Convex Analysis at Infinity

An Introduction to Astral Space

Miro Dudík
Rob Schapire
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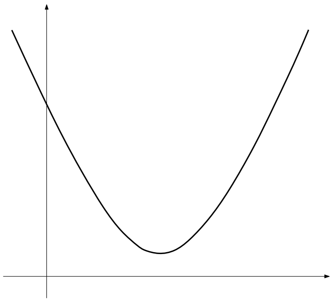
further reading at: aka.ms/astral

Convex functions



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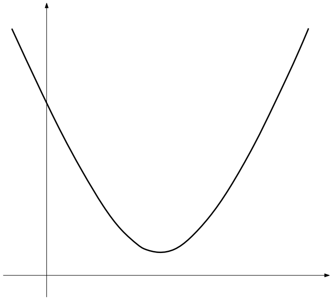

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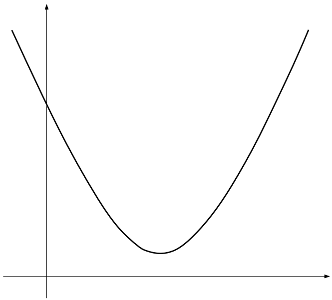
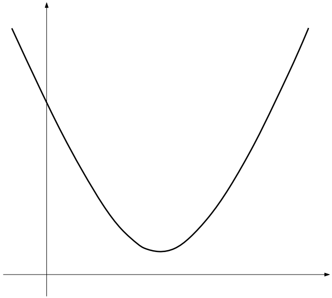


- minimizing convex functions is basis for many methods in machine learning and statistics (and other fields)
  - e.g.: maximum likelihood, maximum entropy, linear regression, logistic regression, boosting, SVM's, ...
- convex functions are really nice!
  - local minimum must be global minimum
  - if gradient $= 0$ then must be global minimum
  - usually easier to find and analyze minimization algorithms
  - beautiful properties

Finite minimizers

- common to assume function actually attains minimum at a finite point



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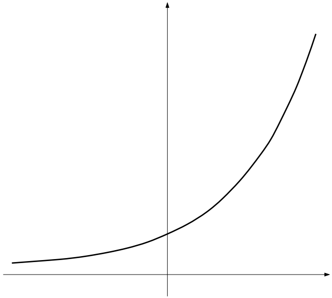

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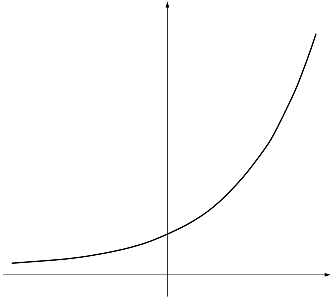

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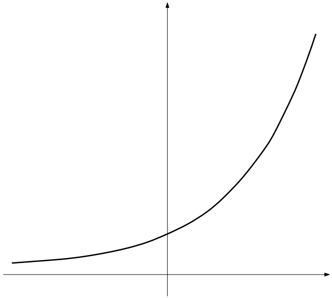

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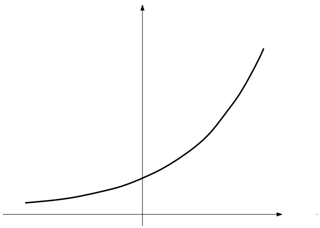
  - analyzing convergence often requires carefully tailored techniques
- this talk: develop theory for studying such minimizers at infinity

Example: exponential function



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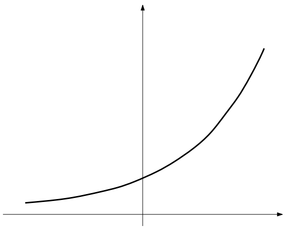

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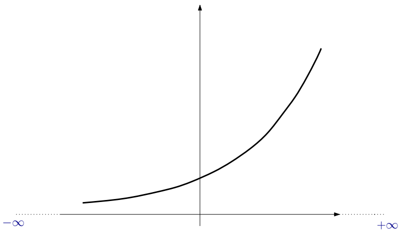

