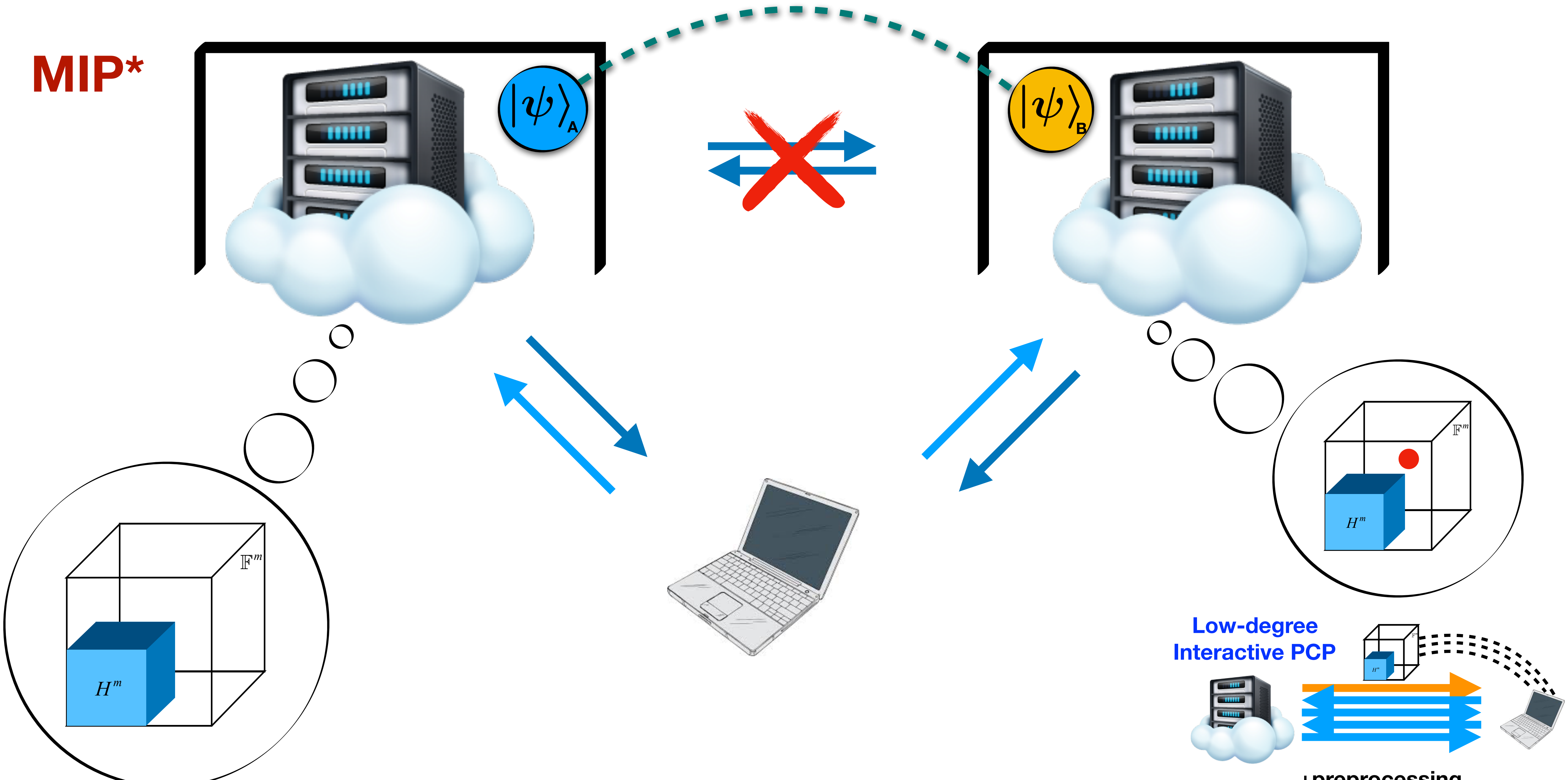

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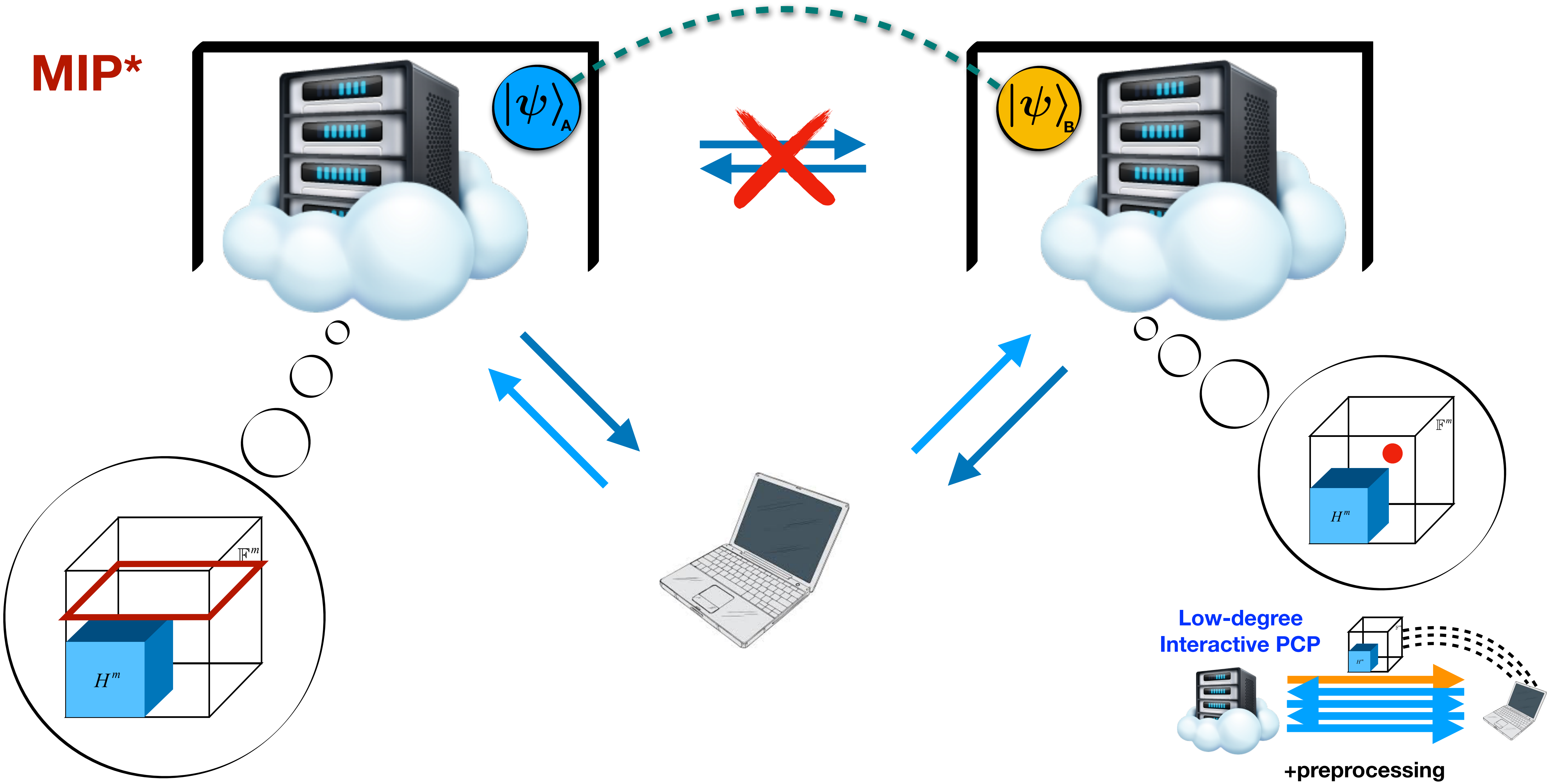

