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Recent Algorithmic Primitives
Linear Combination of Unitaries and Quantum Signal Processing

Robin Kothari
Microsoft Research

Challenges in Quantum Computation, Simons Institute
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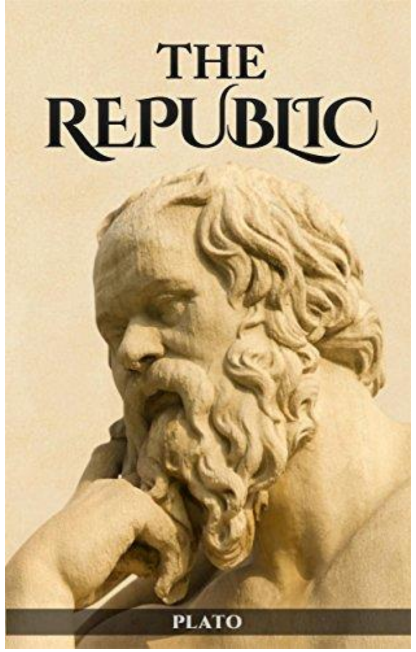
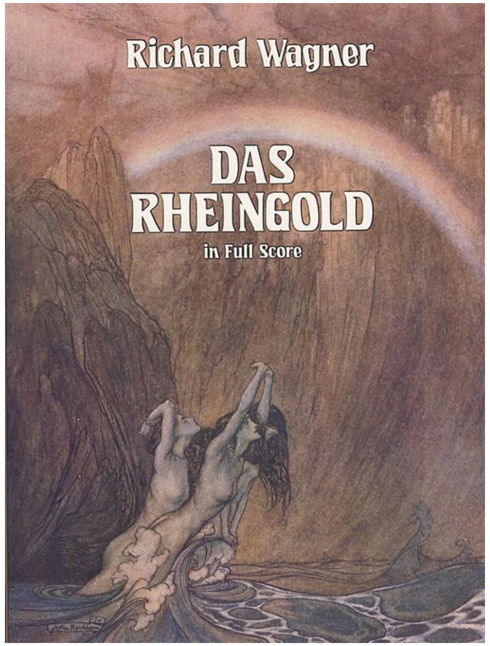
This talk: Focus on algorithmic techniques





vs.

I'll talk about algorithmic primitives of the form:

"We have available an easy-to-implement unitary V, but we want to implement a related unitary U".

Goal of this talk:
Show you some interesting techniques that you might find useful in your research.

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Oblivious Amplitude Amplification (OAA)

Probabilistic implementations

Let V be a unitary such that

$$\forall|\psi\rangle, \qquad V|0^m\rangle|\psi\rangle = \sqrt{p}|0^m\rangle U|\psi\rangle + \sqrt{1-p}|\perp\rangle,$$

where $(|0^m\rangle\langle 0^m| \otimes I)|\perp\rangle = 0.$

Goal: Given a circuit for V, apply U on an arbitrary state $|\psi\rangle$.

$$V = \begin{pmatrix} \sqrt{p}U & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

Terminology: V is "probabilistic implementation" of U with probability p, or V "block-encodes" the operator $\sqrt{p}U$.

Classical repetition

Let V be a unitary such that

$$\forall|\psi\rangle, \qquad V|0^m\rangle|\psi\rangle = \sqrt{p}|0^m\rangle U|\psi\rangle + \sqrt{1-p}|\perp\rangle,$$

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Goal: Given a circuit for V, apply U on an arbitrary state $|\psi\rangle$.

$$V = \begin{pmatrix} \sqrt{p}U & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

Solution 1 (classical repetition)

- Apply V to $|\psi\rangle$, and measure the first m qubits.
- If we observe $|0^m\rangle$, we're done. Otherwise repeat.

