Cut-Preserving Vertex Sparsifiers for Planar and Quasi-Bipartite Graphs

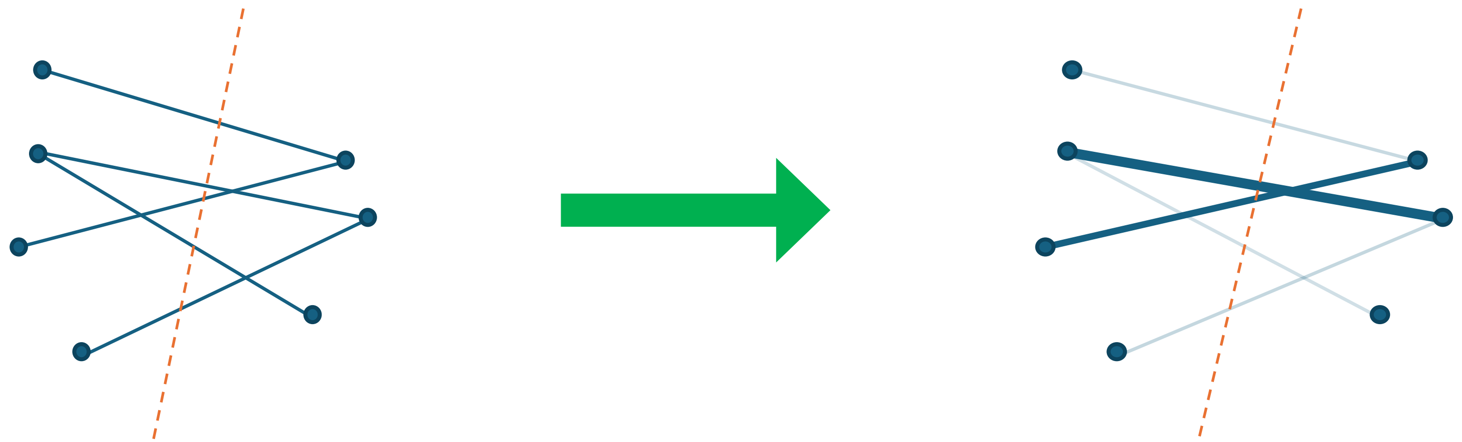
Yu Chen

Joint work with Zihan Tan (Rutgers)

Cut Sparsifier

- Given a graph G, a cut sparsifer G' is a sparse subgraph that (approximately) preserves all cut values in G.

- Importance Sampling:
  - Sample edge e with probability p_e that depends on the importance of e.
  - If e gets sampled, reweight e to $1/p_e$.



Cut Sparsifier

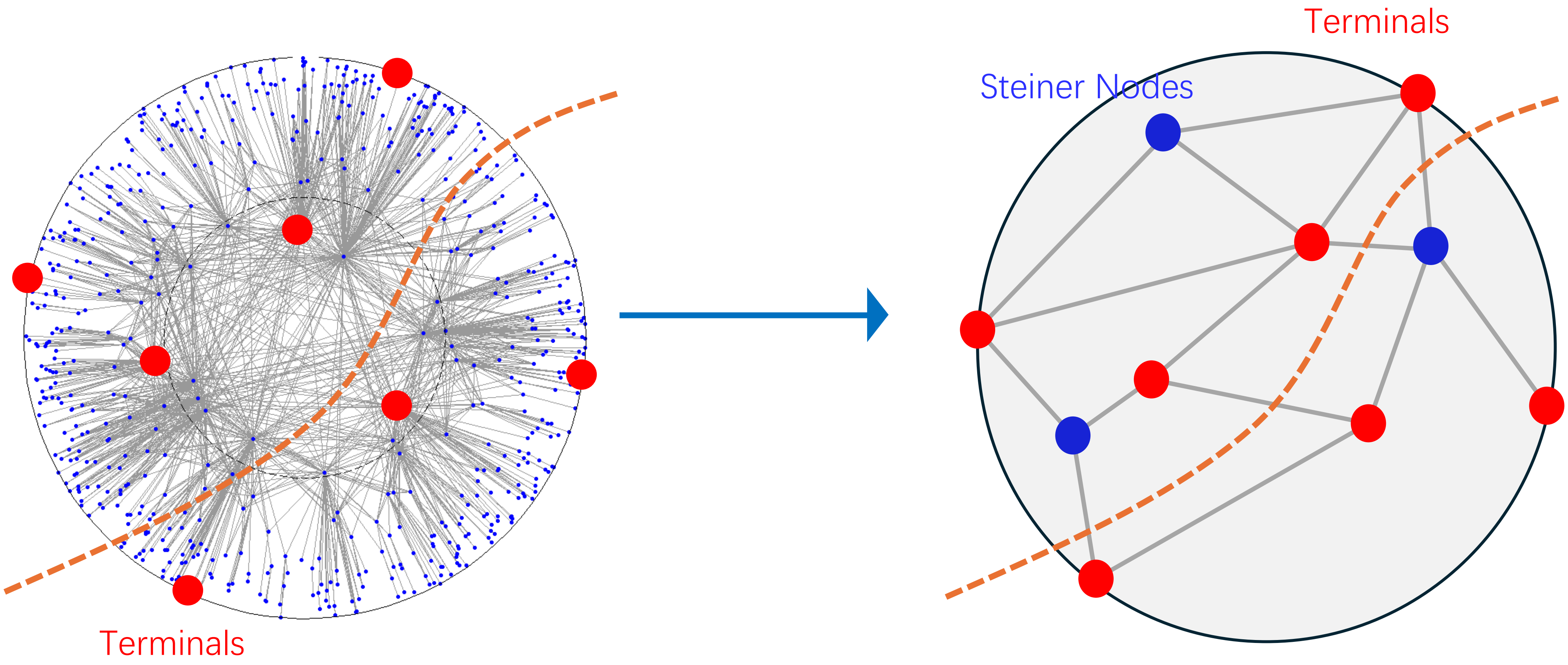
- Given a graph G, a cut sparsifer G' is a sparse subgraph that (approximately) preserves all cut values in G.
- Any graph has a quality-$(1+\varepsilon)$ cut sparsifier with $O(n/\varepsilon^2)$ edges. [BSS12]
- What if n is very large and only k vertices are important?

