Classical Verification of Quantum Computations

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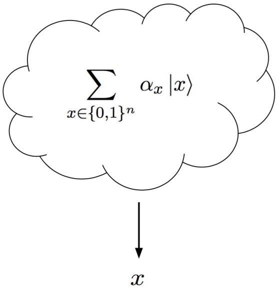
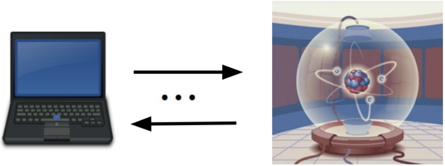
Classical versus Quantum Computers



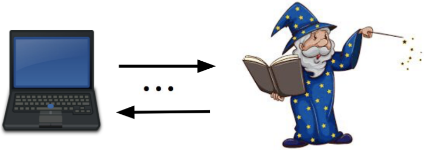
- Can a classical computer verify a quantum computation?
  - Classical output (decision problem)
- Quantum computers compute in superposition
  - Classical description is exponentially large!
- Classical access is limited to measurement outcomes
  - Only n bits of information

Verification through Interactive Proofs

Can a classical computer verify the result of a quantum computation through interaction (Gottesman, 2004)?

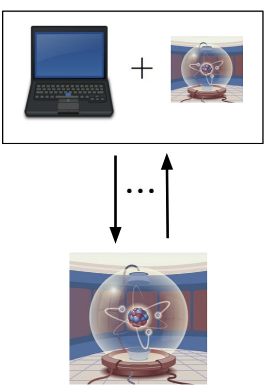


Verification through Interactive Proofs

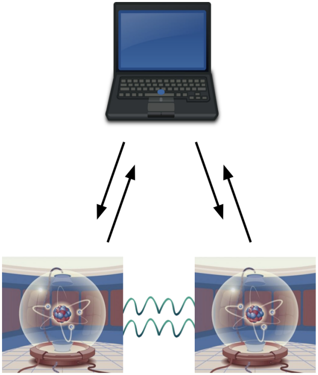


- Classical complexity theory: IP = PSPACE [Shamir92]
- BQP $\subseteq$ PSPACE: Quantum computations can be verified, but only through interaction with a much more powerful prover
- Scaled down to an efficient quantum prover?

Relaxations

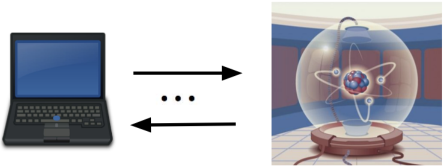


Error correcting codes
[BFK08][ABE08][FK17][ABEM17]



Bell inequalities
[RUV12]

Verification with Post Quantum Cryptography



- In this talk: use post quantum classical cryptography to control the BQP prover
- To do this, require a specific primitive: trapdoor claw-free functions

Core Primitive

- Trapdoor claw-free functions f:
  - Two to one
  - Trapdoor allows for efficient inversion: given y, can output x_0, x_1 such that $f(x_0) = f(x_1) = y$
  - Hard to find a claw (x_0, x_1): $f(x_0) = f(x_1)$
  - Approximate version built from learning with errors in [BCMVV18]

- Quantum advantage: sample y and create a superposition over a random claw

$$\frac{1}{\sqrt{2}}(|x_0\rangle + |x_1\rangle)$$

which allows sampling of a string $d \neq 0$ such that

$$d \cdot (x_0 \oplus x_1) = 0$$

Core Primitive

$$\frac{1}{\sqrt{2}}(|x_0\rangle + |x_1\rangle) \quad \text{or} \quad d \cdot (x_0 \oplus x_1) = 0$$

- Classical verifier can challenge quantum prover
  - Verifier selects f and asks for y
  - Verifier has leverage through the trapdoor: can compute x_0, x_1
- First challenge: ask for preimage of y
- Second challenge: ask for d

Core Primitive

$$\frac{1}{\sqrt{2}}(|x_0\rangle + |x_1\rangle) \quad \text{or} \quad d \cdot (x_0 \oplus x_1) = 0$$

- In [BCMVV18], used to generate randomness:
  - Hardcore bit: hard to hold both d and either x_0, x_1 at the same time
  - Prover must be probabilistic to pass