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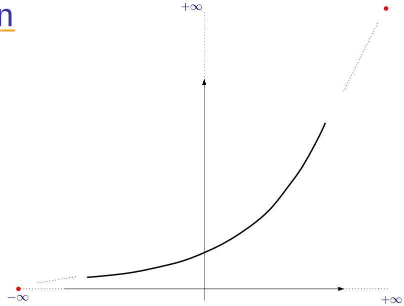

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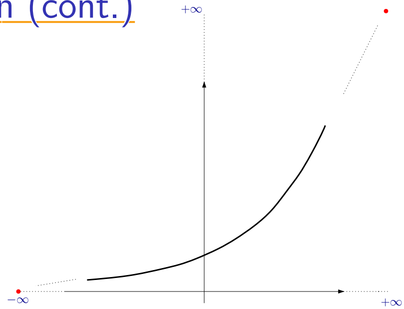
  - extending f to $\overline{\mathbb{R}}$ by setting:
    $f(-\infty) = 0$ and $f(+\infty) = +\infty$

Example: exponential function (cont.)



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  - is continuous over $\overline{\mathbb{R}}$
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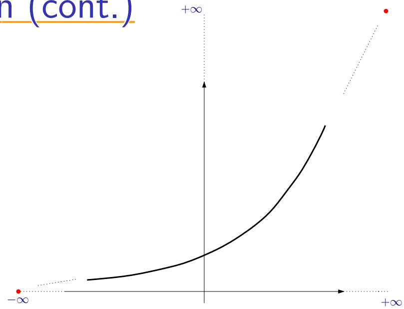

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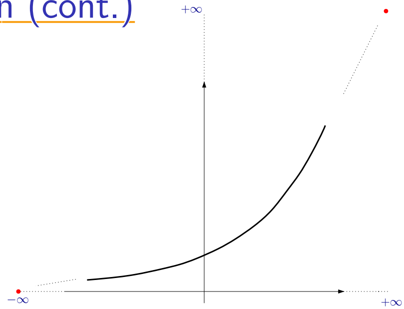

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- what about in $n \geq 2$ dimensions?

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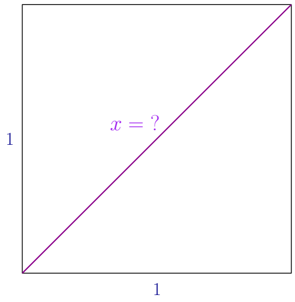
- hope to make study of convex functions more "complete" and "regular"
  - e.g., so every convex function, when extended to new space, has a minimizer
- want compatible with key notions of convex analysis

Analogy

- if only working in $\mathbb{Q} =$ rationals, then no number equals length of diagonal of a unit square



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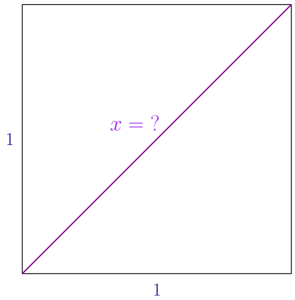

- if only working in $\mathbb{Q} =$ rationals, then no number equals length of diagonal of a unit square
- can approach value with sequences in $\mathbb{Q}$:

  $\frac{1}{1},\ \frac{3}{2},\ \frac{7}{5},\ \frac{17}{12},\ \frac{41}{29},\ \frac{99}{70},\ \frac{239}{169},\ \ldots \to \sqrt{2}$

  $1,\ 1.4,\ 1.41,\ 1.414,\ 1.4142,\ 1.41421,\ \ldots \to \sqrt{2}$

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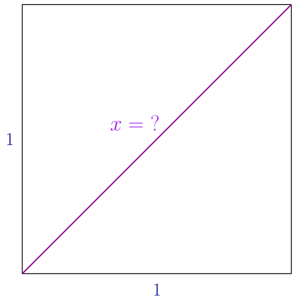

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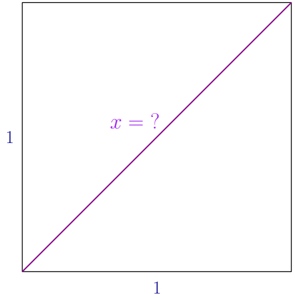

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- so don't "need" reals (can just work over sequences of rationals)
- far preferable to extend $\mathbb{Q}$ to $\mathbb{R}$
  - much more complete, regular, well-structured

Analogy (cont.)