Terminal Cut

- Given a graph G and a set of terminals T, a terminal cut is a partition of the terminals $(S, T - S)$, whose size is defined to be size of the minimum cut that partition S and $T - S$.

- Given a graph G and a set of terminals T, a vertex cut sparsifer G' is a small graph that (approximately) preserves all terminal cut values in G.

Vertex Cut Sparsifier



Without Steiner Nodes

- Given a graph G and k terminals, there is a quality-$O\left(\frac{\log k}{\log\log k}\right)$ cut sparsifier without Steiner nodes. [Moitra09, CLLM10]

- Lower bound $\Omega\left(\frac{\sqrt{\log k}}{\log\log k}\right)$. [MM10, CLLM10]

- How many Steiner nodes do we need to achieve a very good ratio?

Perfect Cut Sparsifier

- Given a graph G and k terminals, there is a quality-1 cut sparsifier with 2^{2^k} vertices. [HKNR98, KR14]
- If an edge is not cut by any terminal cut, then increasing the weight of this edge will not change any terminal cut size.
- If two vertices are on the same side for every terminal cut, then we can contract them.

Perfect Cut Sparsifier

- Given a graph G and k terminals, there is a quality-1 cut sparsifier with 2^{2^k} vertices. [HKNR98, KR14]

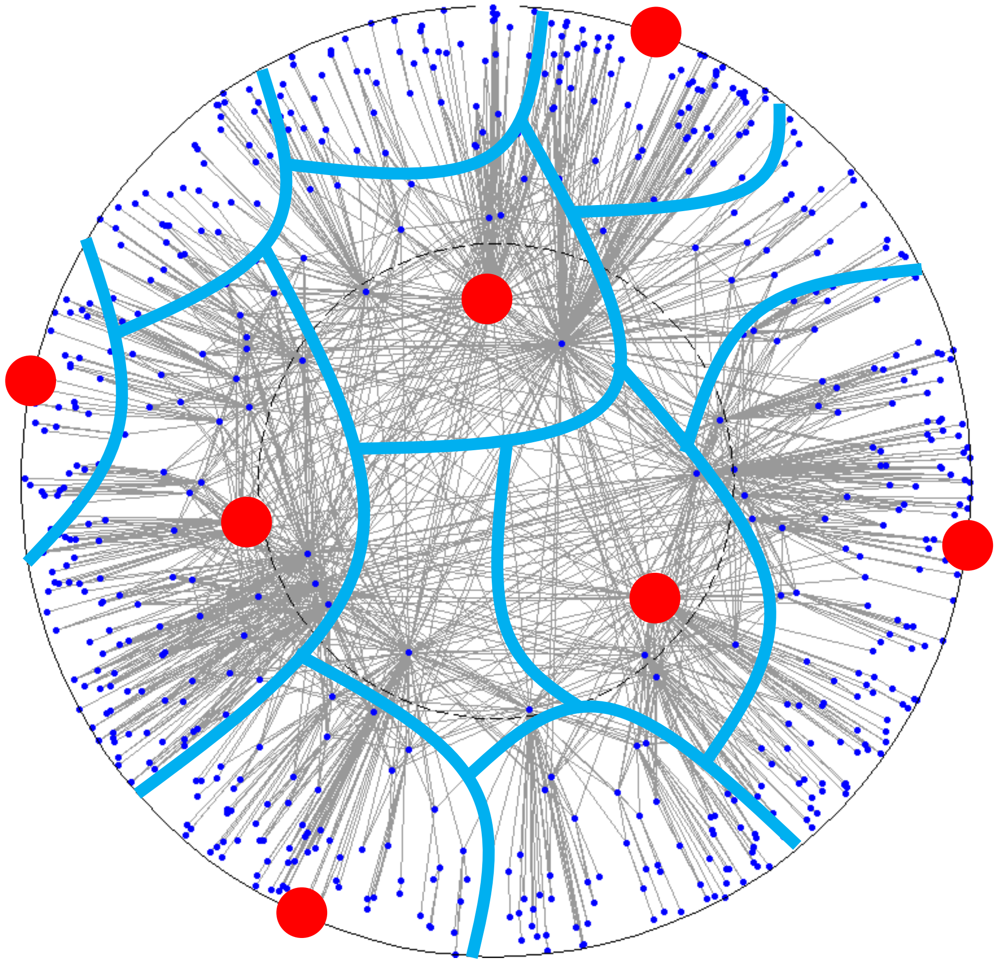
- For any vertex v, define $\pi^v: 2^T \to \{0,1\}$, where $\pi^v(S) = 1$ if v is on the same side as S in the terminal cut $(S, T - S)$, 0 otherwise.

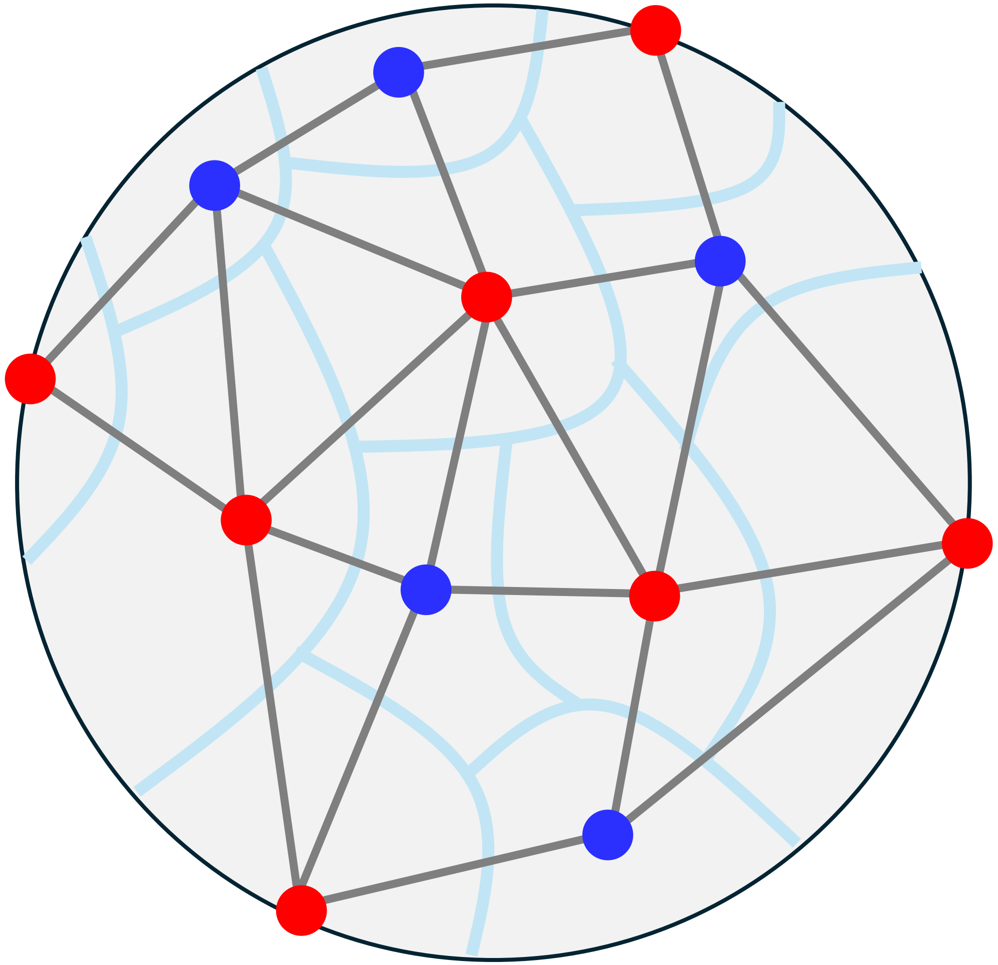
- For any two vertex u, v, if $\pi^u = \pi^v$, then we can contract them.