Core Primitive

$$\frac{1}{\sqrt{2}}(|x_0\rangle + |x_1\rangle) \quad \text{or} \quad d \cdot (x_0 \oplus x_1) = 0$$

- Verification:
  - TCFs are used to constrain prover
  - Use extension of approximate TCF family built in [BCMVV18]
    - Require [BCMVV18] hardcore bit property: hard to hold both d and either (x_0, x_1)
    - Require one more hardcore bit property: there exists d such that for all claws (x_0, x_1), $d \cdot (x_0 \oplus x_1)$ is the same bit and is hard to compute

How to Create a Superposition Over a Claw

$$\frac{1}{\sqrt{2}}(|x_0\rangle + |x_1\rangle)$$

1. Begin with a uniform superposition over the domain:

$$\frac{1}{\sqrt{|\mathcal{X}|}}\sum_{x\in\mathcal{X}} |x\rangle$$

2. Apply the function f in superposition:

$$\frac{1}{\sqrt{|\mathcal{X}|}}\sum_{x\in\mathcal{X}} |x\rangle\,|f(x)\rangle$$

3. Measure the last register to obtain y

Core Primitive

$$\frac{1}{\sqrt{2}}(|x_0\rangle + |x_1\rangle)$$

- Performing a Hadamard transform on the above state results in:

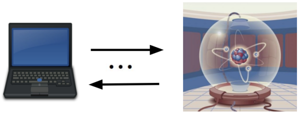
$$\frac{1}{\sqrt{|\mathcal{X}|}}\sum_d((-1)^{d\cdot x_0} + (-1)^{d\cdot x_1})\,|d\rangle$$

- By measuring, obtain a string d such that

$$d\cdot(x_0 \oplus x_1) = 0$$

Verification Outline

Goal: classical verification of quantum computations through interaction



- Define a *measurement protocol*
  - The prover constructs an *n* qubit state ρ of his choice
  - The verifier chooses 1 of 2 measurement bases for each qubit
  - The prover reports the measurement result of ρ in the chosen basis
- Link measurement protocol to verifiability
- Construct and describe soundness of the measurement protocol

Hadamard and Standard Basis Measurements

$$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$$

- Standard: obtain b with probability $|\alpha_b|^2$
- Hadamard:

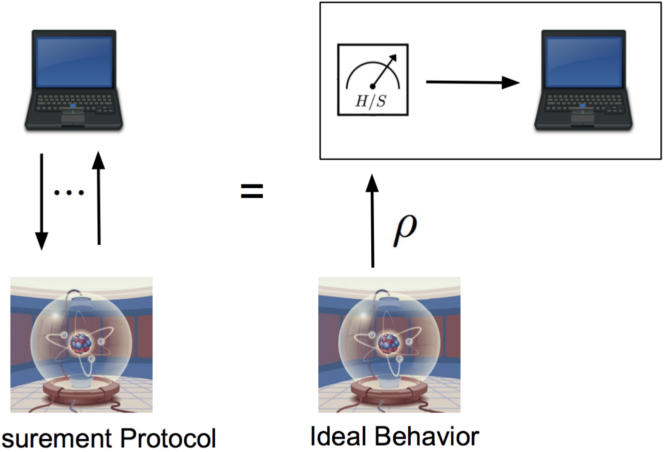
$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H|\psi\rangle = \frac{1}{\sqrt{2}}(\alpha_0 + \alpha_1)|0\rangle + \frac{1}{\sqrt{2}}(\alpha_0 - \alpha_1)|1\rangle$$

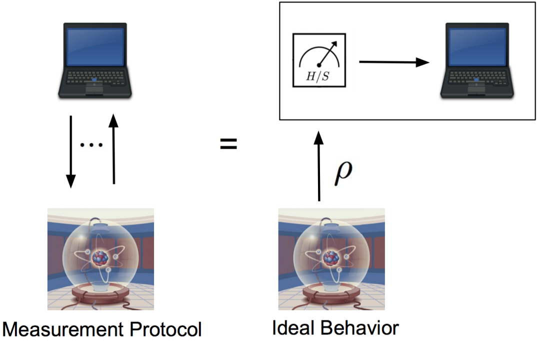
Obtain b with probability $\frac{1}{2}\left|\alpha_0 + (-1)^b \alpha_1\right|^2$