Cost:

- $O(1/p)$ uses of V
- $O(1/p)$ copies of $|\psi\rangle$ ← We may not have multiple copies of $|\psi\rangle$

Amplitude amplification

Let V be a unitary such that

$$\forall|\psi\rangle, \qquad V|0^m\rangle|\psi\rangle = \sqrt{p}|0^m\rangle U|\psi\rangle + \sqrt{1-p}|\perp\rangle,$$

where $(|0^m\rangle\langle 0^m| \otimes I)|\perp\rangle = 0$.

Goal: Given a circuit for V, apply U on an arbitrary state $|\psi\rangle$.

$$V = \begin{pmatrix} \sqrt{p}U & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

Solution 2 (amplitude amplification)

- Repeat $O(1/\sqrt{p})$ times:
  Apply V. Reflect about $|0^m\rangle$. Apply $V^\dagger$. Reflect about $|0^m\rangle|\psi\rangle$.

Cost:

- $O(1/\sqrt{p})$ uses of V and $V^\dagger$
- $O(1/\sqrt{p})$ uses of the reflection about $|\psi\rangle$ ← We may not be able to do this.

Oblivious amplitude amplification

Let V be a unitary such that

$$\forall|\psi\rangle, \qquad V|0^m\rangle|\psi\rangle = \sqrt{p}|0^m\rangle U|\psi\rangle + \sqrt{1-p}|\perp\rangle,$$

where $(|0^m\rangle\langle 0^m| \otimes I)|\perp\rangle = 0$.

Goal: Given a circuit for V, apply U on an arbitrary state $|\psi\rangle$.

$$V = \begin{pmatrix} \sqrt{p}U & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

~~Solution 2 (amplitude amplification)~~ Oblivious amplitude amplification

- Repeat $O(1/\sqrt{p})$ times:
  Apply V. Reflect about $|0^m\rangle$. Apply $V^\dagger$. Reflect about $|0^m\rangle$~~$|\psi\rangle$~~.

Cost:

- $O(1/\sqrt{p})$ uses of V and $V^\dagger$
- ~~$O(1/\sqrt{p})$ uses of the reflection about $|\psi\rangle$ ← We may not be able to do this.~~

Oblivious amplitude amplification

Let V be a unitary such that

$$\forall|\psi\rangle, \qquad V|0^m\rangle|\psi\rangle = \sqrt{p}|0^m\rangle U|\psi\rangle + \sqrt{1-p}|\perp\rangle,$$

where $(|0^m\rangle\langle 0^m| \otimes I)|\perp\rangle = 0.$

Goal: Given a circuit for V, apply U on an arbitrary state $|\psi\rangle$.

$$V = \begin{pmatrix} \sqrt{p}U & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

Oblivious amplitude amplification

- Repeat $O(1/\sqrt{p})$ times:
  Apply V. Reflect about $|0^m\rangle$. Apply $V^\dagger$. Reflect about $|0^m\rangle$.

Cost:

- $O(1/\sqrt{p})$ uses of V and $V^\dagger$

Note: It's very important that U is (close to) unitary for OAA to work!

Oblivious amplitude amplification (OAA)

$$V = \begin{pmatrix} \sqrt{p}U & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

Oblivious amplitude amplification take-home message

A "probabilistic implementation" of U can be converted to an actual implementation of U.

If U is not unitary, use regular amplitude amplification.

Filter by title
AmpAmpByReflectionsPhases
AmpAmpObliviousByOraclePhases
AmpAmpObliviousByReflectionPhases
AmpAmpPhasesFixedPoint
AmpAmpPhasesStandard
AmpAmpRUSByOracle
AmpAmpReflectionPhases
AmpAmpRotationPhases
AmpAmpRotationToReflectionPhases
AndLadder
ApplyLEOperationOnPhaseLE
ApplyLEOperationOnPhaseLEA
ApplyLEOperationOnPhaseLEC
ApplyLEOperationOnPhaseLECA
ApplyMultiControlledC
ApplyMultiControlledCA
ApplyPauli

AmpAmpObliviousByOraclePhases function

Namespace: Microsoft.Quantum.Canon

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In this article
Input
Output
Remarks

Oblivious amplitude amplification by oracles for partial reflections.