Contraction-based

- 2^k terminal cuts, 2^{2^k} possible vectors (profile).

Contraction Based Cut Sparsifier





Perfect Cut Sparsifier

- Given a graph G and k terminals, there is a quality-1 cut sparsifier with ~~2^{2^k}~~ $2^{\binom{k}{k/2}}$ vertices. [HKNR98, KR14]

- For any vertex v, define $\pi^v: 2^T \rightarrow \{0,1\}$, where $\pi^v(S) = 1$ if v is on the same side as S in the terminal cut $(S, T-S)$, 0 otherwise.

- There exist graph such that the vertices have $2^{2^{\Omega(k)}}$ different profiles. [KPZ17]

Perfect Cut Sparsifier

- Given a graph G and k terminals, there is a quality-1 cut sparsifier with ~~2^{2^k}~~ $2^{\binom{k}{k/2}}$ vertices. [HKNR98, KR14]
- There exist graphs such that any contracted-based quality-1 cut sparsifier has $2^{2^{\Omega(k)}}$ vertices. [KPZ17]
- There exist graphs such that any quality-1 cut sparsifier has $2^{\Omega(k)}$ vertices. [KPZ17]

Perfect Cut Sparsifier

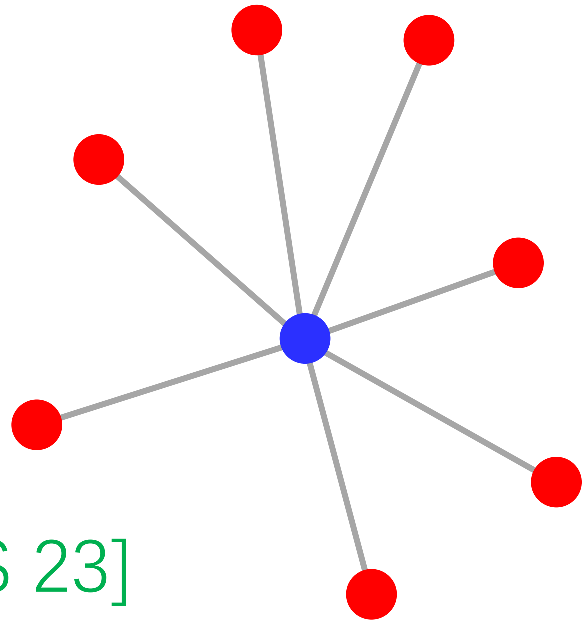
- Given a graph G and k terminals, there is a quality-1 cut sparsifier with ~~2^{2^k}~~ $2^{\binom{k}{k/2}}$ vertices. [HKNR98, KR14]
- There exist graphs such that any contracted-based quality-1 cut sparsifier has $2^{2^{\Omega(k)}}$ vertices. [KPZ17]
- There exist planar graphs such that any quality-1 cut sparsifier has $2^{\Omega(k)}$ vertices. [KPZ17]

Can we use importance sampling?

- What if we consider quality-$(1+\varepsilon)$ cut sparsifier?

Quasi-Bipartite Graph

- In a quasi-bipartite graph, there is no edges between non-terminal vertices.
- The profile of each vertex is independent.
- Sample vertices depend on its importance.
- $\tilde{O}(k/\varepsilon^2)$ size quality-$(1+\varepsilon)$ cut sparsifier. [JLLS 23]



Previous Works

Graph Type	Quality	Size	Contraction-Based?	Work
General	1	2^{2^k}	Yes	[HKNR98,KR14]
General	1	$2^{2^{\Omega(k)}}$	Yes	[KPZ17]
General	1	$2^{\Omega(k)}$	No	[KPZ17,KR14]
Planar	1	$2^{O(k)}$	No	[KR13, KR17]
Planar	1	$2^{\Omega(k)}$	No	[KPZ17]
Quasi-Bipartite	$1+\varepsilon$	$\widetilde{O}(k/\varepsilon^2)$	No	[JLLS23]

Previous Works

Graph Type	Quality	Size	Contraction-Based?	Work
General	1	2^{2^k}	Yes	[HKNR98,KR14]
General	1	$2^{2^{\Omega(k)}}$	Yes	[KPZ17]
General	1	$2^{\Omega(k)}$	No	[KPZ17,KR14]
Planar	1	$2^{O(k)}$	No	[KR13, KR17]
Planar	1	$2^{\Omega(k)}$	No	[KPZ17]
Planar	$1+\varepsilon$	$O(k \cdot Poly(\log k/\varepsilon))$	No	This work
Quasi-Bipartite	$1+\varepsilon$	$\widetilde{O}(k/\varepsilon^2)$	No	[JLLS23]