Measurement Protocol Definition

Measurement protocol: interactive protocol which forces the prover to behave as the verifier's trusted measurement device

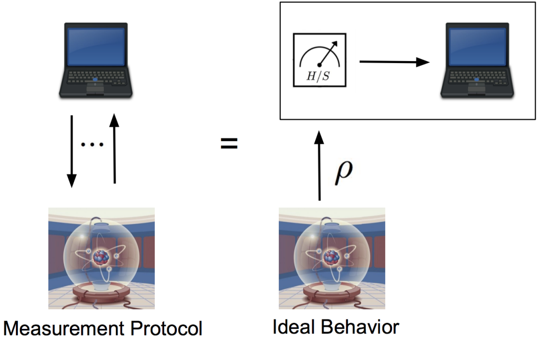


Measurement Protocol Definition



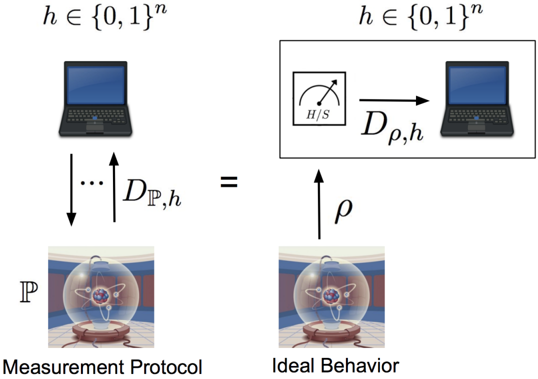
- Key issue: adaptivity; what if ρ changes based on measurement basis?
  - Maybe the prover never constructs a quantum state, and constructs classical distributions instead

Measurement Protocol Soundness



- Soundness: if the verifier accepts, there exists a quantum state *independent of the verifier's measurement choice* underlying the measurement results

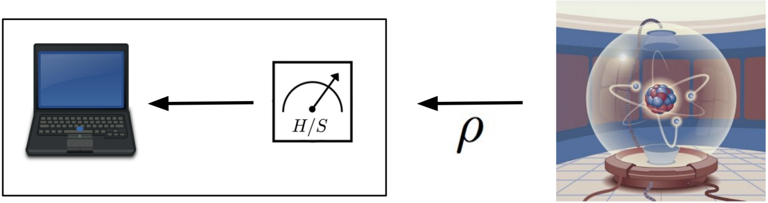
Measurement Protocol Soundness



- Soundness: if $\mathbb{P}$ is accepted with high probability, there exists a state ρ such that for all h, $D_{\rho,h}$ and $D_{\mathbb{P},h}$ are computationally indistinguishable.

Using the Measurement Protocol for Verification

- The measurement protocol implements the following model:



- Prover sends qubits of state ρ and verifier measures

- Next: show that quantum computations can be verified in the above model

Quantum Analogue of NP

- To verify an efficient classical computation, reduce to a 3-SAT instance, ask for satisfying assignment and verify that it is satisfied

$$\begin{aligned} \text{3-SAT} &\iff \text{Local Hamiltonian} \\ n \text{ bit variable assignment } x &\iff n \text{ qubit quantum state} \\ \text{Number of unsatisfied clauses} &\iff \text{Energy} \end{aligned}$$

- To verify an efficient quantum computation, reduce to a local Hamiltonian instance H, ask for ground state and verify that it has low energy
  - If the instance is in the language, there exists a state with low energy

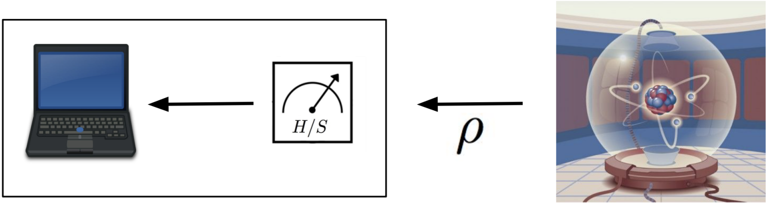
Quantum Analogue of NP

$$\begin{aligned} \text{3 SAT} &\iff \text{Local Hamiltonian} \\ \text{Assignment} &\iff \text{Quantum state} \\ \text{Number of unsatisfied clauses} &\iff \text{Energy} \end{aligned}$$