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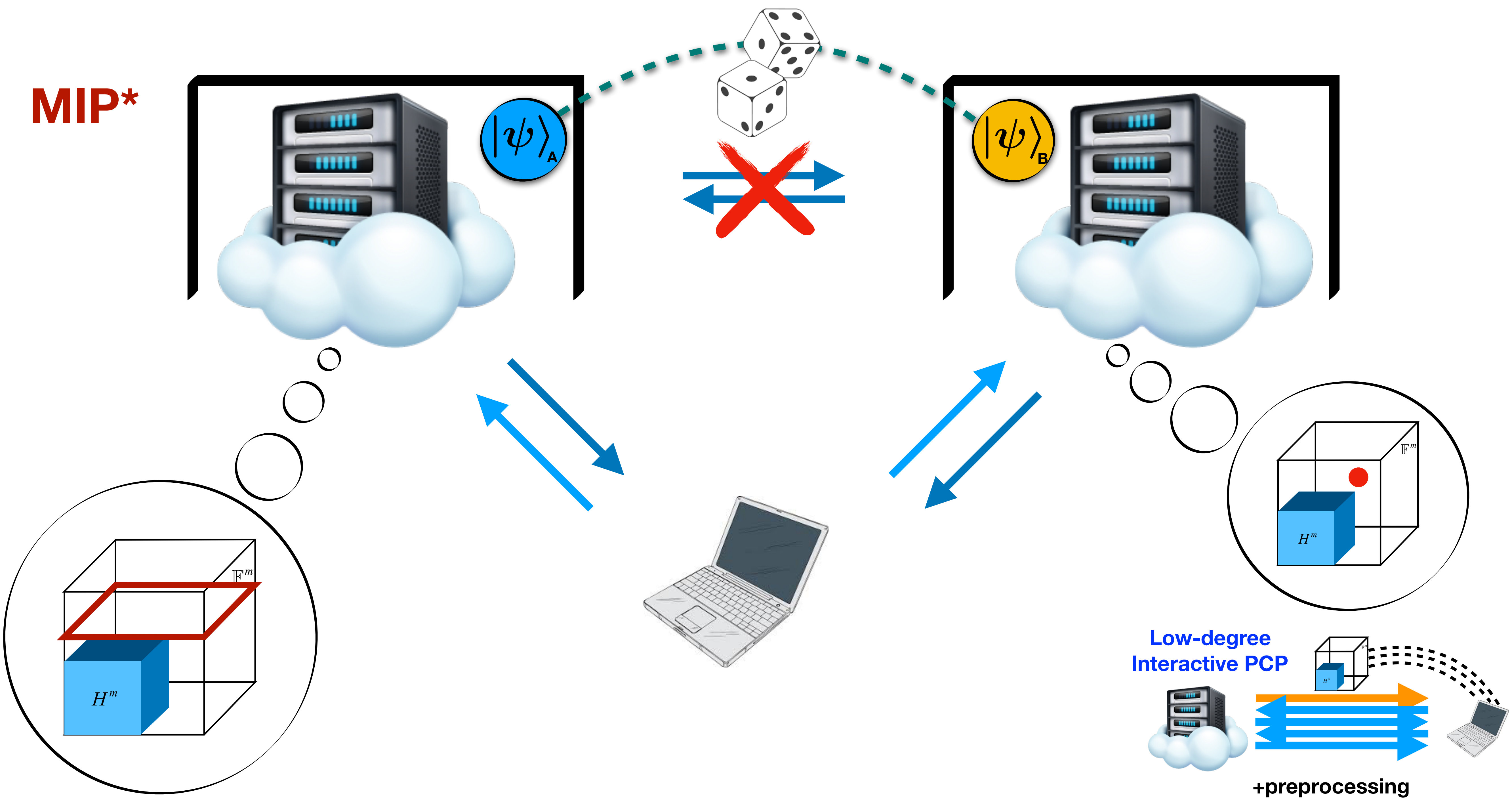
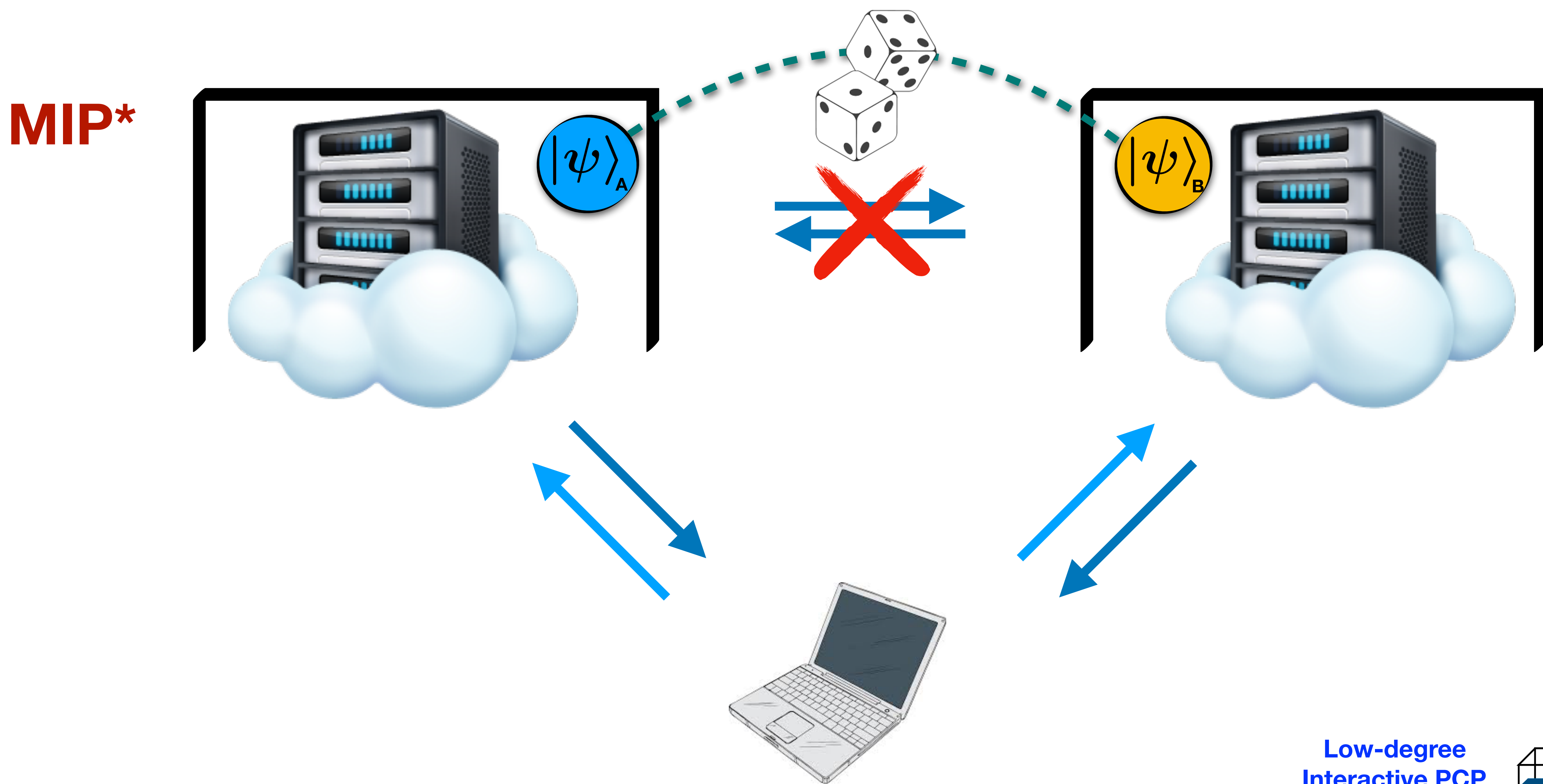
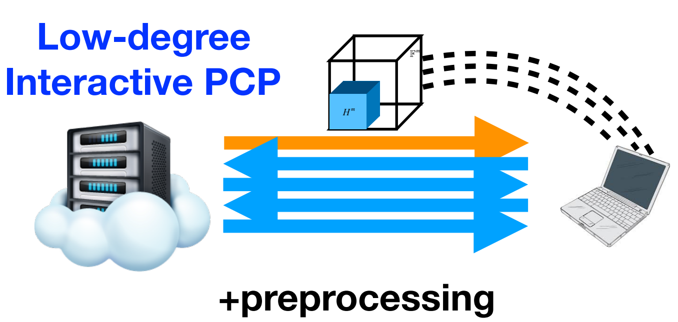


Overview of the Lifting Lemma

w.p. 1/2: **Interactive PCP emulation**





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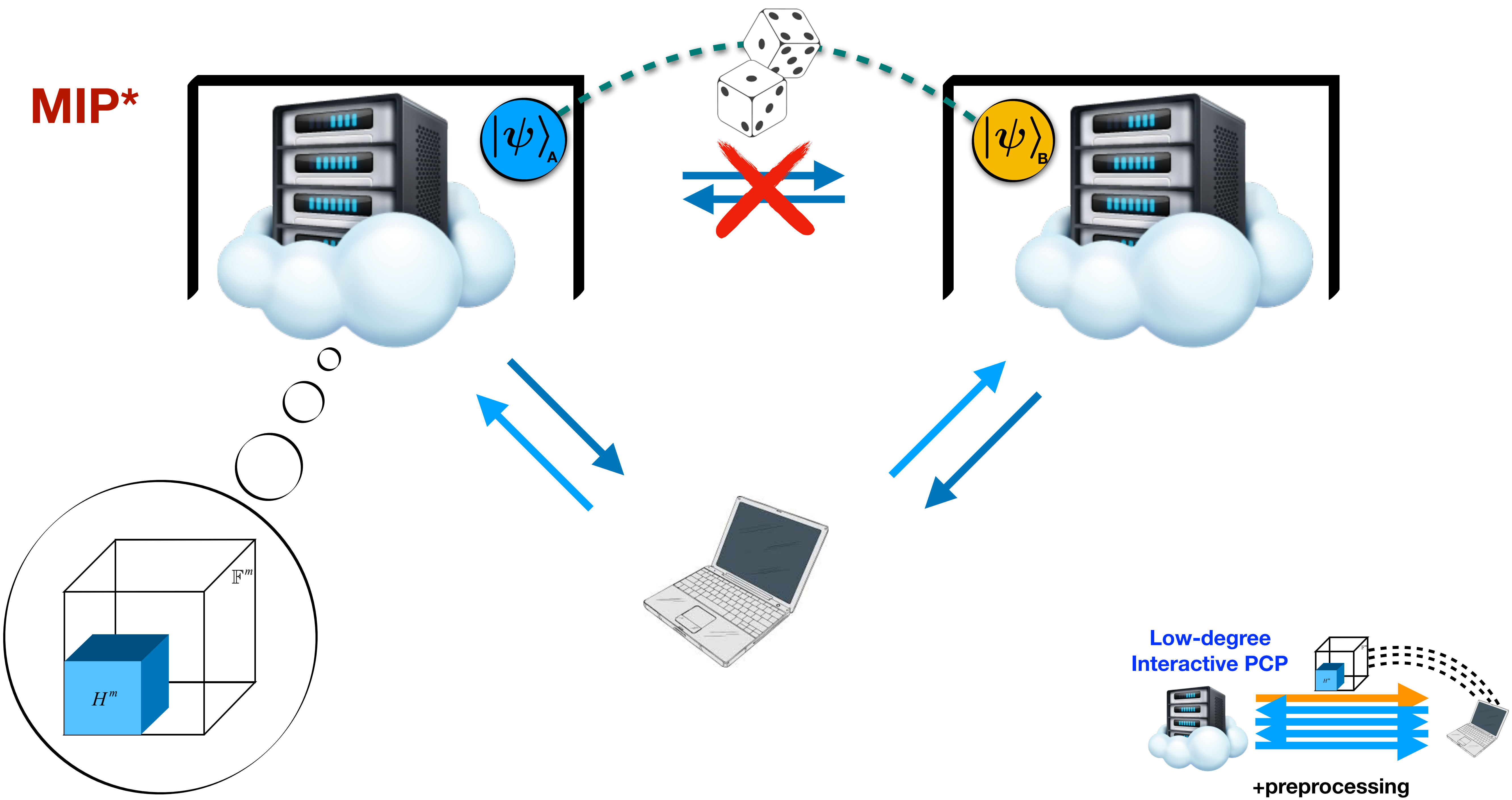

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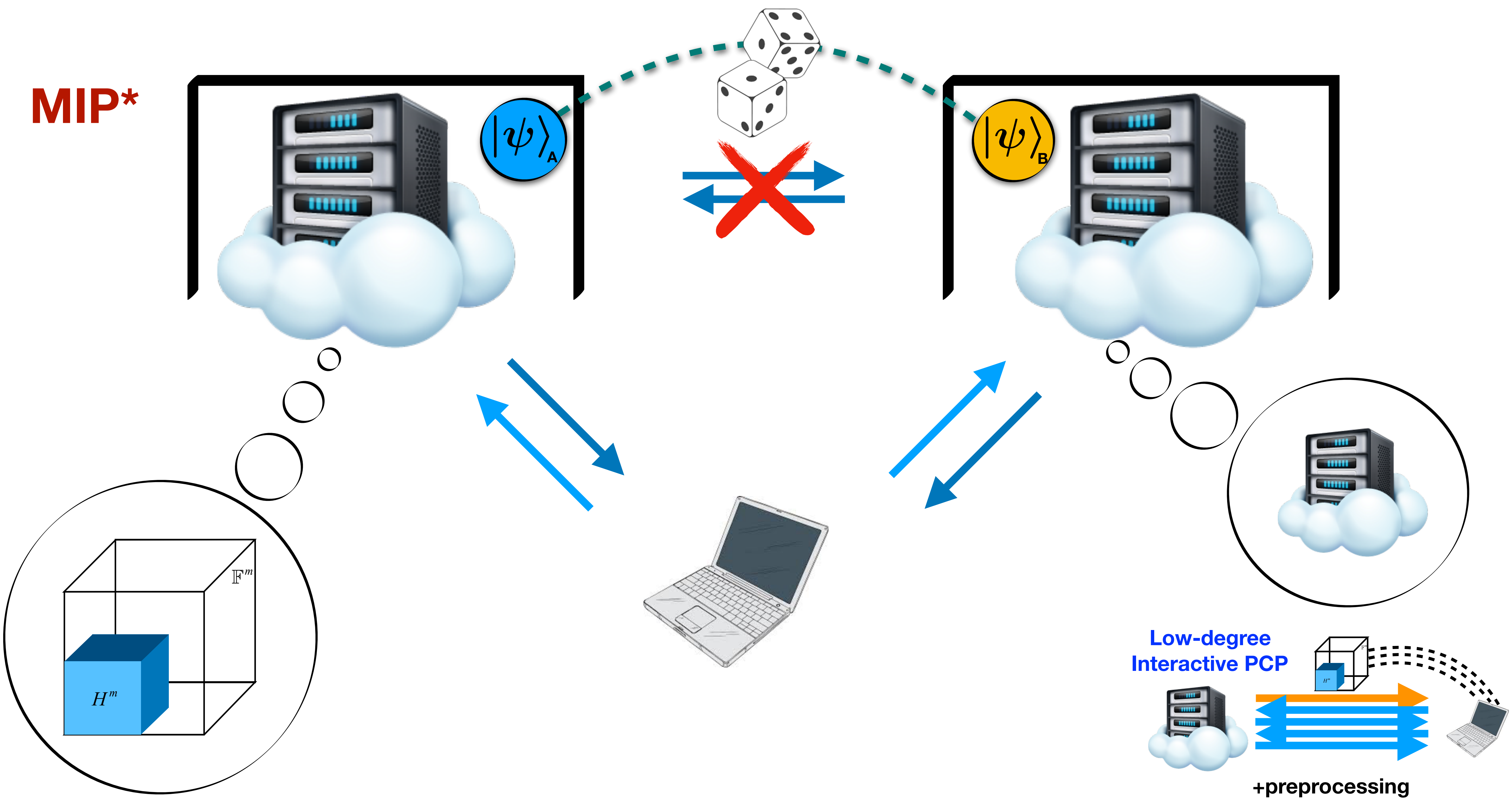