Q# Copy

```
function AmpAmpObliviousByOraclePhases (phases : AmpAmpReflectionPhases, ancillaOracle :
DeterministicStateOracle, signalOracle : ObliviousOracle, idxFlagQubit : Int) : ((Qubit[],
Qubit[]) => () : Adjoint, Controlled)
```

Input

`phases` AmpAmpReflectionPhases

Phases of partial reflections

`ancillaOracle` DeterministicStateOracle

Unitary oracle A that prepares ancilla start state

`signalOracle` ObliviousOracle

Unitary oracle O of type `ObliviousOracle` that acts jointly on the ancilla and system register

`idxFlagQubit` Int

Index to single-qubit flag register

Output

An operation that implements oblivious amplitude amplification based on partial reflections.

Linear Combination of Unitaries (LCU)

A linear combination of unitaries

Let B be a linear combination of easy-to-implement unitaries:

$$B = \sum_i \alpha_i W_i \,.$$

Goal: Implement B given the ability to implement $\mathrm{select}W = \sum_i |i\rangle\langle i| \otimes W_i$.

For example, let $B = W_0 + W_1$.

$$\begin{aligned}\mathrm{select}W|+\rangle|\psi\rangle &= \frac{1}{\sqrt{2}}(|0\rangle W_0|\psi\rangle + |1\rangle W_1|\psi\rangle) \\ &= \frac{1}{2}(|+\rangle(W_0 + W_1)|\psi\rangle + |-\rangle(W_0 - W_1)|\psi\rangle) \\ &= \frac{1}{2}|+\rangle B|\psi\rangle + \frac{1}{2}|-\rangle(W_0 - W_1)|\psi\rangle\end{aligned}$$

This is a probabilistic implementation of B.

A linear combination of unitaries

Let B be a linear combination of easy-to-implement unitaries:

$$B = \sum_i \alpha_i W_i \,.$$

Goal: Implement B given the ability to implement $\mathrm{select}W = \sum_i |i\rangle\langle i| \otimes W_i$.

More generally, we can implement a unitary V that block-encodes $B/\|\alpha\|_1$.

Define $|A\rangle = \frac{1}{\sqrt{\|\alpha\|_1}} \sum_i \sqrt{\alpha_i}\, |i\rangle$.

$$\mathrm{select}W |A\rangle|\psi\rangle = \frac{1}{\sqrt{\|\alpha\|_1}} \sum_i \sqrt{\alpha_i}\, |i\rangle W_i |\psi\rangle$$

$$= \frac{1}{\|\alpha\|_1} |A\rangle B |\psi\rangle + |A^\perp\rangle|\cdots\rangle.$$

This is a probabilistic implementation of B.

Linear combination of unitaries (LCU method)

$$B = \sum_i \alpha_i W_i$$

Linear combination of unitaries take-home message

If B can be expressed as a linear combination of easy-to-implement unitaries, then we can probabilistically implement B.

Then use OAA/AA to get an actual implementation of B.

Application to Hamiltonian simulation

Local Hamiltonian simulation problem: Given a local Hamiltonian $H = \sum_j H_j$, implement the unitary e^{-iHt}.

Step 1: Represent H as a linear combination of unitaries

We have $H = \sum_j H_j$, where H_j acts on $O(1)$ qubits. Write H_j in the Pauli basis.

Step 2: Represent e^{-iHt} as a linear combination of unitaries

Say $H = \sum_i \beta_i P_i$, where P_i are unitary. Then

$$e^{-iHt} = I - iHt + \frac{(iHt)^2}{2!} + \cdots = I - it\left(\sum_i \beta_i P_i\right) + \frac{(it)^2}{2!}\left(\sum_i \beta_i P_i\right)^2 + \cdots$$

is a linear combination of unitaries!

Step 3: Apply LCU and OAA.

This is the "Truncated Taylor Series" algorithm [Berry-Childs-Cleve-K-Somma15].