To verify that a state has low energy with respect to $H = \sum_i H_i$:

- Each H_i acts on at most 2 qubits
- To measure with respect to H_i, only Hadamard/ standard basis measurements are required [BL08]

Verification with a Quantum Verifier



- Prover sends each qubit of ρ to the quantum verifier
- The quantum verifier chooses H_i at random and measures, using only Hadamard/ standard basis measurements [MF2016]
- Measurement protocol can be used in place of the measurement device to achieve verifiability

Measurement Protocol Construction

- Use a TCF with more structure: pair f_0, f_1 which are injective with the same image
- Given f_0, f_1, the honest quantum prover entangles a single qubit of his choice with a claw (x_0, x_1) $(y = f_0(x_0) = f_1(x_1))$.

$$|\psi\rangle \quad \rightarrow \quad \sum_{b \in \{0,1\}} \alpha_b \, |b\rangle \, |x_b\rangle = \mathrm{Enc}(|\psi\rangle)$$

- Once y is sent to the verifier, the verifier now has leverage over the prover's state: he knows x_0, x_1 but the prover does not

Measurement Protocol Construction

- The verifier generates a TCF f_0, f_1 and the trapdoor
- Given f_0, f_1, the honest quantum prover entangles a single qubit of his choice with a claw (x_0, x_1) $(y = f_0(x_0) = f_1(x_1))$.

$$|\psi\rangle = \sum_{b\in\{0,1\}} \alpha_b |b\rangle \quad \rightarrow \quad \sum_{x\in\mathcal{X}} \sum_{b\in\{0,1\}} \alpha_b |b\rangle |x\rangle |f_b(x)\rangle$$

$$\xrightarrow{f_b(x) = y} \sum_{b\in\{0,1\}} \alpha_b |b\rangle |x_b\rangle = \mathrm{Enc}(|\psi\rangle)$$

- Given y, the verifier uses the trapdoor to extract x_0, x_1

Measurement Protocol Testing

- Upon receiving y, the verifier chooses either to test or to delegate measurements

- If a test round is chosen, the verifier requests a preimage (b, x_b) of y

- The honest prover measures his encrypted state in the standard basis:

$$\text{Enc}(|\psi\rangle) = \sum_{b \in \{0,1\}} \alpha_b |b\rangle |x_b\rangle$$

- Point: the verifier now knows the prover's state must be in a superposition over preimages

Delegating Hadamard Basis Measurements

- Prover needs to apply a Hadamard transform:

$$\mathrm{Enc}(|\psi\rangle) = \sum_{b\in\{0,1\}} \alpha_b |b\rangle |x_b\rangle \longrightarrow H(\sum_{b\in\{0,1\}} \alpha_b |b\rangle) = H|\psi\rangle$$

- Issue: x_0, x_1 prevent interference, and prevent the application of a Hadamard transform

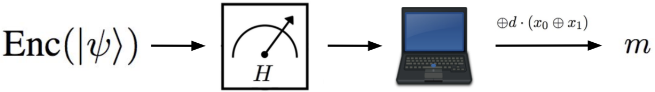
- Solution: apply the Hadamard transform to the entire encoded state, and measure the second register to obtain d

Delegating Hadamard Basis Measurements

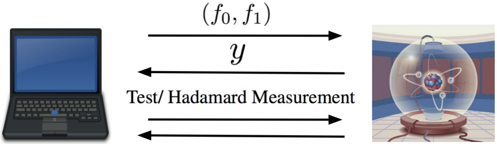
- This results in a different encoding (X is the bit flip operator):

$$\mathrm{Enc}(|\psi\rangle) \xrightarrow{H} X^{d\cdot(x_0 \oplus x_1)} H |\psi\rangle$$

- Verifier decodes measurement result b by XORing $d \cdot (x_0 \oplus x_1)$
- Protocol with honest prover:



Measurement Protocol So Far



- Soundness: there exists a quantum state *independent of the verifier's measurement choice* underlying the measurement results