Algebraic Zero Knowledge

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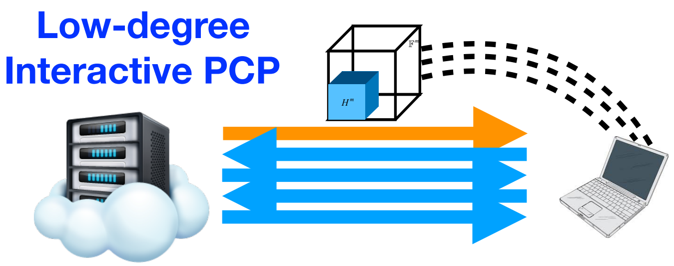
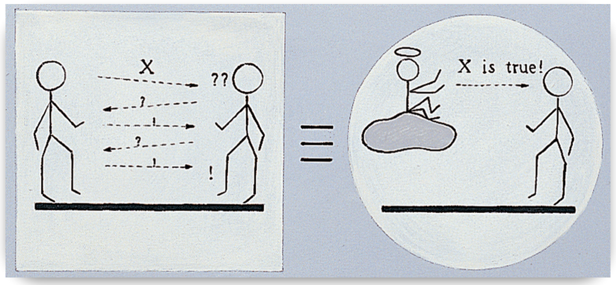
Theorem: There exists a ZERO KNOWLEDGE low-degree interactive PCP for NEXP





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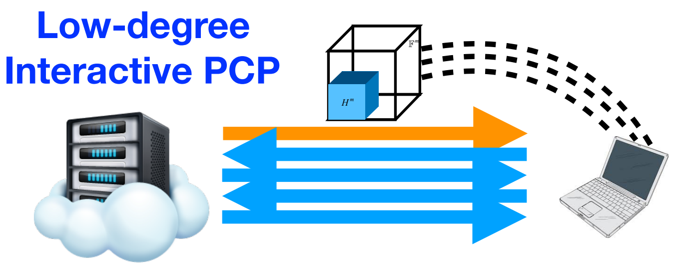

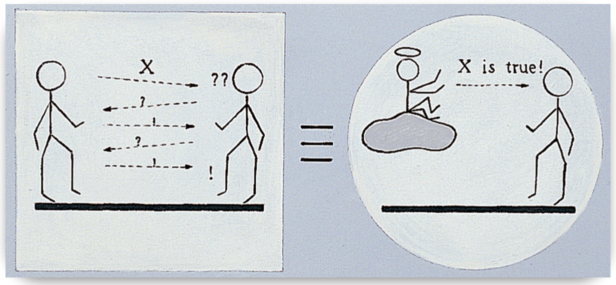

Previous ZK techniques are **Incompatible** with algebraic lifting

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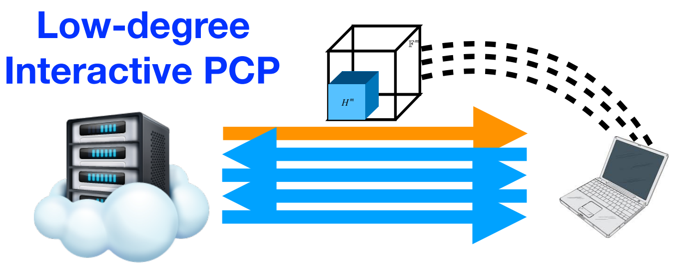
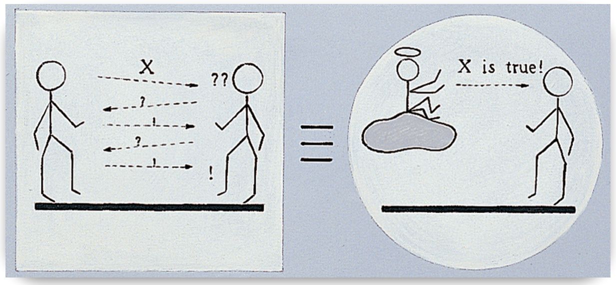




Strong ZK sumcheck

Algebraic Commitment scheme

Weak ZK sumcheck [BCFGRS17]

Previous ZK techniques are **Incompatible** with algebraic lifting

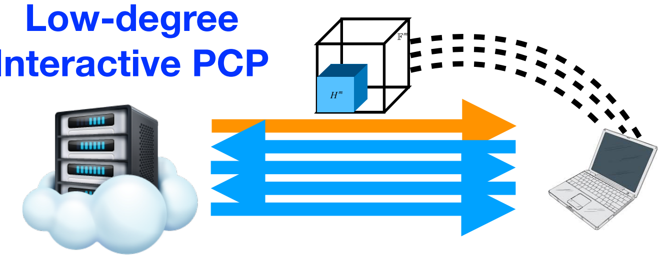
Structural results on Reed-Muller subcube sums

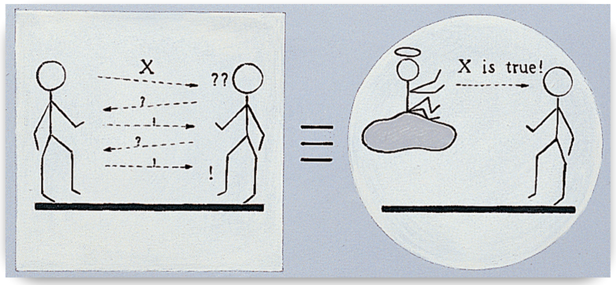
Succinct constraint detection for Reed-Muller [BCFGRS17]

Derandomized PIT for sums of products of Reed-Solomon [RS05]

Algebraic Zero Knowledge

Theorem: There exists a ZERO KNOWLEDGE low-degree interactive PCP for NEXP





Strong ZK sumcheck

Algebraic Commitment scheme

Weak ZK sumcheck [BCFGRS17]

Previous ZK techniques are **Incompatible** with algebraic lifting

Structural results on Reed-Muller subcube sums

Succinct constraint detection for Reed-Muller [BCFGRS17]

Derandomized PIT for sums of products of Reed-Solomon [RS05]

Algebraic Commitment Scheme

First some high-level motivation

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IPCP model

Algebraic Commitment Scheme

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Goal: commit to a message $\beta \in \mathbb{F}$

perfectly **HIDING** the message

in a statistically **BINDING** way

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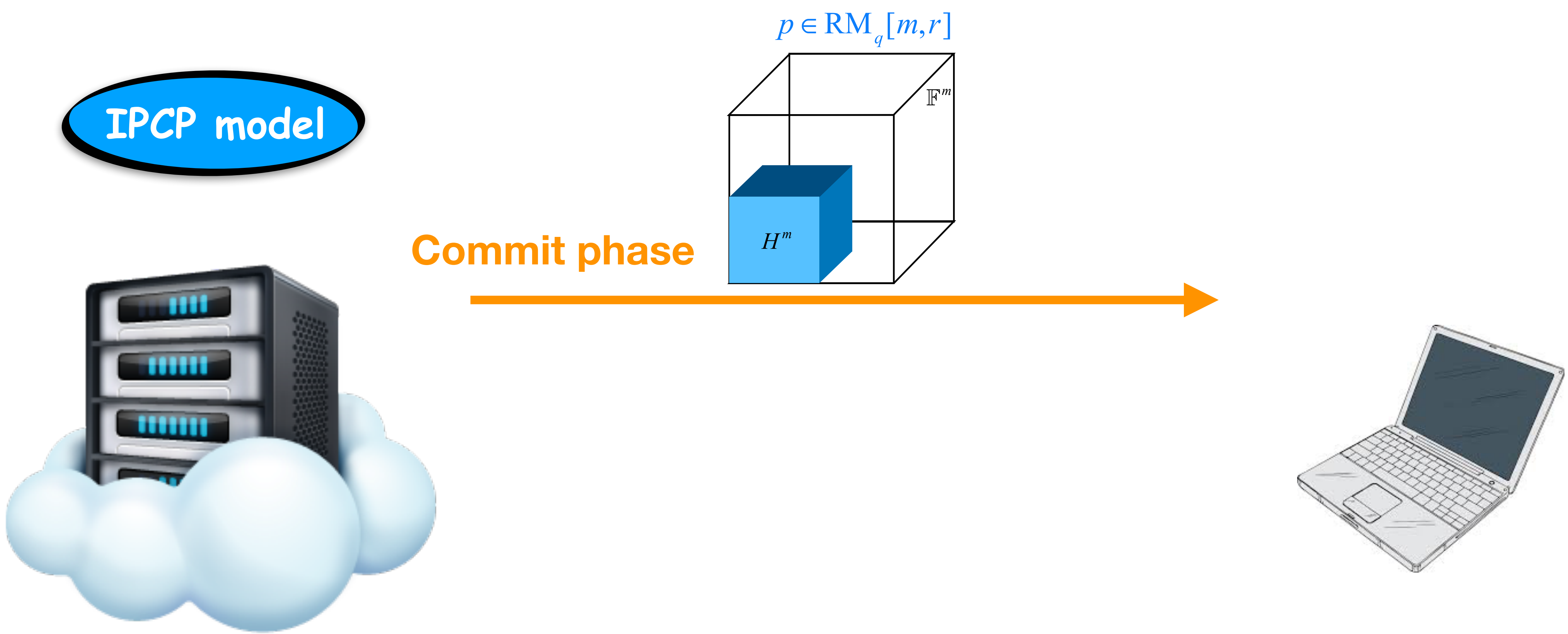

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How: send a random polynomial
p s.t. $\sum_{\alpha \in H^m} p(\alpha) = \beta$