Previous Works

Graph Type	Quality	Size	Contraction-Based?	Work
General	1	$2^{2^{k}}$	Yes	[HKNR98,KR14]
General	1	$2^{2^{\Omega(k)}}$	Yes	[KPZ17]
General	1	$2^{\Omega(k)}$	No	[KPZ17,KR14]
Planar	1	$2^{O(k)}$	No	[KR13, KR17]
Planar	1	$2^{\Omega(k)}$	No	[KPZ17]
Planar	$1+\varepsilon$	$O(k \cdot Poly(\log k/\varepsilon))$	No	This work
Quasi-Bipartite	1	2^{k^2}	No	[DKV24]
Quasi-Bipartite	1	$2^{O(k^2 \log k)}$	Yes	This work
Quasi-Bipartite	$1+\varepsilon$	$\widetilde{O}(k/\varepsilon^2)$	No	[JLLS23]

Previous Works

Graph Type	Quality	Size	Contraction-Based?	Work
General	1	2^{2^k}	Yes	[HKNR98,KR14]
General	1	$2^{2^{\Omega(k)}}$	Yes	[KPZ17]
General	1	$2^{\Omega(k)}$	No	[KPZ17,KR14]
Planar	1	$2^{O(k)}$	No	[KR13, KR17]
Planar	1	$2^{\Omega(k)}$	No	[KPZ17]
Planar	$1+\varepsilon$	$O(k \cdot Poly(\log k / \varepsilon))$	No	This work
Quasi-Bipartite	1	2^{k^2}	No	[DKV24]
Quasi-Bipartite	1	$2^{O(k^2 \log k)}$	Yes	This work
Quasi-Bipartite	$1+\varepsilon$	$\widetilde{O}(k/\varepsilon^2)$	No	[JLLS23]
Quasi-Bipartite	$1+\varepsilon$	$k^{O(1/\varepsilon^2)}$, $k^{\widetilde{\Omega}(1/\varepsilon)}$	Yes	This work

Previous Works

Graph Type	Quality	Size	Contraction-Based?	Work
General	1	2^{2^k}	Yes	[HKNR98,KR14]
General	1	$2^{2^{\Omega(k)}}$	Yes	[KPZ17]
General	1	$2^{\Omega(k)}$	No	[KPZ17,KR14]
Planar	1	$2^{O(k)}$	No	[KR13, KR17]
Planar	1	$2^{\Omega(k)}$	No	[KPZ17]
Planar	$1+\varepsilon$	$O(k \cdot Poly(\log k/\varepsilon))$	No	This work
Quasi-Bipartite	1	2^{k^2}	No	[DKV24]
Quasi-Bipartite	1	$2^{O(k^2 \log k)}$	Yes	This work
Quasi-Bipartite	$1+\varepsilon$	$\widetilde{O}(k/\varepsilon^2)$	No	[JLLS23]
Quasi-Bipartite	$1+\varepsilon$	$k^{O(1/\varepsilon^2)}$, $k^{\widetilde{\Omega}(1/\varepsilon)}$	Yes	This work

Quality-1 Cut Sparsifier for Quasi-Bipartite Graphs

Perfect Cut Sparsifier for Quasi-Bipartite Graph

- For any vertex v, define $\pi^v: 2^T \rightarrow \{0,1\}$, where $\pi^v(S) = 1$ if v is on the same side as S in the terminal cut $(S, T - S)$, 0 otherwise.

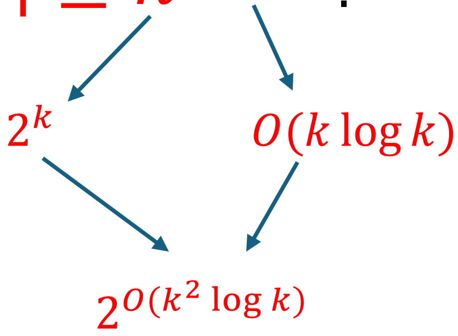
- **Lemma:** In a Quasi-Bipartite Graph, only $2^{O(k^2 \log k)}$ profiles are possible.

- View π^v as a set of terminal cuts. All possible profile $\Pi(\mathrm{G})$ is a set family.

- **Lemma:** VC-dimension of $\Pi(\mathrm{G})$ is $O(k \log k)$.

Shattering Sets and VC-dimension

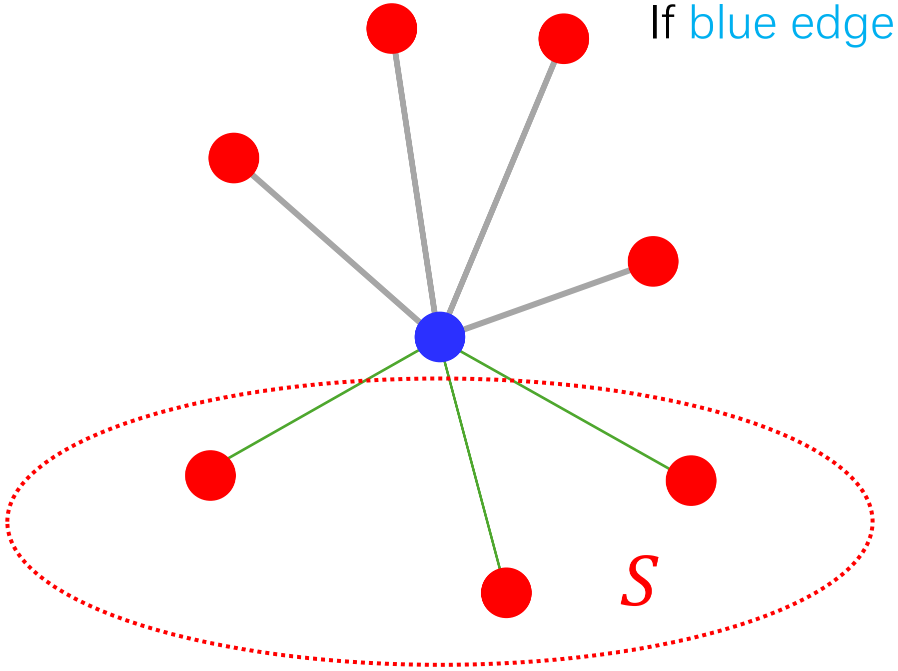
- A set family $\mathcal{F}$ shatters a set U if for any $U' \subseteq U$, there is a set $F \in \mathcal{F}$ such that $F \cap U = U'$.
- VC-dimension of $\mathcal{F}$ is defined as the size of maximum U such that $\mathcal{F}$ shatters a set U.
- Sauer-Shelah Lemma: $|\mathcal{F}| \leq n^{VC(\mathcal{F})}$.