- Necessary condition: messages required to delegate standard basis must be computationally indistinguishable

- To delegate standard basis measurements: only need to change the first message

Delegating Standard Basis Measurements

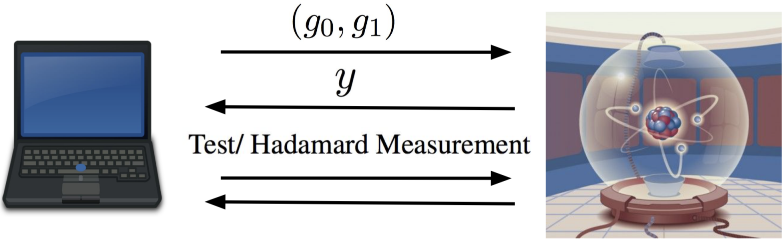
- Let g_0, g_1 be trapdoor injective functions: the images of g_0, g_1 do not overlap
  - The functions (f_0, f_1) and (g_0, g_1) are computationally indistinguishable
- If prover encodes with g_0, g_1 rather than f_0, f_1, this acts as a standard basis measurement:

$$\sum_{b\in\{0,1\}} \alpha_b |b\rangle \rightarrow \sum_{b\in\{0,1\},x} \alpha_b |b\rangle |x\rangle |g_b(x)\rangle$$

- With use of trapdoor, standard basis measurement b can be obtained from $y = g_b(x)$

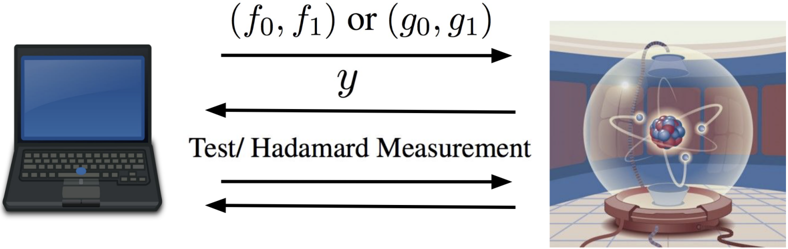
Delegating Standard Basis Measurements

- Protocol is almost the same, except f_0, f_1 is replaced with g_0, g_1



- Verifier ignores Hadamard measurement results; only uses y to recover standard basis measurement

Measurement Protocol Recap

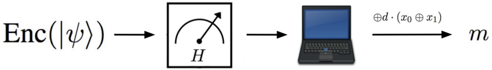


- Goal: use the prover as a blind, verifiable measurement device
- Verifier selects basis choice; sends claw free function for Hadamard basis and injective functions for standard basis
- Verifier either tests the structure of the state or requests measurement results

Soundness Intuition: Example of Cheating Prover

- Recall adaptive cheating strategy: prover fixes two bits, b_H and b_S, which he would like the verifier to stores as his Hadamard/ standard basis measurement results

- Assume there is a claw (x_0, x_1) and a string d for which the prover knows both x_{b_S} and $d \cdot (x_0 \oplus x_1)$



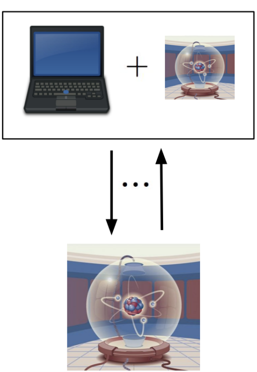
- How to cheat:
  - To compute y: prover evaluates received function on x_{b_S} ($y = g_{b_S}(x_{b_S})$ or $y = f_{b_S}(x_{b_S})$).
  - When asked for a Hadamard measurement: prover reports d and $b_H \oplus d \cdot (x_0 \oplus x_1)$

Hardcore Bit Properties

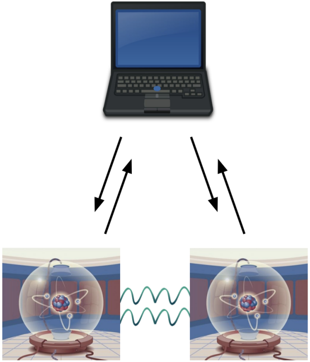
Soundness rests on two hardcore bit property of TCFs:

1. For all $d \neq 0$ and all claws (x_0, x_1), it is computationally difficult to compute both $d \cdot (x_0 \oplus x_1)$ and either x_0 or x_1.
2. There exists a string d such that for all claws (x_0, x_1), the bit $d \cdot (x_0 \oplus x_1)$ is the same and computationally indistinguishable from uniform.