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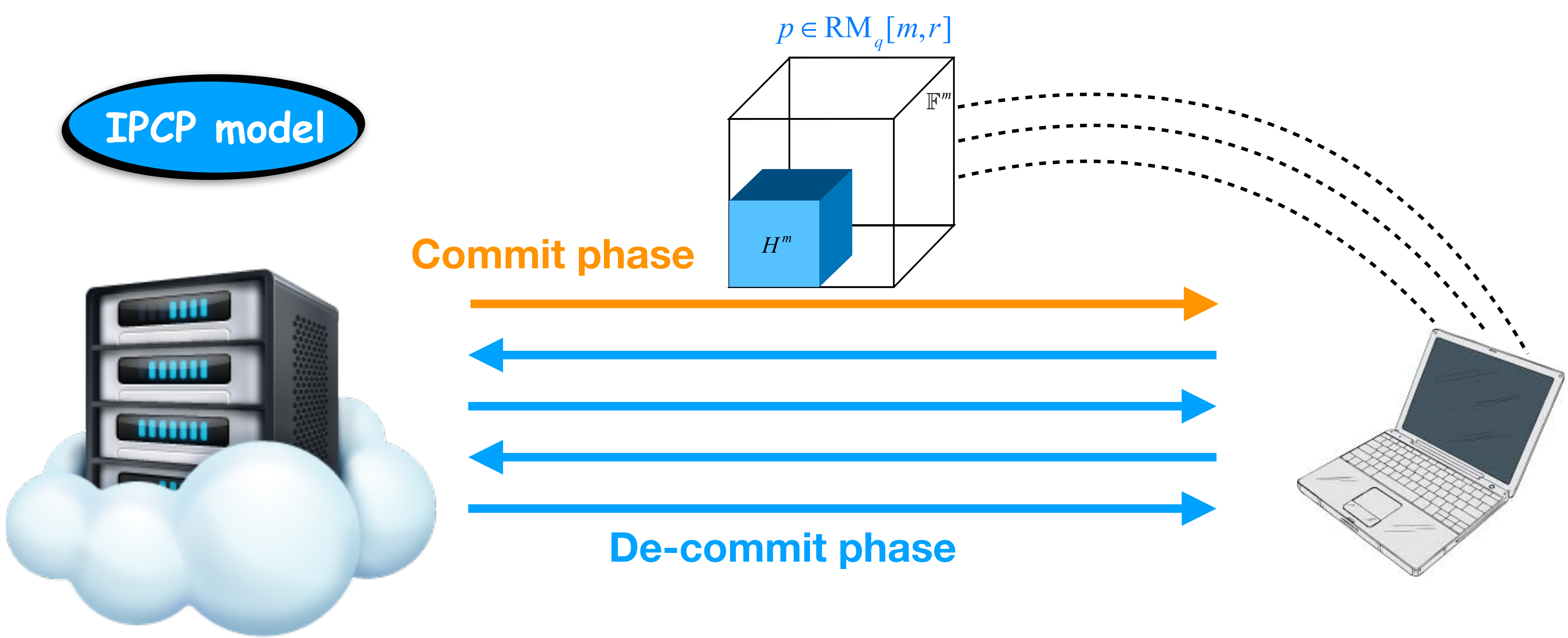

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perfectly **HIDING** the message
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How: send a random polynomial
p s.t. $\sum_{\alpha \in H^m} p(\alpha) = \beta$
de-commit via interaction

Warmup: Subcube Sums of Reed-Muller

$$\mathrm{RM}_q[m,r] = \{\langle p(\alpha)\rangle \mid p \in \mathbb{F}_q^{\leq r}[X_1,\ldots,X_m]\}$$

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Low-degree extension perspective

$H \subseteq \mathbb{F}$ $\quad |H| < r$

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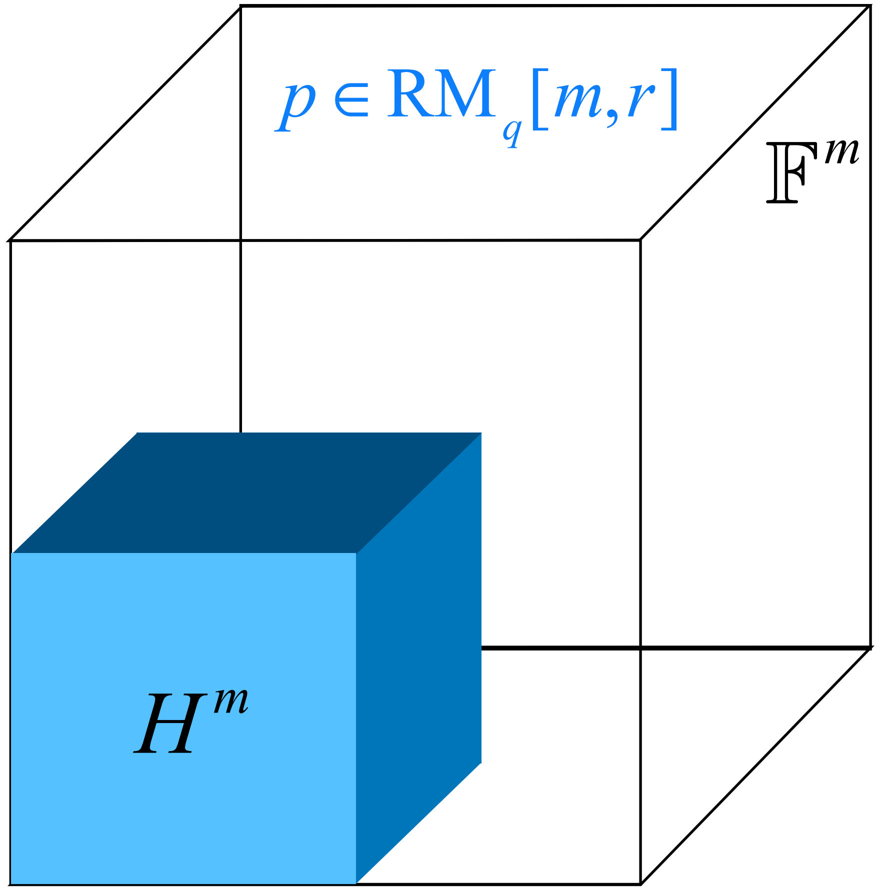

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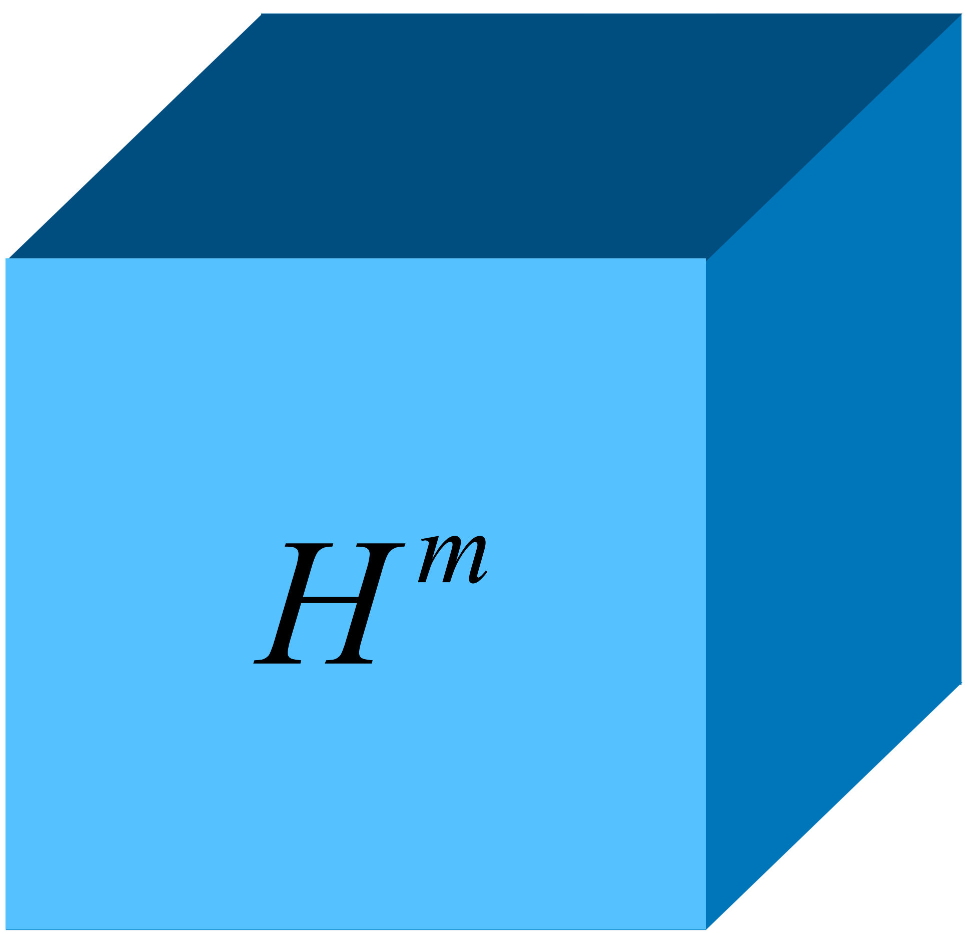

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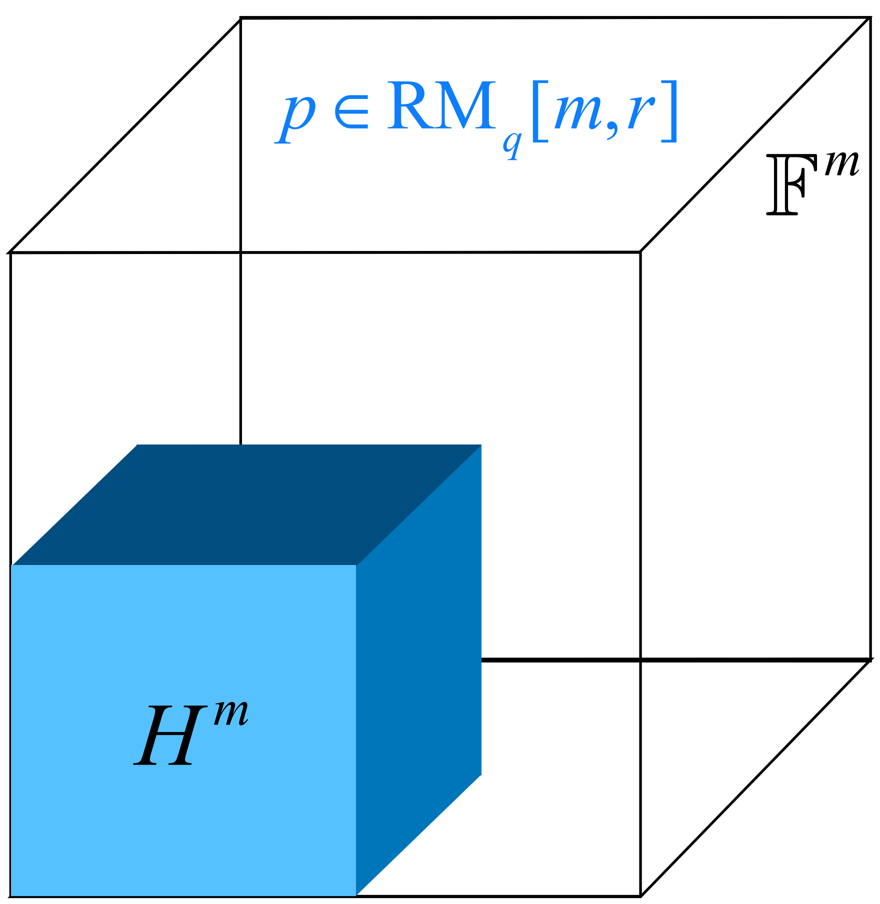