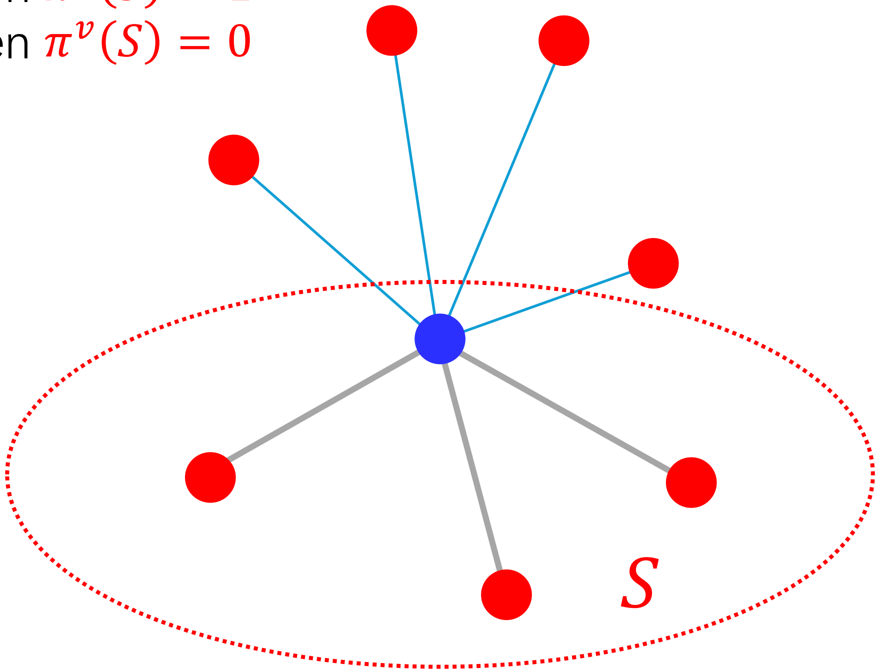


Profiles for Quasi-Bipartite Graph

- For any vertex v, define $\pi^v: 2^T \rightarrow \{0,1\}$, where $\pi^v(S) = 1$ if v is on the same side as S in the terminal cut $(S, T - S)$, 0 otherwise.

If blue edge < green edge, then $\pi^v(S) = 1$
If blue edge > green edge, then $\pi^v(S) = 0$





Profiles for Quasi-Bipartite Graph

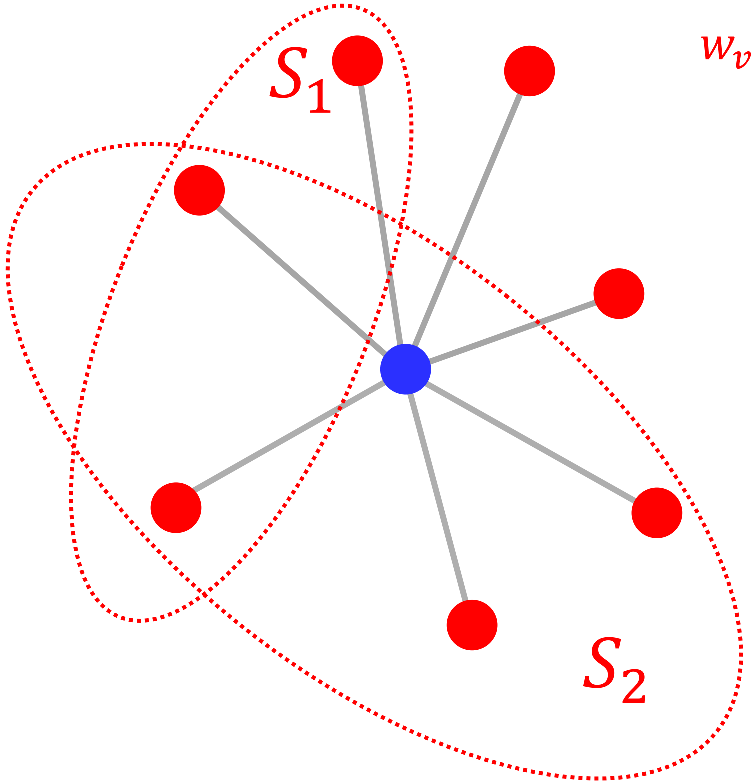
- For any vertex v, define $\pi^v: 2^T \to \{0,1\}$, where $\pi^v(S) = 1$ if v is on the same side as S in the terminal cut $(S, T - S)$, 0 otherwise.

- Define $w_v(S)$ as the total weight of edges between v and S.

- $\pi^v(S) = 1$ iff $w_v(S) > w_v(T)/2$.

Profiles for Quasi-Bipartite Graph

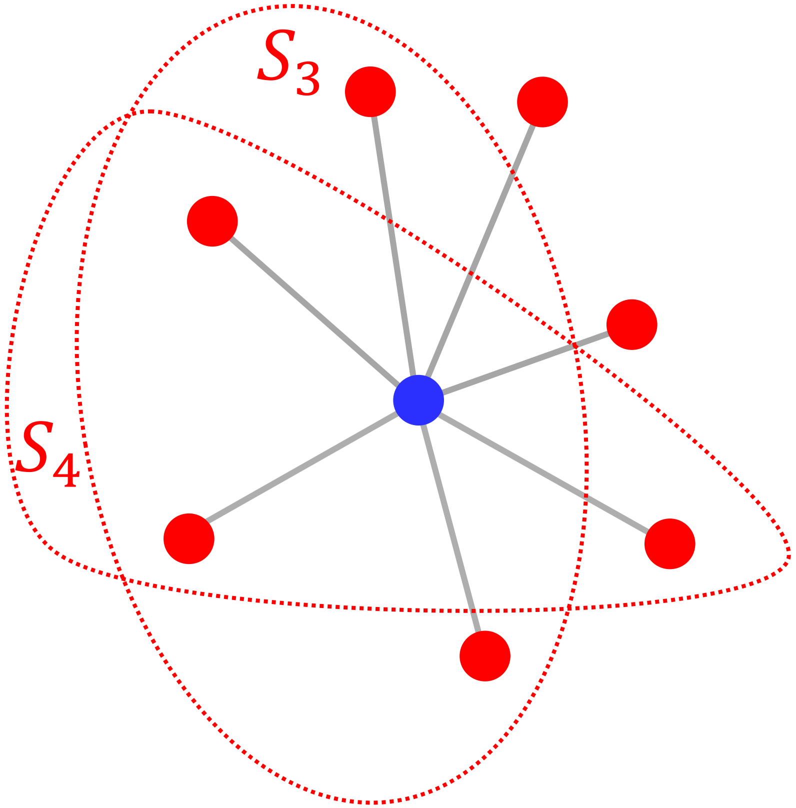
- $\pi^v(S) = 1$ iff $w_v(S) > w_v(T)/2$.