Other applications

Quantum linear systems algorithm: Given a Hermitian matrix A, and state $|b\rangle$, the goal is to produce the state $|x\rangle = \frac{A^{-1}|b\rangle}{\|A^{-1}|b\rangle\|}$.

Solution: Represent A^{-1} as $A^{-1} = \sum_t \alpha_t e^{-iAt}$ [Childs-K-Somma16]. Apply LCU + AA.

Other applications:

- Solving differential equations [Berry-Childs-Ostrander-Wang17]
- Preparing Gibbs states [Chowdhury-Somma16] (and solving SDPs and LPs on a quantum computer [Apeldoorn-Gilyen-Gribling-de Wolf17])
- Hamiltonian simulation for other Hamiltonians (e.g., sparse Hamiltonians using quantum walks [Berry-Childs-K15], quantum chemistry [Babbush-Wiebe-McClean-McClain-Neven-Chan18])

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Quantum Signal Processing (QSP)

Eigenvalue transformation

Let W be an easy-to-implement unitary with $W = e^{i\theta_i}|\theta_i\rangle\langle\theta_i|$.

Problem: Implement $A = f(W) = \sum_i f\left(e^{i\theta_i}\right)|\theta_i\rangle\langle\theta_i|$, where f is a continuous function.

E.g., we have $W = \begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix}$; we want $A = \begin{pmatrix} f\left(e^{i\theta_1}\right) & 0 & 0 \\ 0 & f(e^{i\theta_2}) & 0 \\ 0 & 0 & f(e^{i\theta_3}) \end{pmatrix}$.

Can we implement $f\left(e^{i\theta}\right) = e^{ik\theta}$ for some integer k? (easy, just use W^k)

Can we implement $f\left(e^{i\theta}\right) = \theta^{-1}$? (arises in quantum linear systems solvers)

Can we implement $f\left(e^{i\theta}\right) = e^{i\cos(\theta)}$? (arises in Hamiltonian simulation)

Eigenvalue transformation

Let W be an easy-to-implement unitary with $W = e^{i\theta_i}|\theta_i\rangle\langle\theta_i|$.

Problem: Implement $A = f(W) = \sum_i f\left(e^{i\theta_i}\right)|\theta_i\rangle\langle\theta_i|$, where f is a continuous function.

Some solutions:

1. Use phase estimation on W. (Has poor scaling with precision.)
2. Express $A = \sum_i a_i W^i$ and use LCU.
3. Use Quantum Signal Processing [Low-Chuang16].

Setting up the "Signal"

Let W be an easy-to-implement unitary with $W = e^{i\theta_i}|\theta_i\rangle\langle\theta_i|$.

Problem: Implement $A = f(W) = \sum_i f\left(e^{i\theta_i}\right)|\theta_i\rangle\langle\theta_i|$, where f is a continuous function.

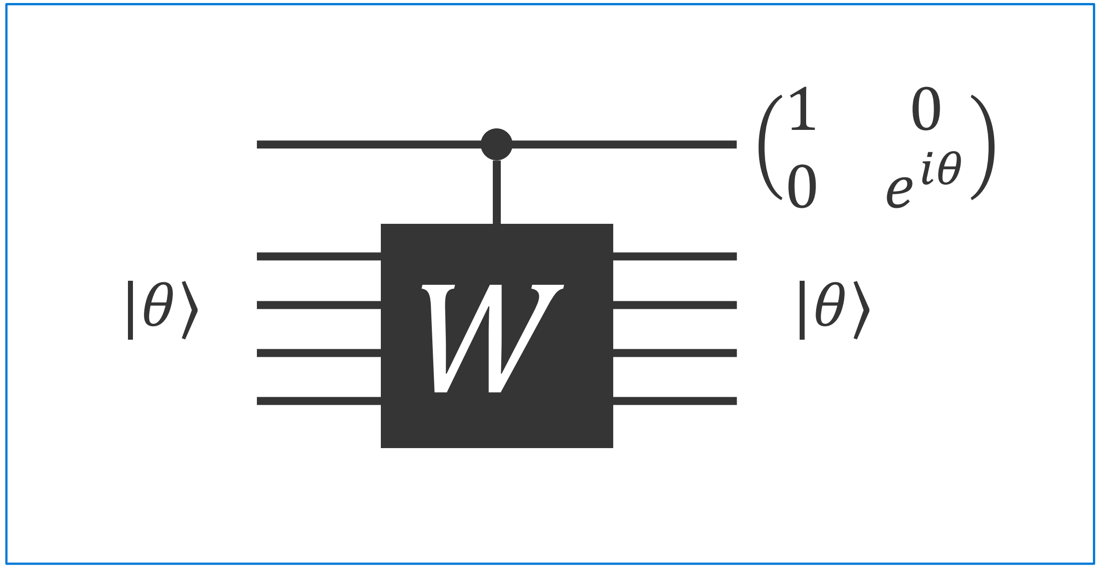
Consider the controlled-W operator:

$$\text{c-}W|0\rangle|\theta_i\rangle = |0\rangle|\theta_i\rangle$$

$$\text{c-}W|1\rangle|\theta_i\rangle = e^{i\theta_i}|1\rangle|\theta_i\rangle$$

In the subspace with the second register equal to $|\theta\rangle$,

c-W is $\begin{pmatrix}1 & 0\\ 0 & e^{i\theta}\end{pmatrix}$.

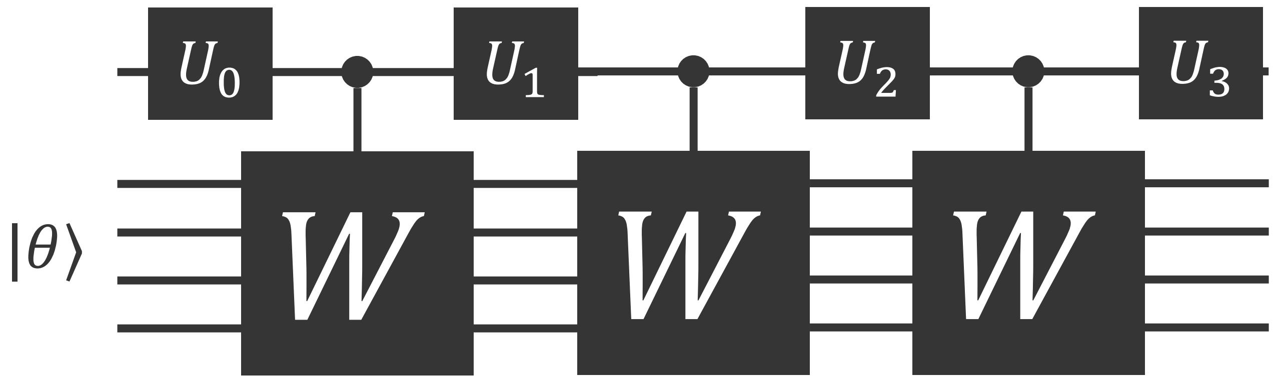


Signal processing

Let W be an easy-to-implement unitary with $W = e^{i\theta_i}|\theta_i\rangle\langle\theta_i|$.

Problem: Implement $A = f(W) = \sum_i f\left(e^{i\theta_i}\right)|\theta_i\rangle\langle\theta_i|$, where f is a continuous function.

Consider the following circuit:



$$\begin{pmatrix} g_{00}(\theta) & g_{01}(\theta) \\ g_{10}(\theta) & g_{11}(\theta) \end{pmatrix}$$

$$= U_0 \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix} U_1 \cdots U_{k-1} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix} U_k$$

Question: What matrices $\begin{pmatrix} g_{00}(\theta) & g_{01}(\theta) \\ g_{10}(\theta) & g_{11}(\theta) \end{pmatrix}$ can we get like this?

QSP fully answers this question

Signal processing

Let W be an easy-to-implement unitary with $W = e^{i\theta_i}|\theta_i\rangle\langle\theta_i|$.

Problem: Implement $A = f(W) = \sum_i f\left(e^{i\theta_i}\right)|\theta_i\rangle\langle\theta_i|$, where f is a continuous function.