- in same way, can continue to use sequences to study minimizers of convex functions
  - might be much nicer to study minimizers at infinity as mathematical objects in their own right
  - can hope larger space would be more complete, regular, and revealing of structure

This work

- introduce astral space, extension of $\mathbb{R}^n$ with points at infinity
- extend functions on $\mathbb{R}^n$ to astral space
- study key properties and topics extended to astral space, especially from convex analysis

Outline

- what can minimizers at infinity look like?
- constructing astral space
- what are astral points like?
- extending functions to astral space
- convergence of iterative algorithms

Notation

- $n =$ dimension
- scalars (in $\mathbb{R}$): $x, y, \ldots$
- vectors (in $\mathbb{R}^n$): $\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}, \ldots$
  - as tuple: $\boldsymbol{x} = (x_1, \ldots, x_n)$

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- all sequences indexed by $t = 1, 2, \ldots$
- limits and convergence always as $t \to +\infty$
- $(\boldsymbol{x}_t)$ is sequence $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots$

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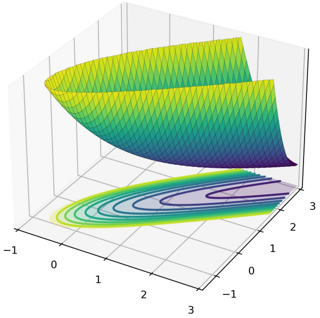
- what can such "minimizers at infinity" look like?
- in $n = 1$ dimensions, can only converge to $\pm\infty$
- in $n \geq 2$ dimensions, many possibilities
  - for example...

Example: Diagonal valley



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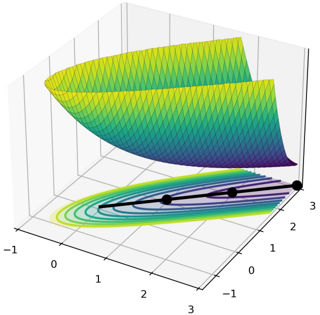

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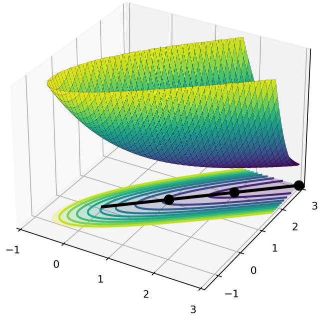

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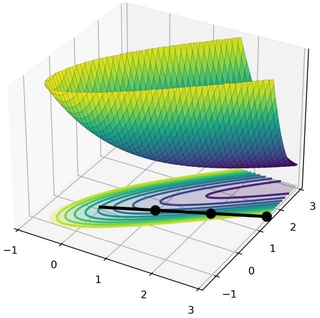

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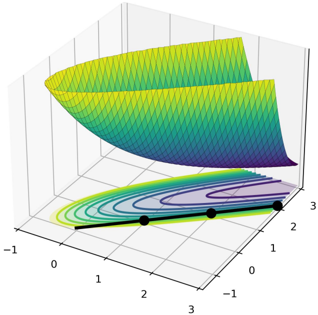

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  - offset also matters

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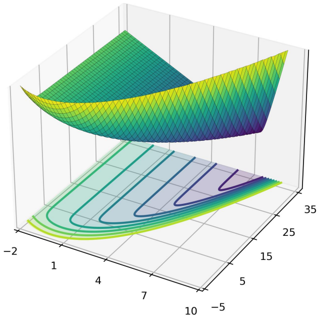

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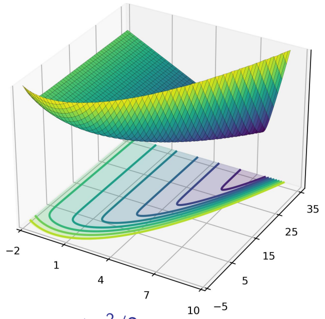