$w_v(S_1) + w_v(S_2) = w_v(S_3) + w_v(S_4)$

It is not possible that
$\pi^v(S_1) = \pi^v(S_2) = 1$ and
$\pi^v(S_3) = \pi^v(S_4) = 0$

Π cannot shatter $\{S_1, S_2, S_3, S_4\}$



Profiles for Quasi-Bipartite Graph

- If two set families $\mathcal{S}_1$, $\mathcal{S}_2$ satisfy:
  - $|\mathcal{S}_1| = |\mathcal{S}_2|$.
  - $\sum_{S \in \mathcal{S}_1} S = \sum_{S \in \mathcal{S}_2} S$
- Then $\sum_{S \in \mathcal{S}_1} w_v(S) = \sum_{S \in \mathcal{S}_2} w_v(S)$
- It is not possible that $\pi^v(S) = 1$ for all $S \in \mathcal{S}_1$ and $\pi^v(S) = 0$ for all $S \in \mathcal{S}_2$

- Π cannot shatter $\mathcal{S}_1 \cup \mathcal{S}_2$.

Profiles for Quasi-Bipartite Graph

- If Π shatters $\mathcal{S}$, then for all $\mathcal{S}' \subseteq \mathcal{S}$ such that $|\mathcal{S}'| = |\mathcal{S}|/2$, $\sum_{S \in \mathcal{S}'} S$ are different from each other.

- There are $\binom{|\mathcal{S}|}{|\mathcal{S}|/2}$ such subsets,
- There are at most $|\mathcal{S}|^k$ possible values of $\sum_{S \in \mathcal{S}'} S$.

- $\binom{|\mathcal{S}|}{|\mathcal{S}|/2} \leq |\mathcal{S}|^k$
- $|\mathcal{S}| = O(k \log k)$

Quality-$(1+\varepsilon)$ Cut Sparsifier for Quasi-Bipartite Graphs

Imaginal Vertex

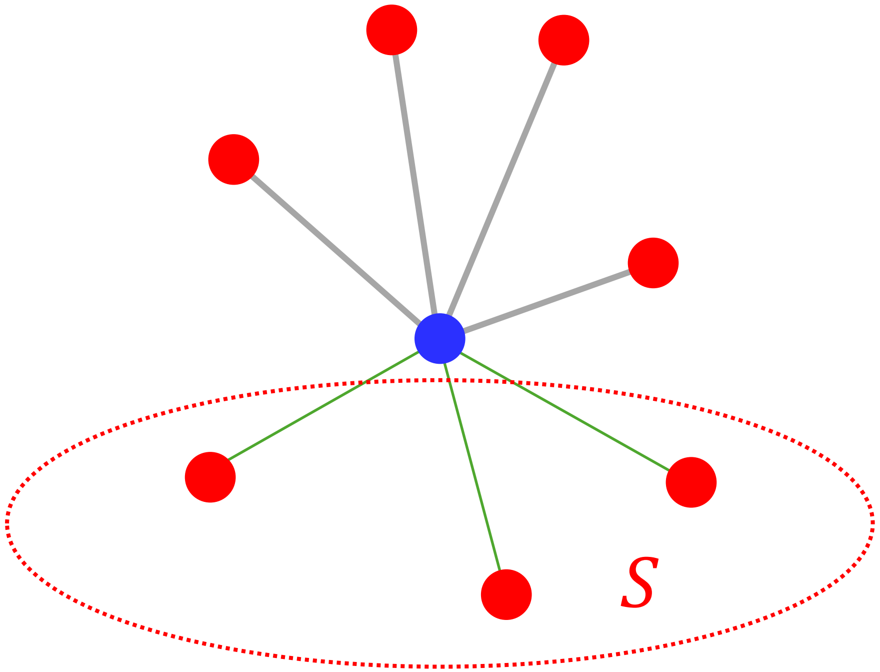
- Each vertex v will randomly choose an imaginal vertex v'.
- The number of possible imaginal vertices is small.
- We call the profile of v' as the virtual profile of v.
- Vertices with the same virtual profile will be contracted together.

Idea

- The contribution of a vertex v to a terminal cut S will change only when $\pi^v(S) \neq \pi^{v'}(S)$.

- In expectation, the contribution of v to each terminal cut will go up by a factor of $(1+\varepsilon)$.

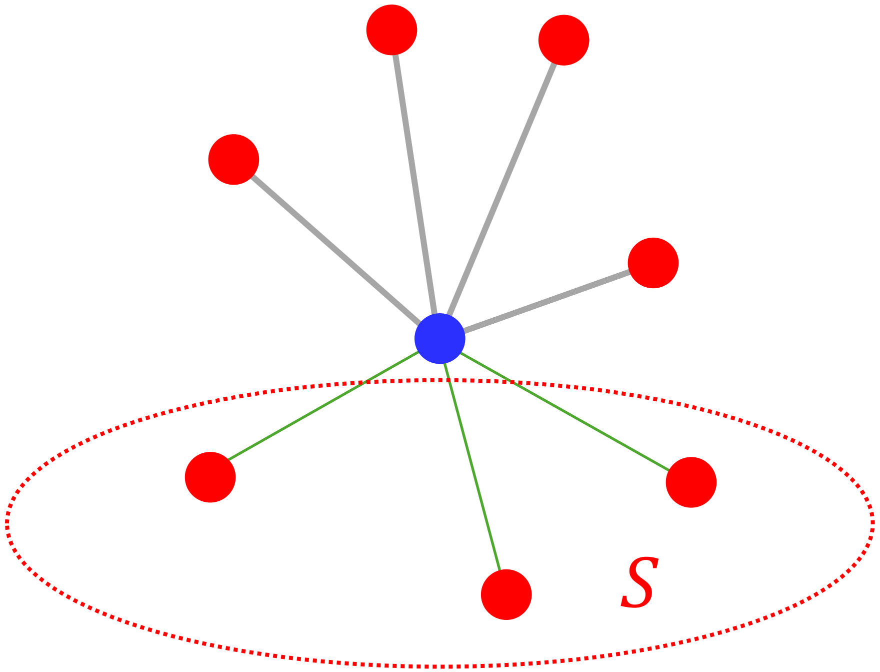
- We then prove concentration for the size of each terminal cut.