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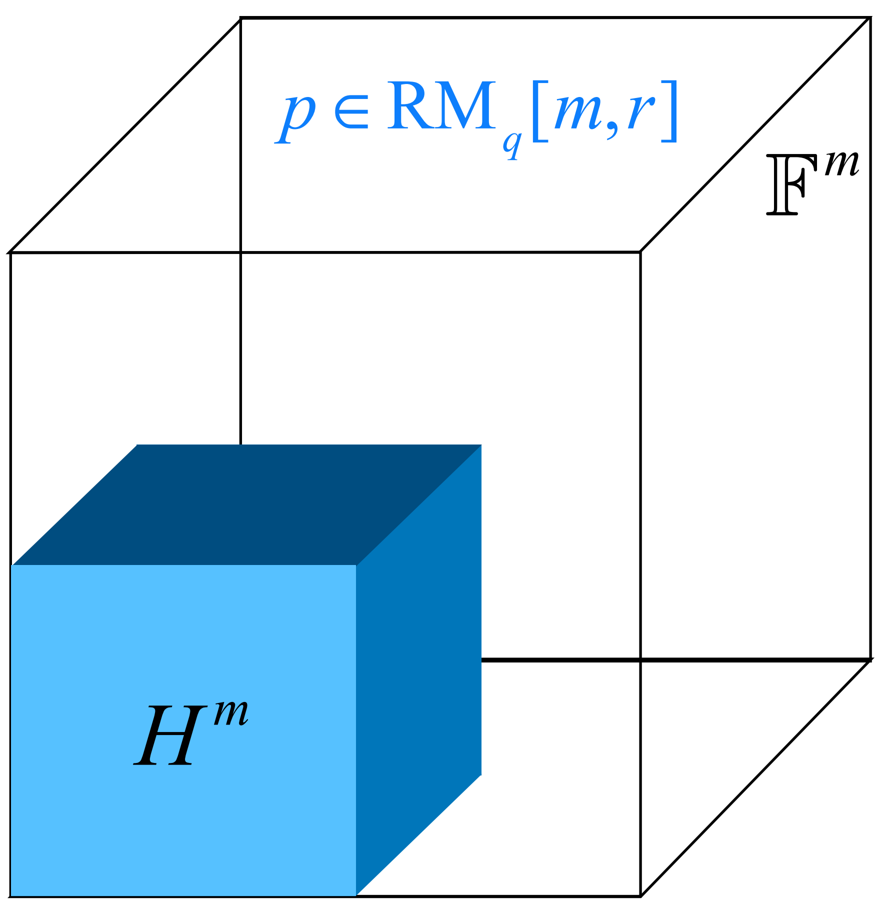

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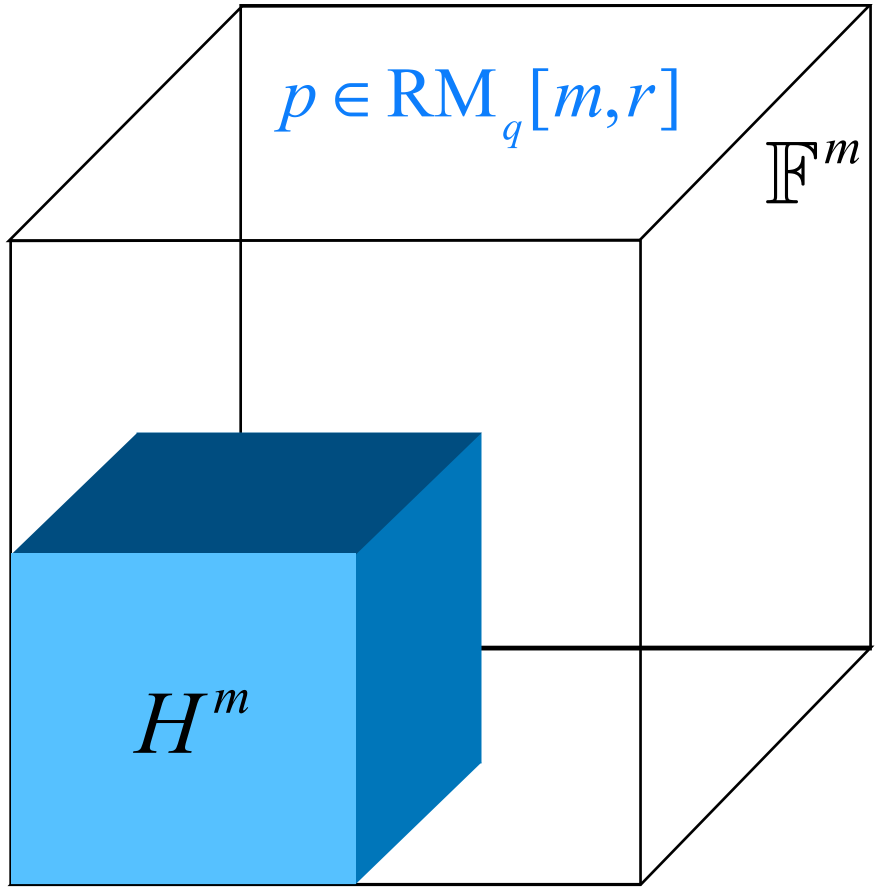

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Algebrization framework

[Aaronson-Wigderson 09]

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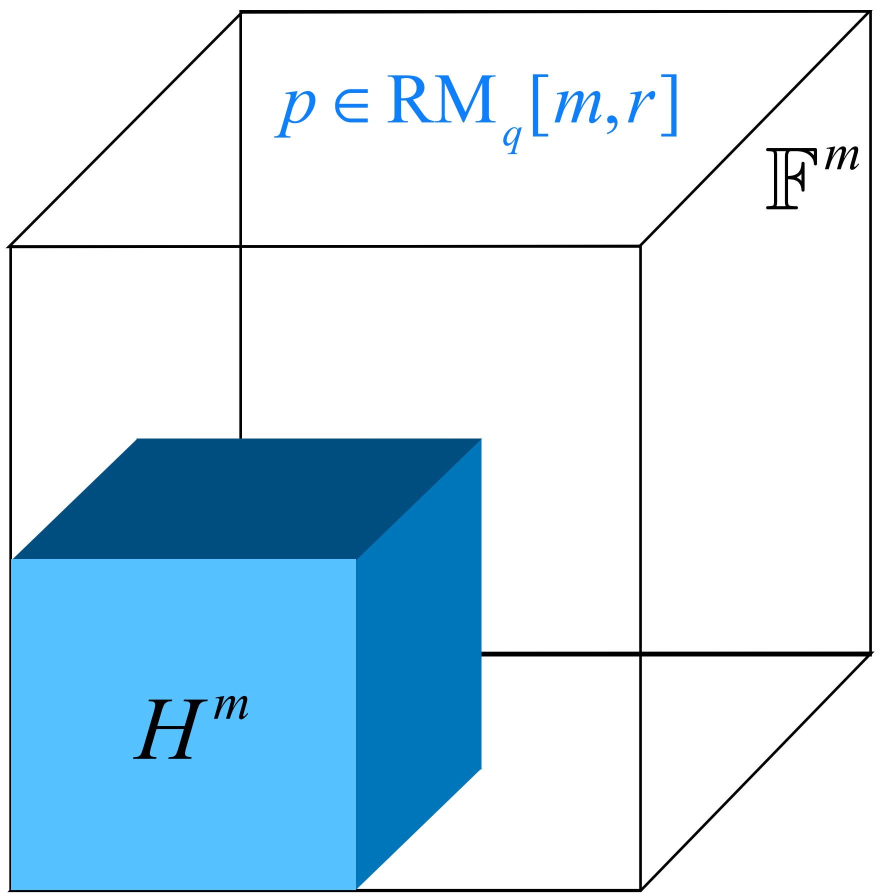

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Is $\sum_{\alpha \in H^m} p(\alpha)$ **still hard to compute?**

For r=1, H={0,1}
(multilinear extension)

$$p(2^{-1},\ldots,2^{-1}) = 2^{-k} \sum_{\alpha \in H^m} p(a)$$

NO!

[JKRS09]

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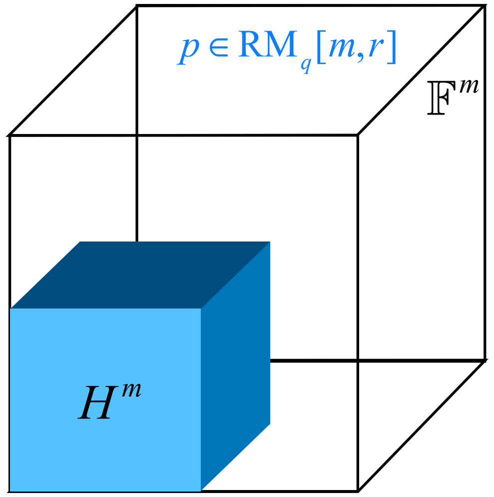

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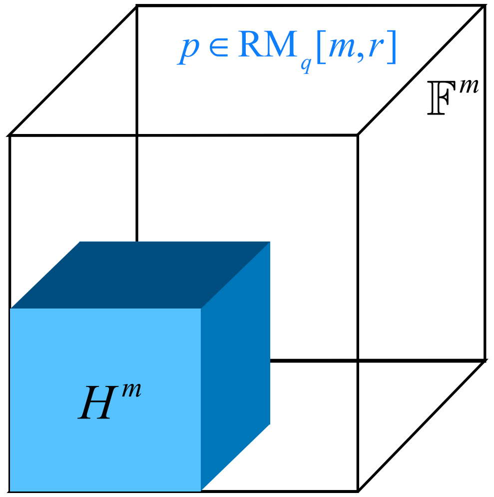

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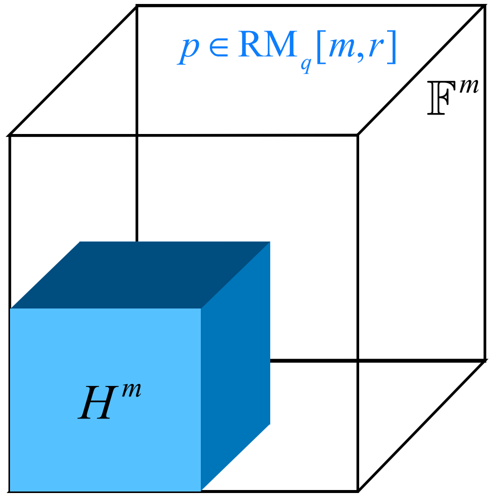

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Approach: Reduction from **communication complexity**

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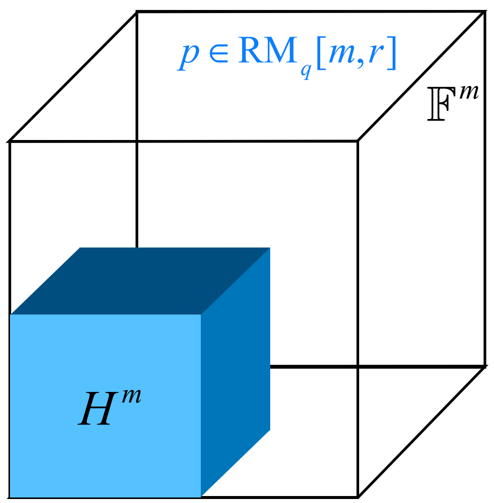