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- e.g., in $\mathbb{R}^2$, let

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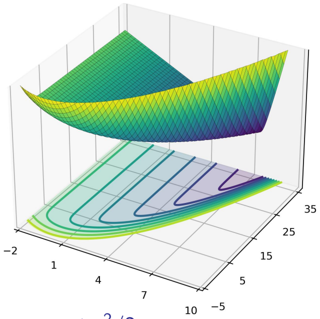

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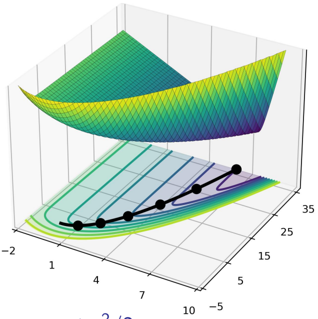

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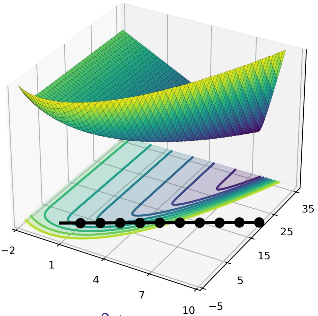

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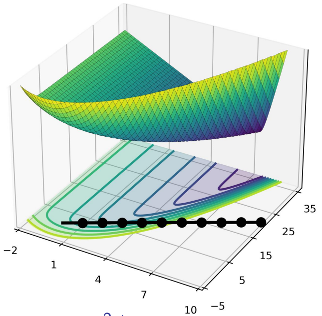

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- no minimizing sequence along straight ray
- how to construct space capturing such minimizers at infinity?

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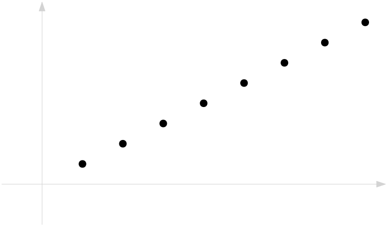
- idea: add "new" points to $\mathbb{R}^n$ that can be limits of such sequences
- key questions:
  - which sequences should have limits?
  - when should two sequences have same limit?
- once answered, can construct space:
  - add "new" points to be limits of each group of sequences that should all have same limit

Examples and intuitions



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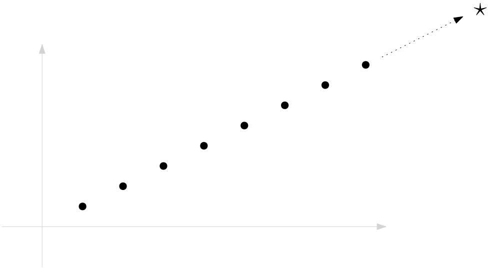

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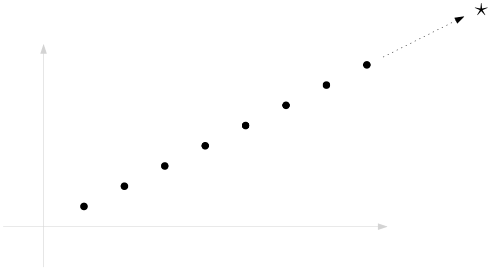

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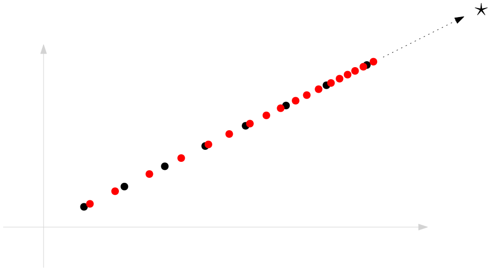
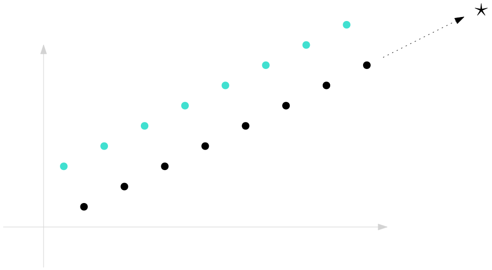
- what sequences should have same limit?