Choosing Imaginal Vertex

- We randomly choose $\Theta(1/\varepsilon^2)$ terminal, and the probabilities are proportional to the edge weights.

- The imaginal vertex v' connects to the chosen terminals, the weights of the edges to each terminal are the same and the total weight equals $w_v(T)$.

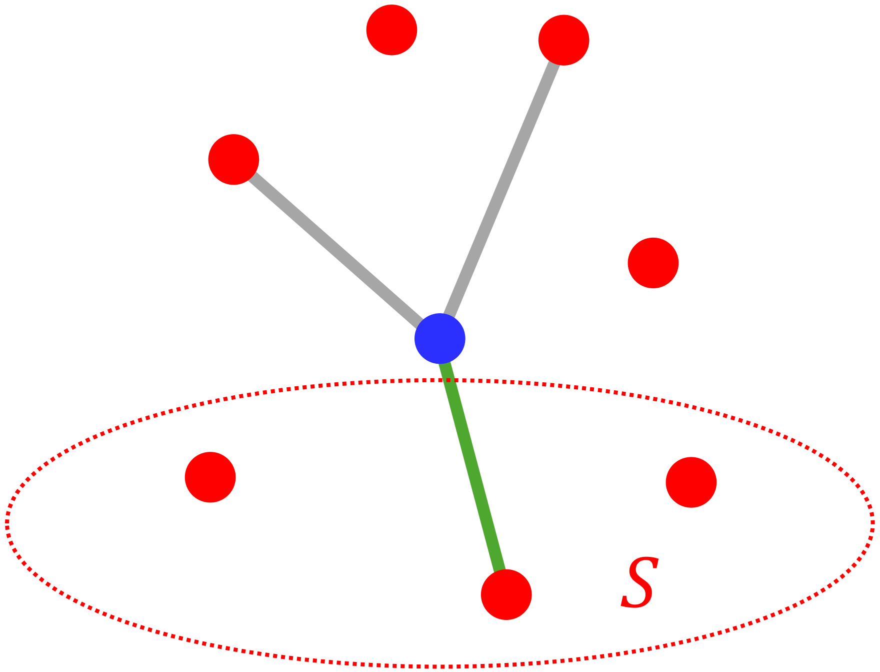


Choosing Imaginal Vertex

- We randomly choose $\Theta(1/\varepsilon^2)$ terminal, and the probabilities are proportional to the edge weights.

- The imaginal vertex v' connects to the chosen terminals, the weights of the edges to each terminal are the same and the total weight equals $w_v(T)$.

- If $w_v(S)$ far away from $w_v(T-S)$, the probability of $\pi^v(S) \neq \pi^{v'}(S)$ is very small.
- If $w_v(S)$ is close to $w_v(T-S)$, then the contribution of v does not change a lot even if $\pi^v(S) \neq \pi^{v'}(S)$



Concentration

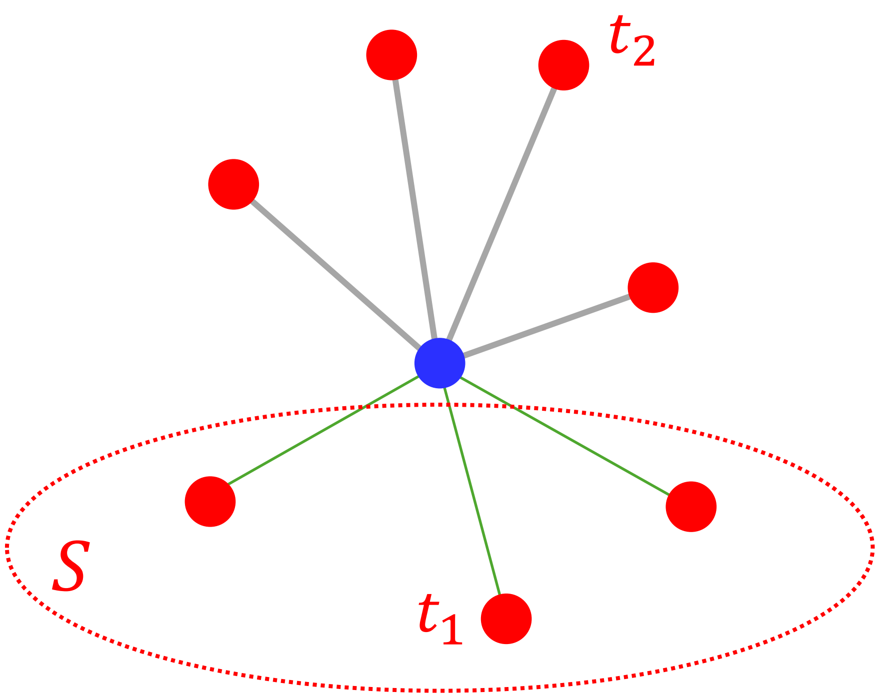
- Terminal cut size = sum of the contribution of all vertices.
- Difficulty: very few vertices contribute most of the weight.
- If a vertex contributes at least $\Omega(1/k\varepsilon^2)$ fraction of some terminal cut size, we say the vertex is important, and does not choose imaginal vertex.
- **Lemma:** the number of important vertices is polynomial.

Important Vertex

- Important cut: for any pair of terminals (t_1, t_2), we say the minimal terminal cut that separates t_1 and t_2 as important cut.