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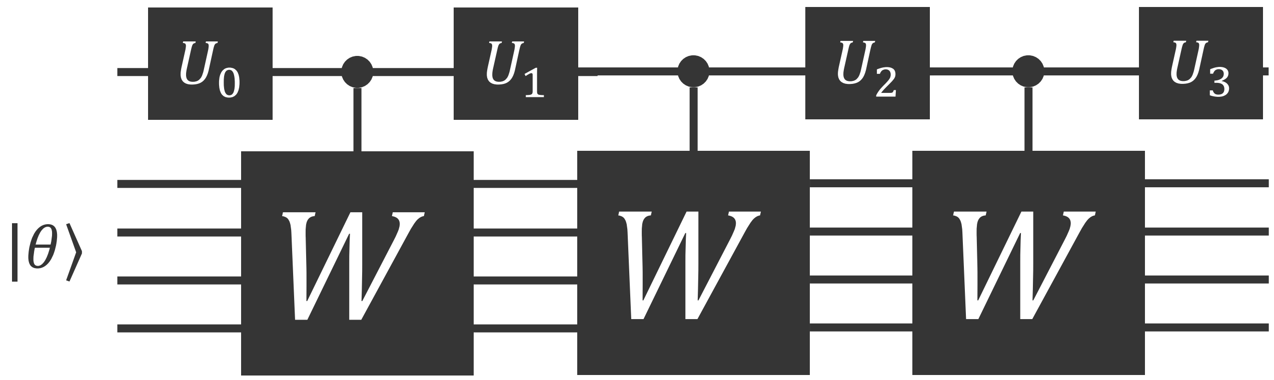

$$\begin{pmatrix} g_{00}(\theta) & g_{01}(\theta) \\ g_{10}(\theta) & g_{11}(\theta) \end{pmatrix}$$

If we choose U_i such that $\begin{pmatrix} g_{00}(\theta) & g_{01}(\theta) \\ g_{10}(\theta) & g_{11}(\theta) \end{pmatrix} = \begin{pmatrix} f(e^{i\theta}) & * \\ * & * \end{pmatrix}$, then we're done!

Quantum signal processing (QSP)

$$A = f(W)$$

Quantum signal processing take-home message

If A can be written as a (reasonable) function of an easy-to-implement unitary W, then we can implement A.

Recap

$$V = \begin{pmatrix} \sqrt{p}U & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

Oblivious amplitude amplification

$$B = \sum_i \alpha_i W_i$$

Linear combination of unitaries

$$A = f(W)$$

Quantum signal processing

Quantum signal processing

$$W(x) := \begin{bmatrix} x & i\sqrt{1-x^2} \\ i\sqrt{1-x^2} & x \end{bmatrix} = e^{i\arccos(x)\sigma_x}.$$

Theorem 3. *Let $k \in \mathbb{N}$; there exists $\Phi = \{\phi_0, \phi_1, \ldots, \phi_k\} \in \mathbb{R}^{k+1}$ such that for all $x \in [-1,1]$:*

$$e^{i\phi_0\sigma_z} \prod_{j=1}^{k} \left(W(x) e^{i\phi_j\sigma_z} \right) = \begin{bmatrix} P(x) & iQ(x)\sqrt{1-x^2} \\ iQ^*(x)\sqrt{1-x^2} & P^*(x) \end{bmatrix} \tag{3}$$

if and only if $P, Q \in \mathbb{C}[x]$ such[2] *that*

(i) $\deg(P) \leq k$ *and* $\deg(Q) \leq k-1$

(ii) P *has parity-*$(k \mod 2)$ *and* Q *has parity-*$(k-1 \mod 2)$

(iii) $\forall x \in [-1,1]\colon |P(x)|^2 + (1-x^2)|Q(x)|^2 = 1.$

[Gilyén-Su-Low-Wiebe18]

Thanks!

Microsoft Quantum Development Kit:
www.microsoft.com/quantum