Examples and intuitions



- any sequence following ray to infinity should have a limit
  e.g., $\boldsymbol{x}_t = (2t, t) = t\boldsymbol{v}$ where $\boldsymbol{v} = (2, 1)$
- what sequences should have same limit?
  - e.g., if change rate converging to infinity

Example: parallel sequences



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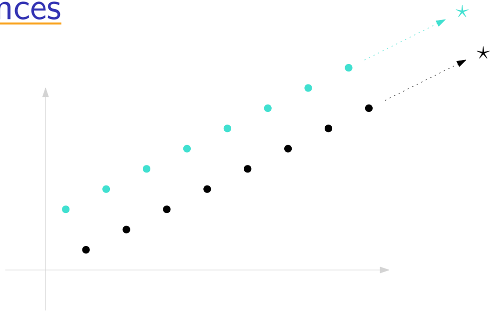

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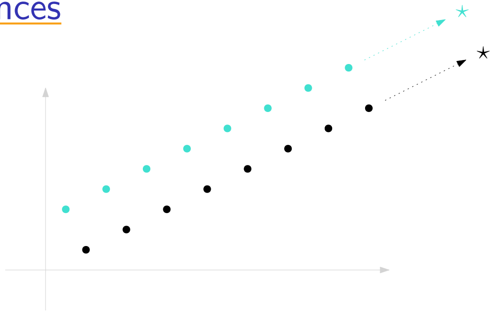

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- should two sequences have same limit?
- we believe no because:
  - offset matters for minimization
  - in applications, often care about such offsets, not just overall direction of minimization

A basic principle

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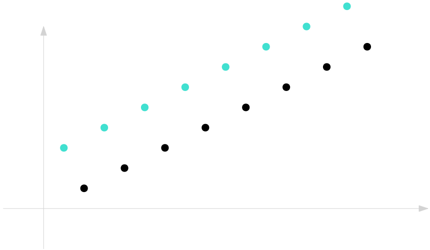

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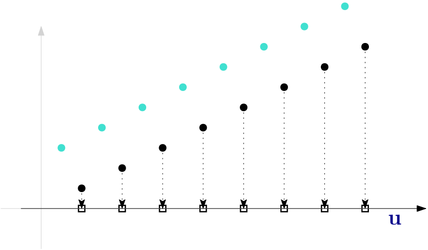

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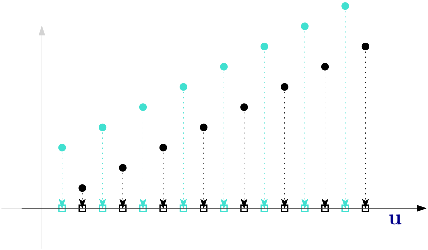

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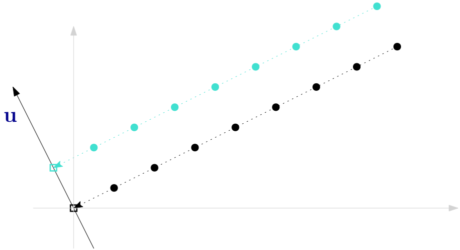

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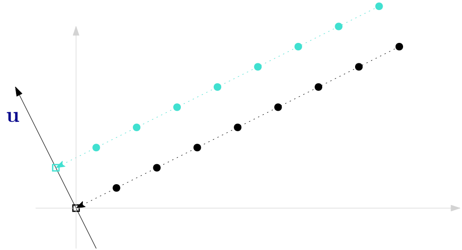

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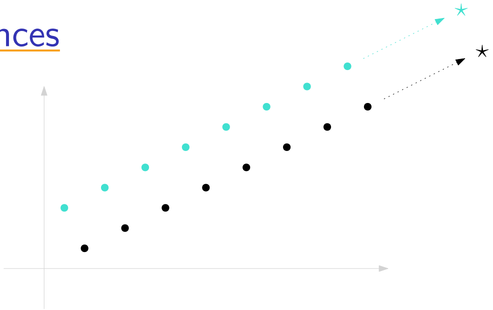