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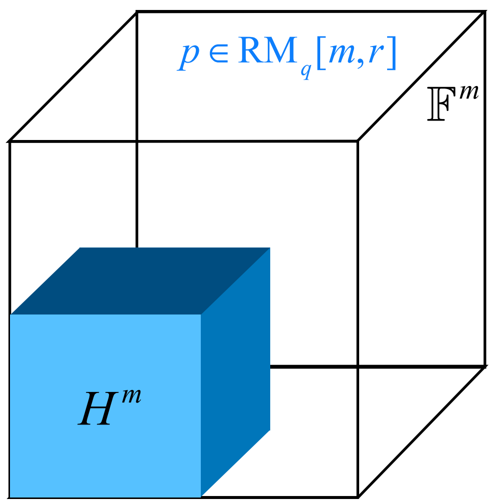

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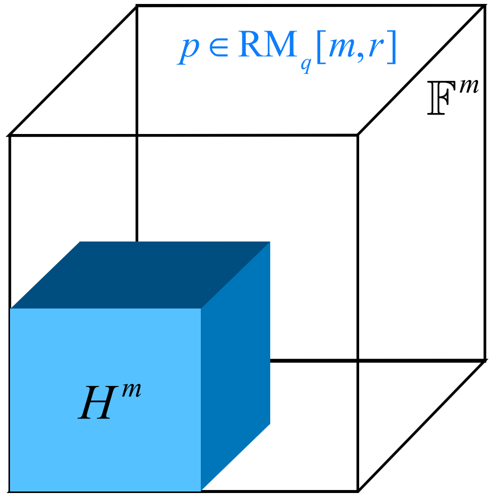

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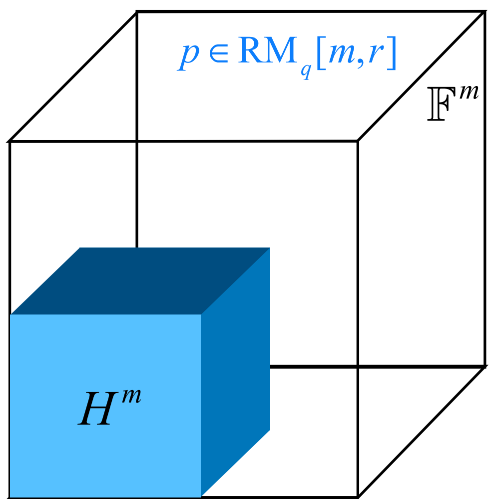

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Approach: Reduction from **communication complexity**

$x \in \{0,1\}^n$ → $y \in \{0,1\}^n$

$\Omega(n)$ **communication required to decide unique-disjointness:** $\exists! \quad x_i = y_i = 1$

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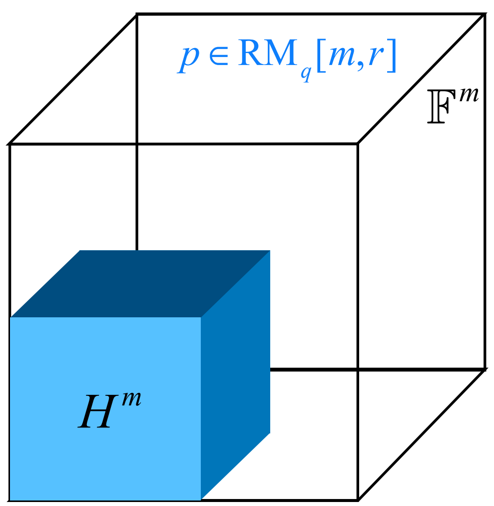

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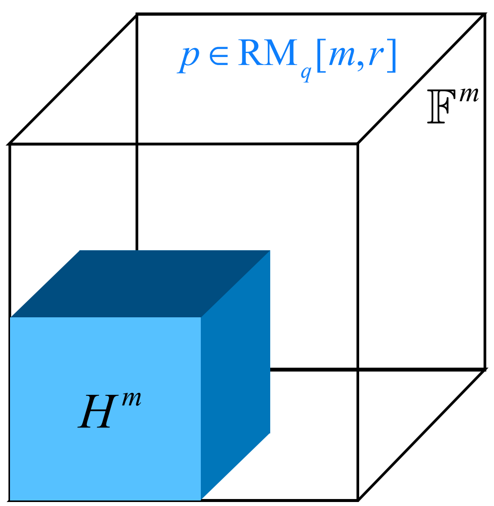

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Construct a protocol for unique disjointness!

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The protocol

Towards contradiction:

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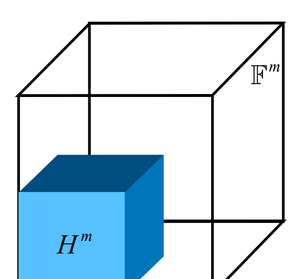

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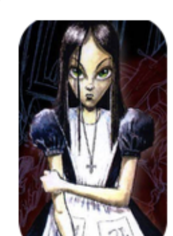

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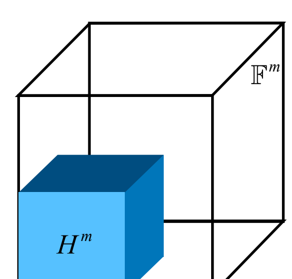

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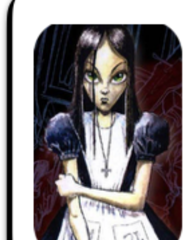

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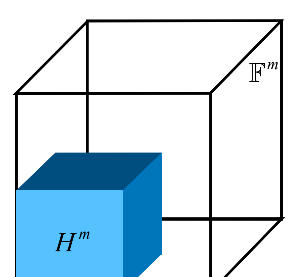

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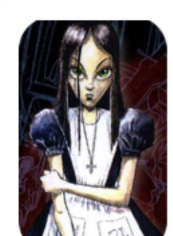

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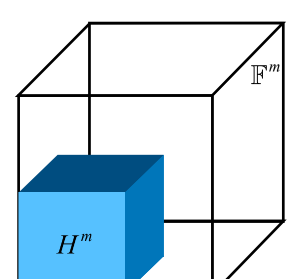

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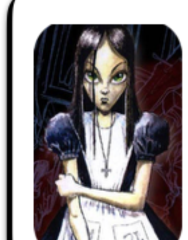

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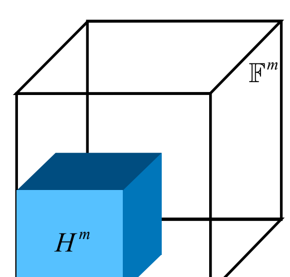

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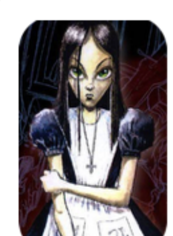

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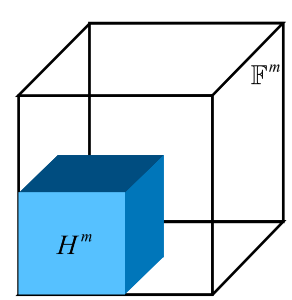

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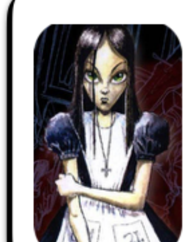

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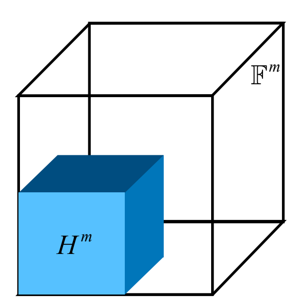

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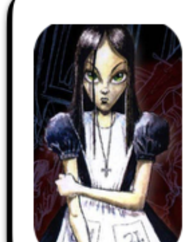

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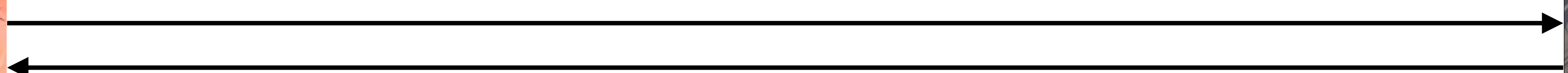

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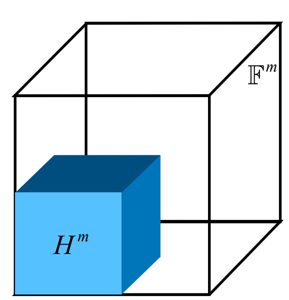

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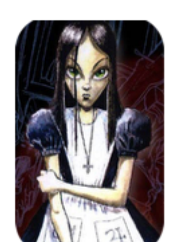

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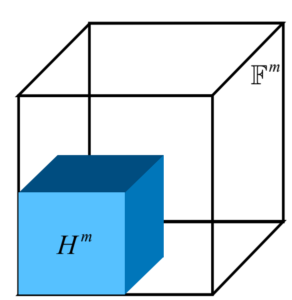

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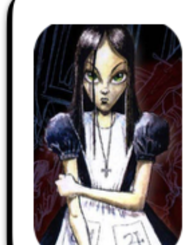

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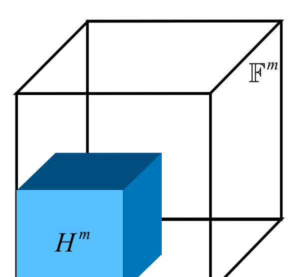

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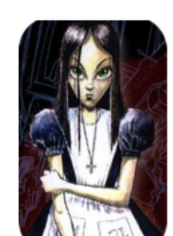

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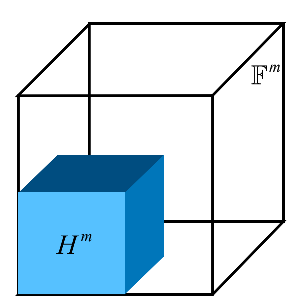

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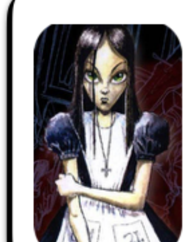

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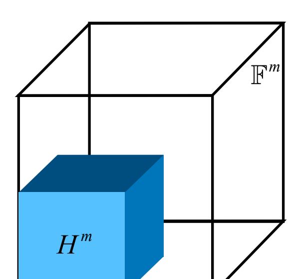

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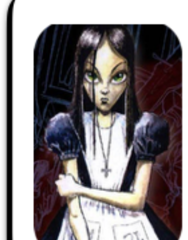

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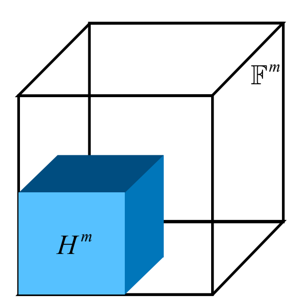

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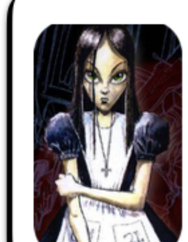

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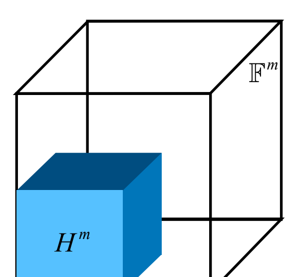