Suppose $\min\{w_v(S), w_v(T-S)\} = \alpha \cdot size(S)$

Exist $t_1 \in S$, $t_2 \notin S$, $w_v(t_1), w_v(t_2) \geq \frac{\alpha}{k} \cdot size(S)$



Important Vertex

- Important cut: for any pair of terminals (t_1, t_2), we say the minimal terminal cut that separates t_1 and t_2 as important cut.

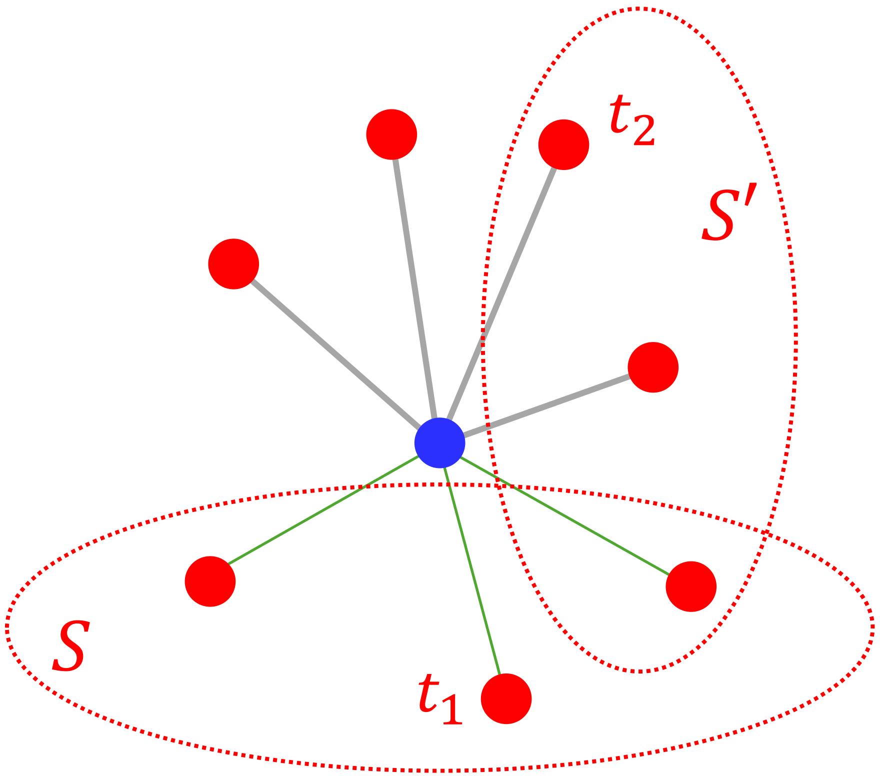
Suppose $\min\{w_v(S), w_v(T-S)\} = \alpha \cdot size(S)$

Exist $t_1 \in S$, $t_2 \notin S$, $w_v(t_1), w_v(t_2) \geq \frac{\alpha}{k} \cdot size(S)$

Let S' be the minimum terminal cut separates t_1 and t_2

$size(S') \leq size(S)$

$\min\{w_v(S'), w_v(T-S')\} = \frac{\alpha}{k} \cdot size(S')$



Important Vertex

- Important cut: for any pair of terminals (t_1, t_2), we say the minimal terminal cut that separates t_1 and t_2 as important cut.

Suppose $\min\{w_v(S), w_v(T-S)\} = \alpha \cdot size(S)$

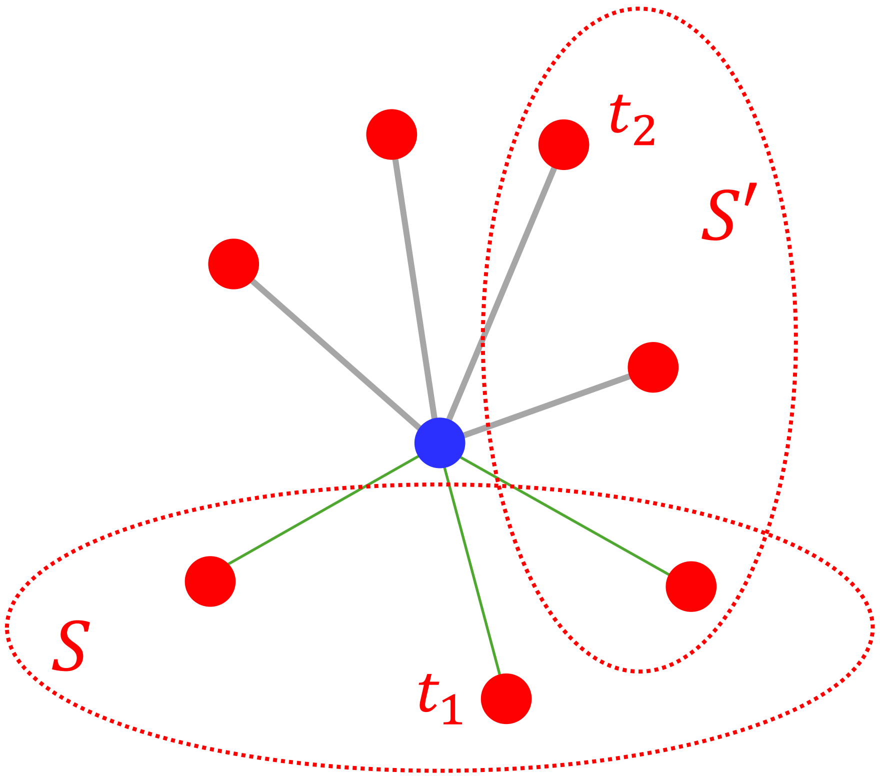
Exist $t_1 \in S$, $t_2 \notin S$, $w_v(t_1), w_v(t_2) \geq \frac{\alpha}{k} \cdot size(S)$

Let S' be the minimum terminal cut separates t_1 and t_2

$size(S') \leq size(S)$

$\min\{w_v(S'), w_v(T-S')\} = \frac{\alpha}{k} \cdot size(S')$

Lemma: Any important vertex contributions at least $\Omega(1/k^2\varepsilon^2)$ fraction of the size of some important cut.



Future Direction

- What about quality-$(1 + \varepsilon)$ cut sparsifier for general graph?
  - Can it be polynomial size like Planar graph and Quasi-Bipartite graph?
  - Or can we proof an exponential lower bound?

Thanks for Listening!