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- therefore: require (x_t) and (x'_t) to have different limits

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- note that limits can be in $\overline{\mathbb{R}}$

Astral space

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- in $n = 1$ dimensions, only add $\pm\infty$ so $\overline{\mathbb{R}^1}$ same as $\overline{\mathbb{R}}$

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- astral space is not a vector space, nor a metric space

Outline

- what can minimizers at infinity look like?
- constructing astral space
- what are astral points like?
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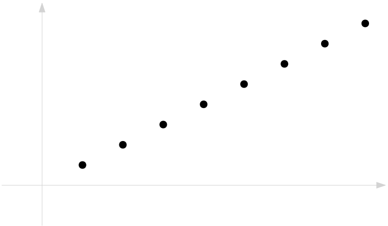
Astrons



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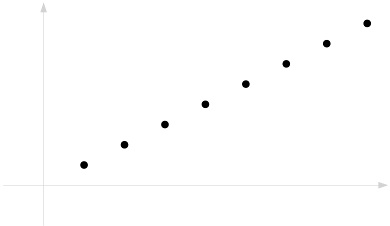

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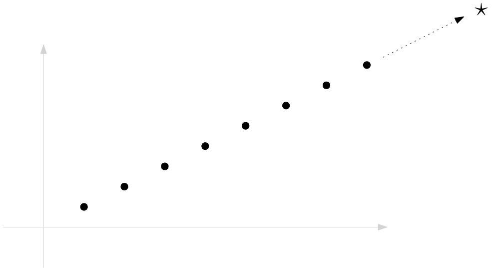

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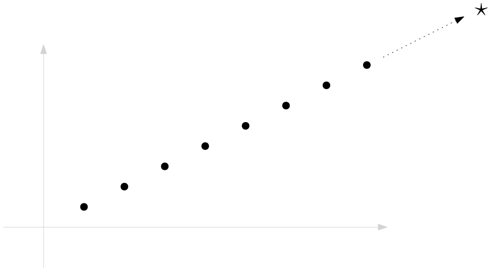

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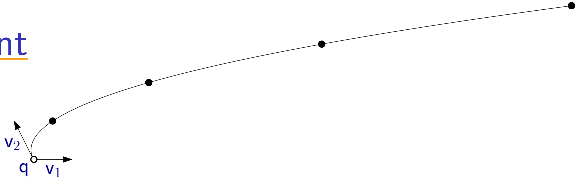
- turn out to be building blocks for all astral points

Example astral point



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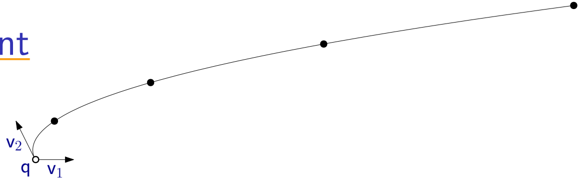

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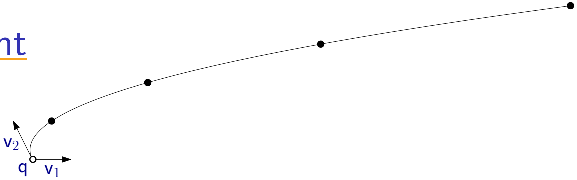

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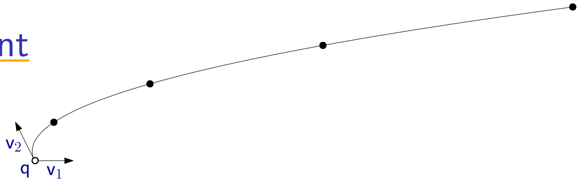

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- turns out, can write in form:

$$\overline{\mathbf{x}} = \underbrace{\omega\mathbf{v}_1 \mathbin{\dot{+}} \omega\mathbf{v}_2}_{\text{astrons}} \mathbin{\dot{+}} \mathbf{q}$$

- operation $\dot{+}$ is leftward addition:
  - similar to vector addition but not commutative
  - gives kind of "dominance" to term on left