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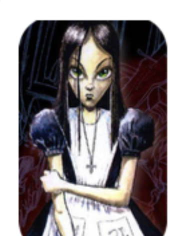

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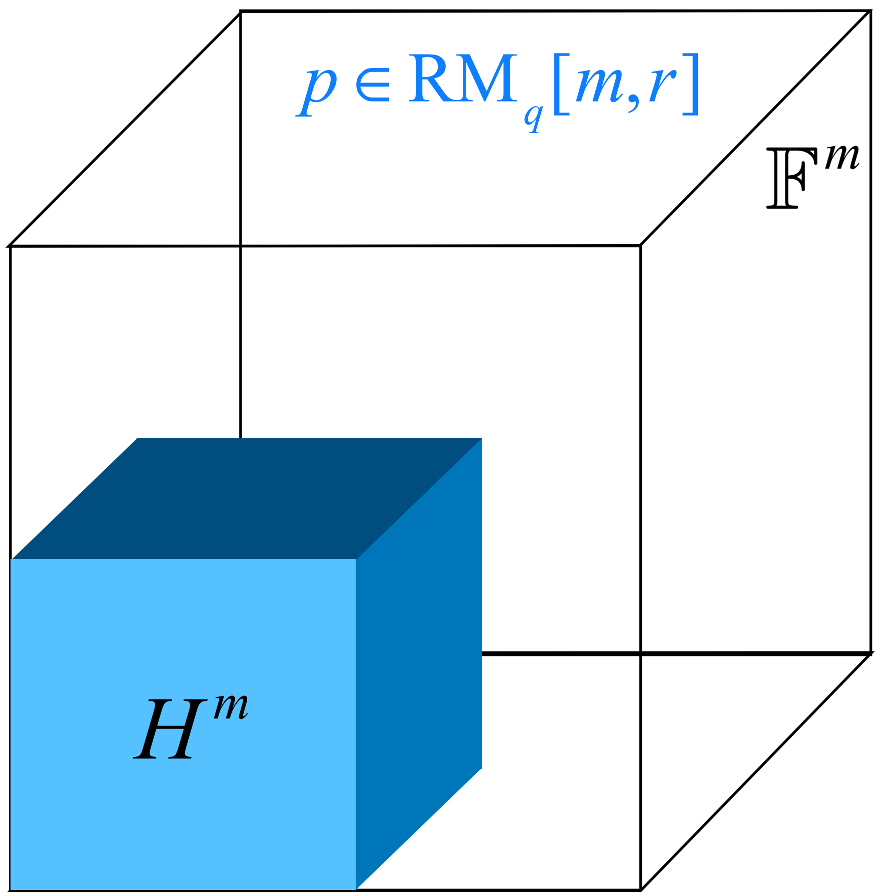

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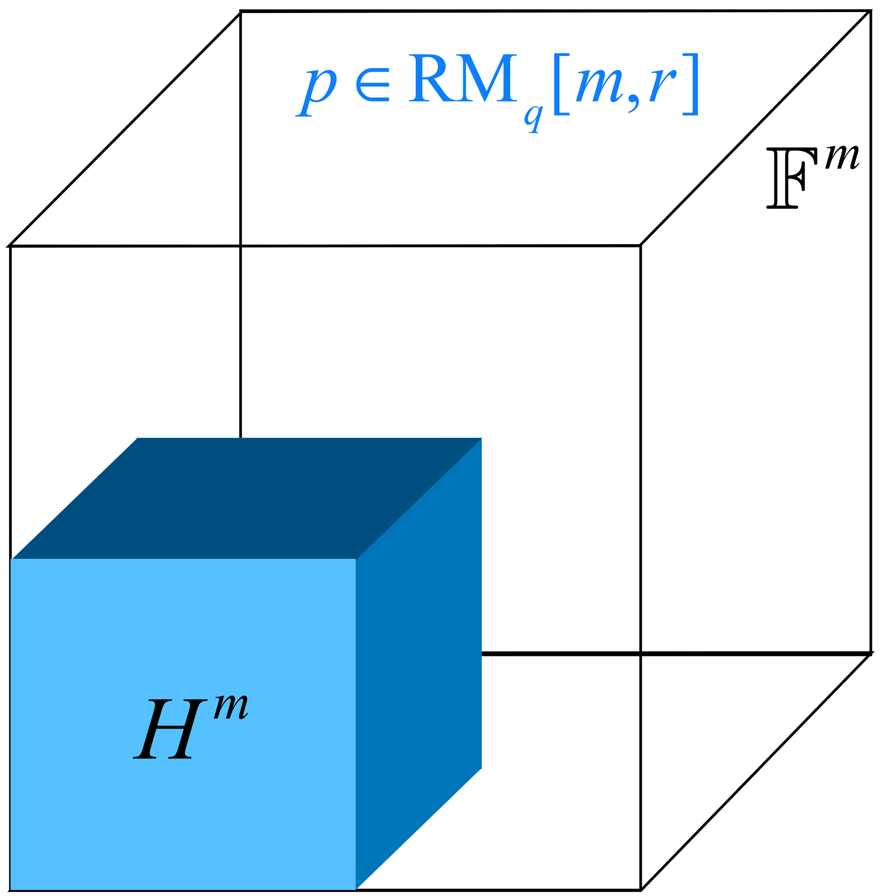

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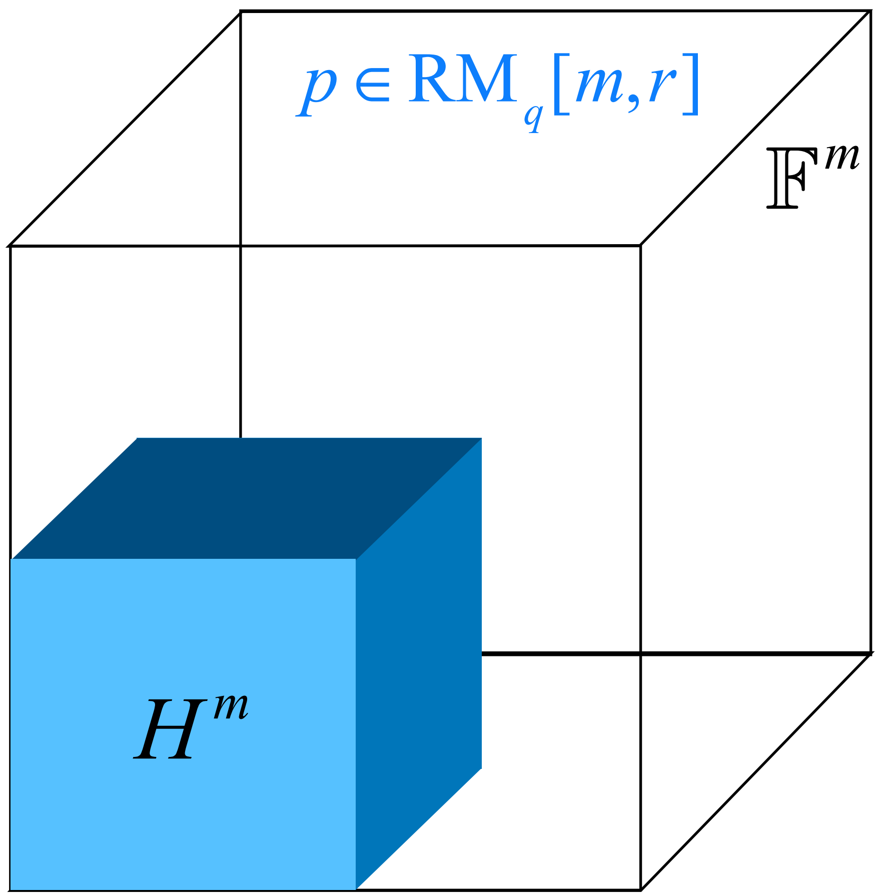

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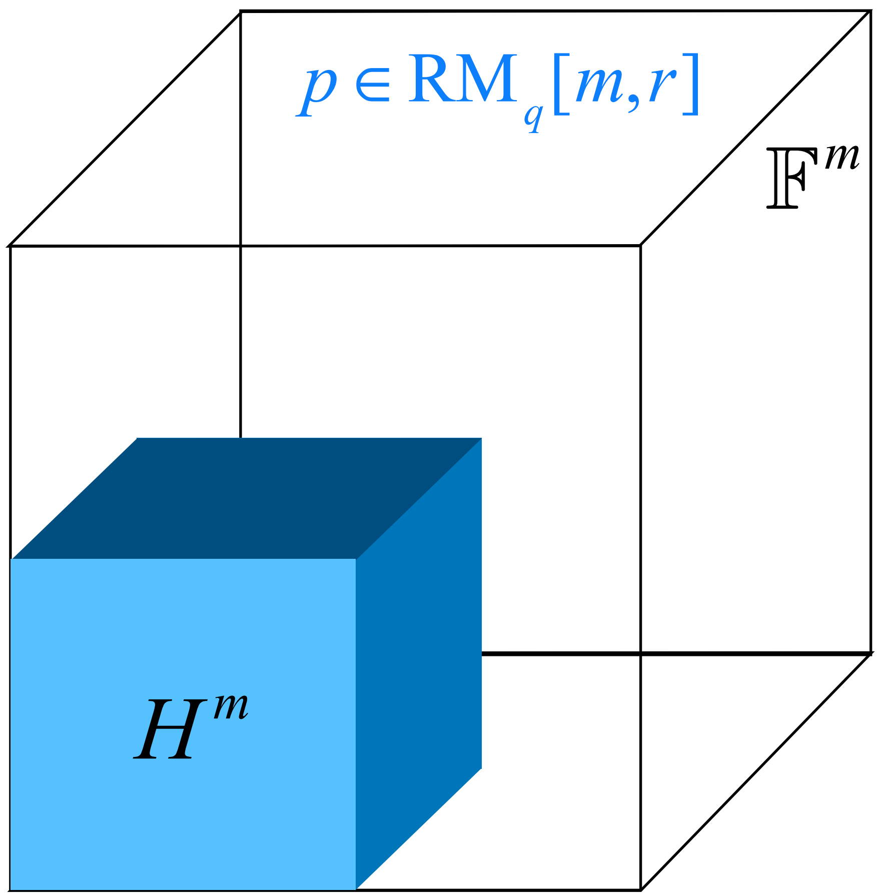