How to Prove Soundness



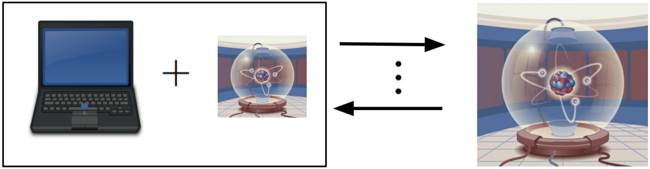
[BFK08][ABE08][FK17][ABEM17]



[RUV12]

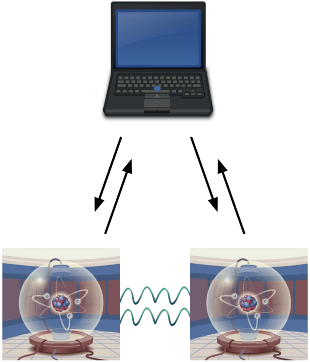
Key step: enforcing structure in prover's state

How to Prove Soundness: Quasi Classical Verifier



Verifier sends qubits encoded with secret error correcting code to the prover.

How to Prove Soundness: Two Provers

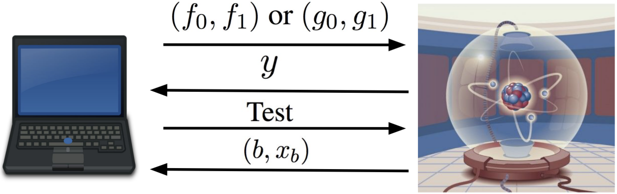


Verifier plays CHSH with the provers and checks for a Bell inequality violation. If prover passes, he must be holding Bell pairs.

How to Prove Soundness: Measurement Protocol

Enforcing structure?

- No way of using previous techniques
- Use test round of measurement protocol as starting point



At some point in time, prover's state must be of the form:

$$\sum_{b\in\{0,1\}} \alpha_b \left|b\right\rangle \left|x_b\right\rangle \left|\psi_{b,x_b}\right\rangle \quad \text{or} \quad \left|b\right\rangle \left|x_b\right\rangle \left|\psi_{b,x_b}\right\rangle$$

How to Prove Soundness: Measurement Protocol

Why is this format useful in proving the existence of an underlying quantum state?

$$\sum_{b\in\{0,1\}} \alpha_b \left|b\right\rangle \left|x_b\right\rangle \left|\psi_{b,x_b}\right\rangle \quad \text{or} \quad \left|b\right\rangle \left|x_b\right\rangle \left|\psi_{b,x_b}\right\rangle$$

- Can be used as starting point for prover, followed by deviation from the protocol, measurement and decoding by the verifier
  - Deviation is an arbitrary unitary operator U
  - Verifier's decoding is $d \cdot (x_0 \oplus x_1)$
- The part of the unitary U acting on the first qubit is therefore *computationally randomized*, by both the initial state and the verifier's decoding
  - Pauli twirl technique?

How to Prove Soundness: Measurement Protocol

Why is this format useful in proving the existence of an underlying quantum state?

$$\sum_{b\in\{0,1\}} \alpha_b \left|b\right\rangle \left|x_b\right\rangle \left|\psi_{b,x_b}\right\rangle \quad \text{or} \quad \left|b\right\rangle \left|x_b\right\rangle \left|\psi_{b,x_b}\right\rangle$$

- Difficulty in using Pauli twirl: converting this computational randomness into a form which can be used to simplify the prover's deviation
  - Rely on hardcore bit properties regarding $d \cdot (x_0 \oplus x_1)$

Conclusion

- Verifiable, secure delegation of quantum computations is possible with a classical machine
- Rely on quantum secure trapdoor claw-free functions (from learning with errors)
  - Use TCF to characterize the intial space of the prover
  - Strengthen the claw-free property to complete the characterization and prove the existence of a quantum state

Thanks!