Representing astral points

- in general: every astral point $\overline{\boldsymbol{x}}$ can be written in form

$$\overline{\boldsymbol{x}} = \underbrace{\omega \boldsymbol{v}_1 \mathbin{\dot{+}} \cdots \mathbin{\dot{+}} \omega \boldsymbol{v}_k}_{\text{astrons}} \mathbin{\dot{+}} \underbrace{\boldsymbol{q}}_{\substack{\text{finite}\\ \text{part}}}$$

  for some orthonormal $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$
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  for some orthonormal $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$
  and some $\mathbf{q} \in \mathbb{R}^n$ orthogonal to the $\mathbf{v}_i$'s
- astral rank $= k$ (number of astrons in $\overline{\mathbf{x}}$'s representation)
  - astral rank $= 0 \Rightarrow \overline{\mathbf{x}} \in \mathbb{R}^n$
  - astral rank $= 1 \Rightarrow \overline{\mathbf{x}}$ is limit of sequence along ray

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Extending a function to astral space

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- want to extend to astral space:

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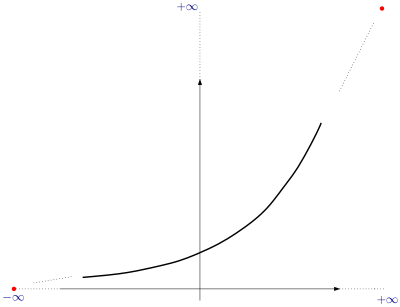
- how to define?

Example: exponential function



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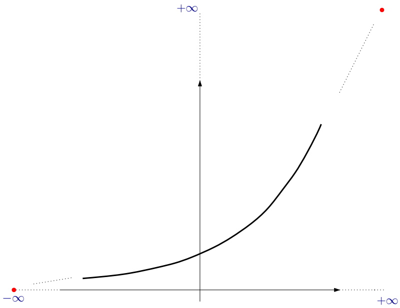

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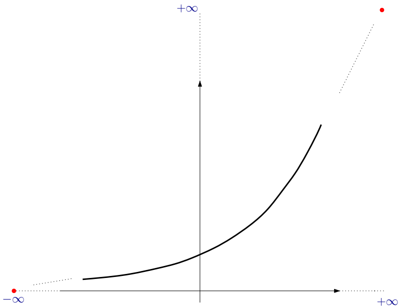

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- only way to extend to $\overline{\mathbb{R}}$ continuously

Continuous extensions

- in general, given convex $f : \mathbb{R}^n \to \overline{\mathbb{R}}$
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- if holds for all $\overline{\mathbf{x}} \in \overline{\mathbb{R}^n}$ then
  $\bar{f}$ is (unique) continuous extension of f to $\overline{\mathbb{R}^n}$

Example: Linear functions

- say $f(\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{u}$ for some $\boldsymbol{u} \in \mathbb{R}^n$ [e.g., $f(x_1, x_2) = 2x_1 - x_2$]

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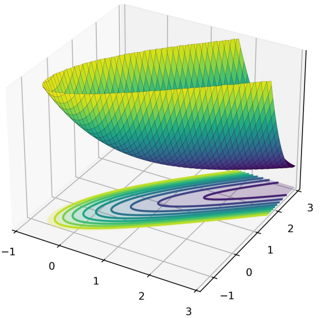
- so: every linear function can be continuously extended to $\overline{\mathbb{R}^n}$

Example: Diagonal valley



- in $\mathbb{R}^2$, recall

$$f(x_1, x_2) = e^{-x_1} + (x_2 - x_1)^2$$

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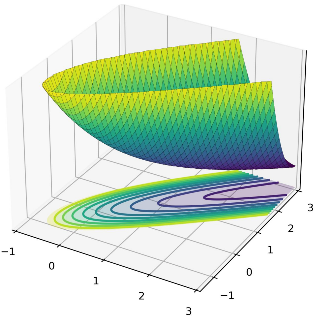