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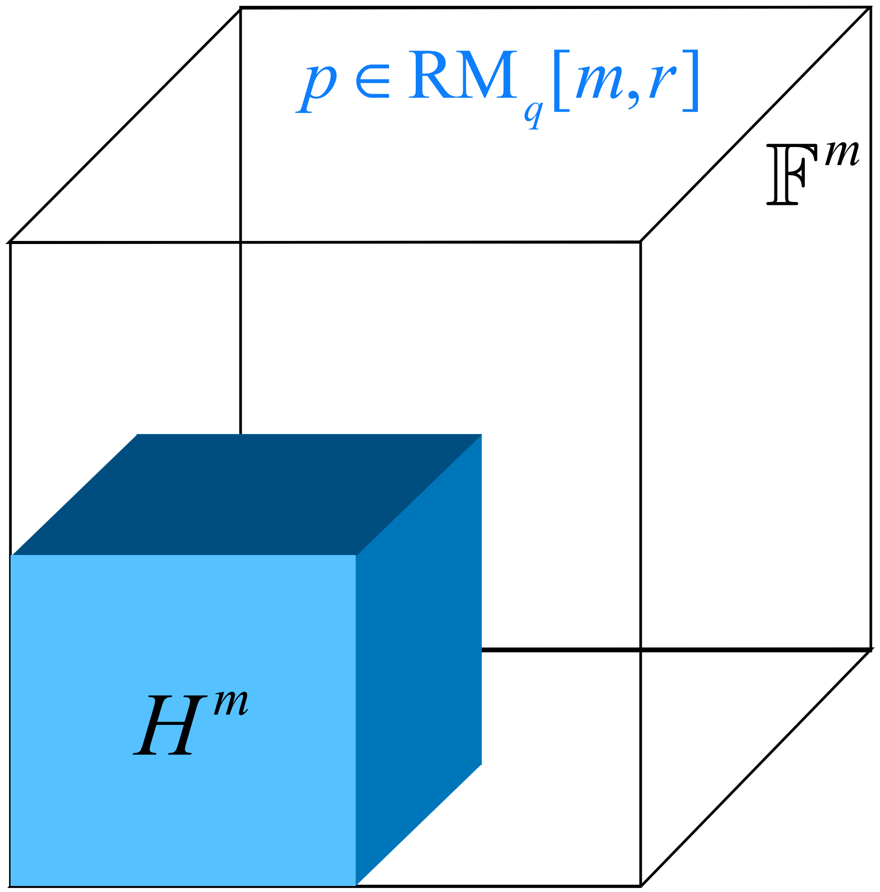

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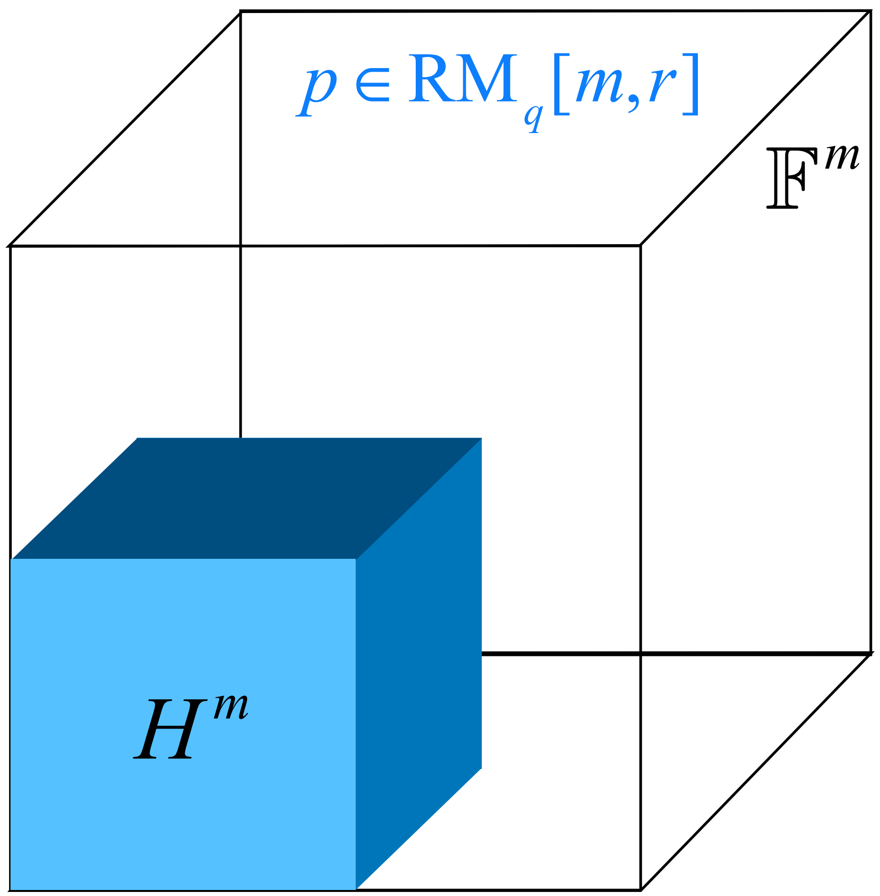

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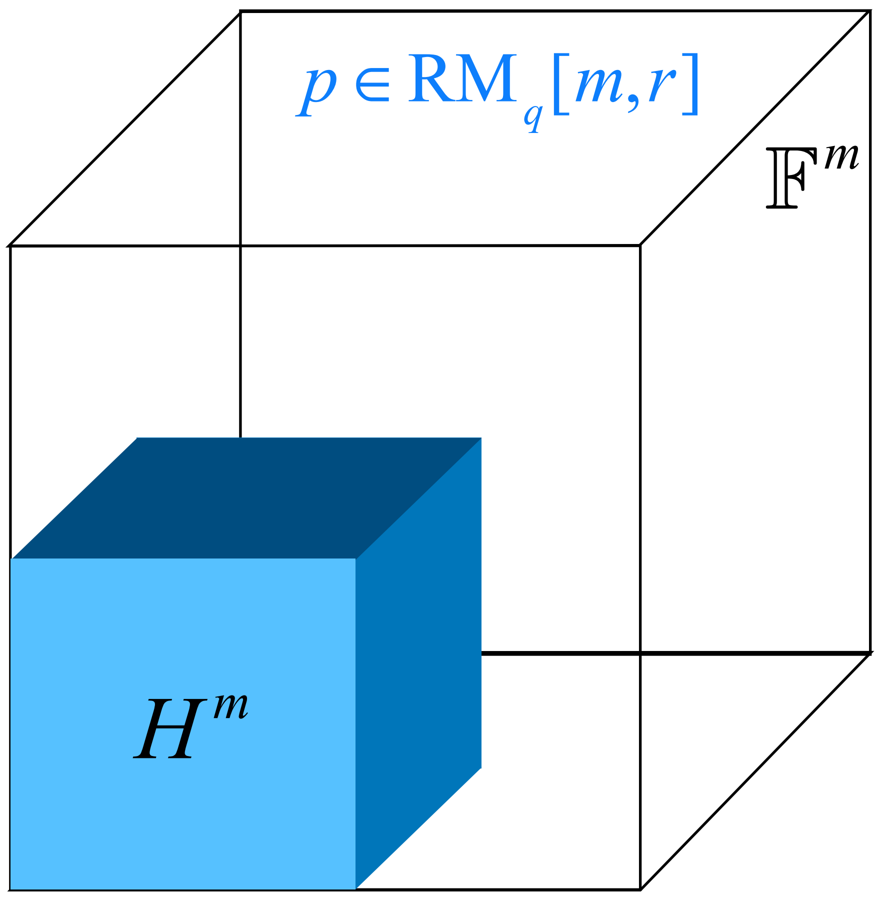

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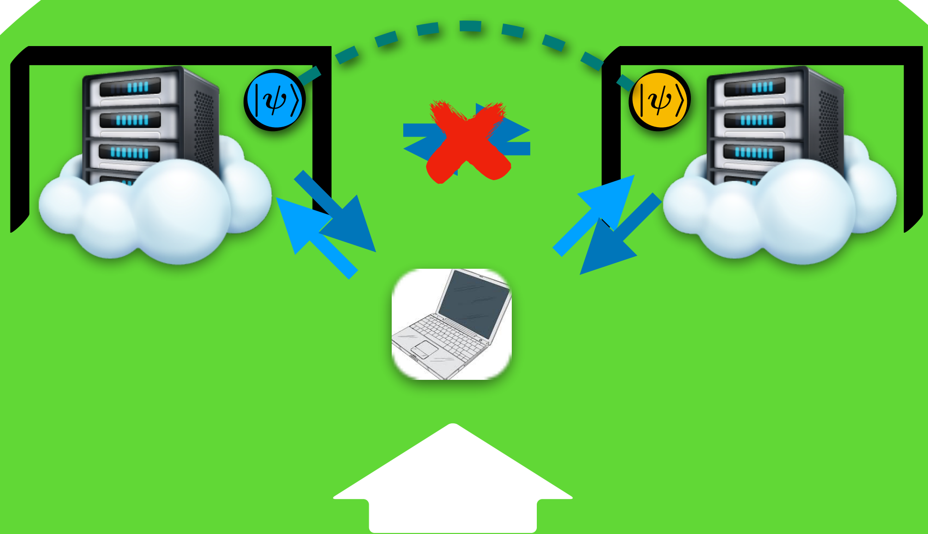

Lifting a richer class of protocols

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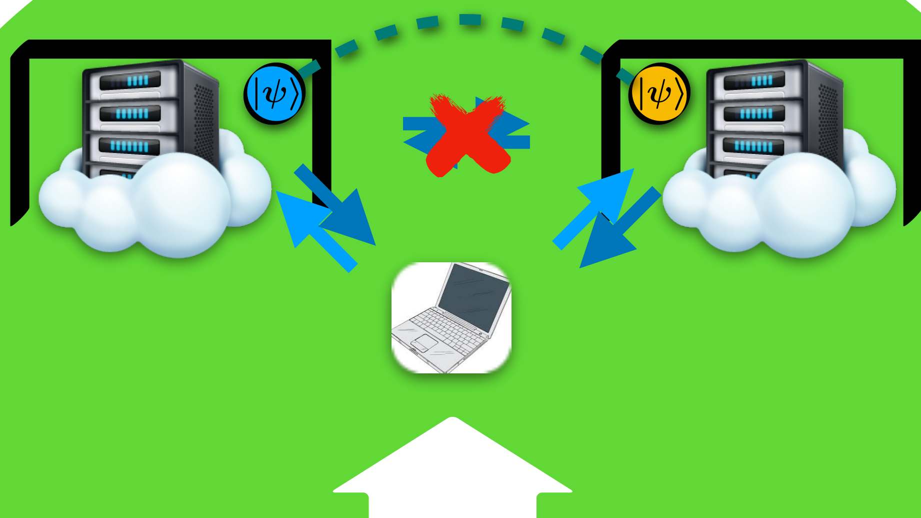

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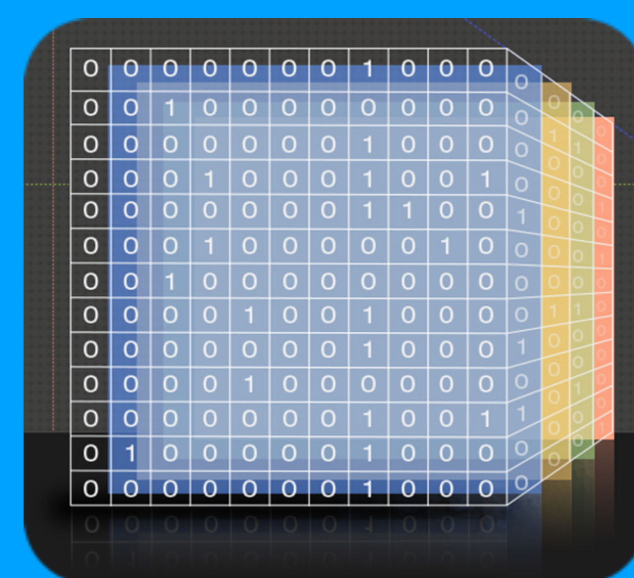

Entanglement-resistant
Tensor code testing

Thank you!