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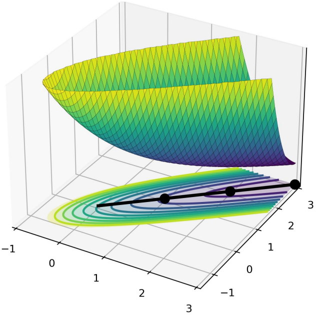

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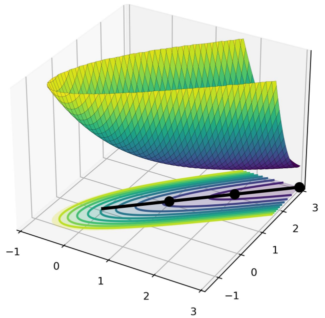

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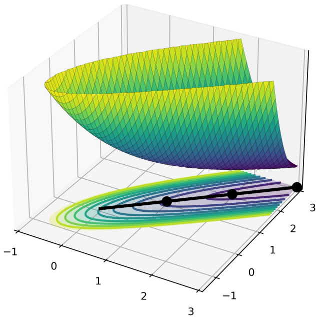

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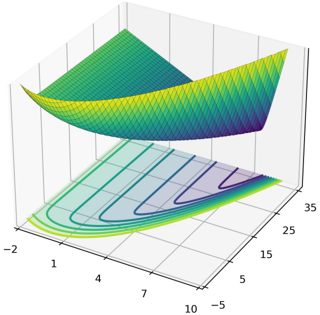

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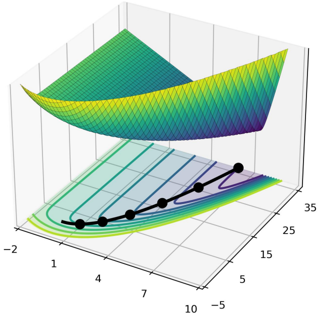

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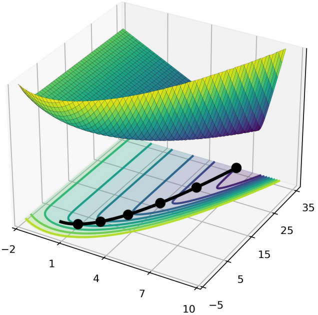

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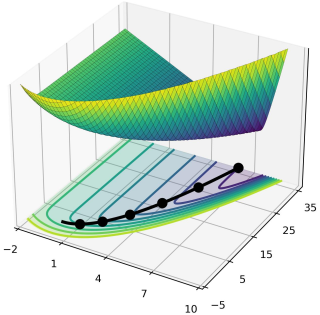

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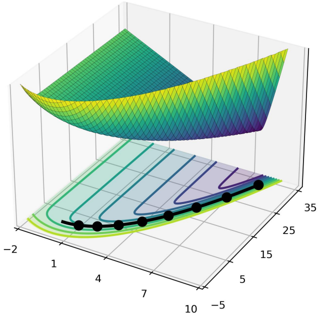

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  but $f(\mathbf{x}'_t) \rightarrow 1 \neq \bar{f}(\overline{\mathbf{x}})$

Outline

- what can minimizers at infinity look like?
- constructing astral space
- what are astral points like?
- extending functions to astral space
- convergence of iterative algorithms

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  - however, $f(\boldsymbol{x}_t) = t \to +\infty$

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- reveals structure and regularity not otherwise apparent

Convergence and astral continuity (cont.)

- can use to prove convergence of standard iterative methods applied to various ML/statistical settings
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- can use to prove convergence of standard iterative methods applied to various ML/statistical settings
  - e.g.: gradient descent, coordinate descent, steepest descent
  - e.g.: logistic regression, boosting, maximum likelihood (which all have continuous extensions)
- don't require finite minimizer
- algorithms operate in $\mathbb{R}^n$, but use astral methods in proofs
  - rely on astral continuity properties
    (without which results do not hold, in general)

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- e.g. AdaBoost minimizes exponential loss
  - finds solution with large-margin property, implying generalization
  - really an astral property of minimizer at infinity (namely, of first astron in representation)

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- far more not covered
  - details at: aka.ms/astral [or arxiv.org/abs/2205.03260] (will eventually be published as a book)